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Normability of gamma spaces

Filip Soudský

Abstract. We give a full characterization of normability of Lorentz spaces Γ^p_w. This result is in fact known since it can be derived from Kamińska A., Maligranda L., *On Lorentz spaces*, Israel J. Funct. Anal. **140** (2004), 285–318. In this paper we present an alternative and more direct proof.

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1. Introduction and the main result

In this paper we present a complete characterization of those parameters p and w, where $p \in (0,1)$ and w is a nonnegative measurable function (weight), for which the corresponding classical Lorentz space Γ^p_w (the precise definition is given below) is normable. By this we mean that the functional $\|\cdot\|_{\Gamma^p_w}$ is equivalent to a norm. We in fact prove two characterizations, quite different in nature. One of them is a certain integrability condition on the weight while the other states that the corresponding space coincides with the space $L^1 + L^\infty$. The proofs are based on a combination of discretization and weighted norm inequalities.

This result is in fact known as it can be derived from Theorem 2.1 in [4] characterizing isomorphic copies of l^p in the space Γ. We present here a new elementary proof which does not go beyond the scope of the classical Lorentz spaces.

We recall that classical Lorentz spaces of type Λ were first introduced by Lorentz in 1951 ([5]) while their modification of type Γ was developed first in 1990 by Sawyer ([6]) in connection with their crucial duality properties. These spaces proved to be extremely useful for a wide range of applications and have been studied ever since by many authors (e.g., [1], [3], [8], [7]). Normability of spaces of type Λ has been characterized long time ago (see [6] and [2]).

The result is a contribution to the long-standing research of functional properties such as linearity, (quasi)-normability etc., of classical Lorentz spaces of various types (see, e.g. [5], [1], [6], [3]).

During the whole paper, the underlying measure space $(\mathcal{R},\mu)$ is always nonatomic and σ-finite with $\mu(\mathcal{R}) = \infty$. We shall also use the symbol $\mathcal{M}(\mathcal{R})$ for

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the set of all real-valued measurable functions defined on $\mathcal{R}$. For a measurable, real-valued function f on such a space, a *non-increasing rearrangement* of f is defined by

$$f^*(t) := \inf\left\{s : \mu\left(\{|f| > s\}\right) \le t\right\},$$

while the *maximal function* of f is given by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s.$$

Throughout all of this paper the expression *weight* will always be used for positive, measurable function defined on $(0,\infty)$.

Definition 1. Let $0 < p < \infty$ and let w be a weight. Set

$$\Lambda^p_w := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Lambda^p_w} := \left(\int_0^\infty f^*(s)^p w(s)\mathrm{d}s\right)^{\frac{1}{p}} < \infty \right\}$$

and

$$\Gamma^p_w := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Gamma^p_v} := \left(\int_0^\infty f^{**}(s)^p w(s)\mathrm{d}s\right)^{\frac{1}{p}} < \infty \right\}.$$

Furthermore in the following text we shall use notation $X := \Gamma^p_w$. In order to avoid the technical difficulties, we shall assume that w is locally integrable and

$$\int_a^\infty w(s)s^{-p}\mathrm{d}s < \infty, \tag{1.1}$$

for all $a > 0$. We may also assume this without loss of generality, since if $w \notin L^1_{\mathrm{loc}}$ or (1.1) is not satisfied, then $\Gamma^p_w = \{0\}$. In the following text function W will be defined as

$$W(t) := \int_0^t w(s)\mathrm{d}s.$$

We recall that the space $L^1 + L^\infty$ consists of all functions $f \in \mathcal{M}(\mathcal{R})$ for which there exists a decomposition $f = g + h$ such that $g \in L^1$ and $f \in L^\infty$, and it is equipped with the norm

$$\|f\|_{L^1+L^\infty} := \int_0^1 f^*(s)\mathrm{d}s.$$

Let us also recall the definition of norm in weighted Lebesgue space on $(0,\infty)$ which shall be also used in the proof, namely

$$\|f\|_{L^p_w} := \left(\int_0^\infty |f(s)|^p w(s)\mathrm{d}s\right)^{\frac{1}{p}}.$$

Remark 1. The equivalence of condition (ii) and (iii) in the following theorem can be obtained from [4, Proposition 1.4], while the equivalence of (i) and (iii) from [4, Theorem 2.1].

Theorem 1. *Let $0 < p < 1$ and let w be a weight. Then the following conditions are equivalent.*

(i) *The space Γ^p_w is normable.*
(ii) *Both $w(s)$ and $w(s)s^{-p}$ are integrable on $(0,\infty)$.*
(iii) *The identity*

$$\Gamma^p_w = L^1 + L^\infty$$

holds in the sense of equivalent norms.

2. Proof of Theorem 1

Lemma 1. *Let X be a linear vector space. Let $\sigma : X \to [0,\infty)$ be a positively homogenous functional. Then the following conditions are equivalent:*

(i) *σ is equivalent to a norm;*
(ii) *there exists a constant C, independent on N, such that*

$$\sigma\left(\sum_{k=1}^N f_k\right) \le C \sum_{k=1}^N \sigma(f_k),$$

for all $f_k \in X$.

Proof of Lemma 1: First let us suppose that (i) holds. Denote the equivalent norm by ϱ. Then we have

$$\sigma\left(\sum_{k=1}^N f_k\right) \le C\varrho\left(\sum_{k=1}^N f_k\right) \le C\sum_{k=1}^N \varrho(f_k) \le C\sum_{k=1}^N \sigma(f_k).$$

Now, suppose that (2) holds. Denote

$$\varrho(f) := \inf\left(\sum_{k=1}^N \sigma(f_k)\right),$$

where the infimum on the right-hand side is taken over all finite decompositions of f, i.e.,

$$\sum_{k=1}^N f_k = f. \tag{2.1}$$

Then obviously

$$\varrho(f) \le \sigma(f),$$

for all $f \in X$. On the other hand, for all f_k satisfying (2.1) we have

$$C\left(\sum_{k=1}^{N} \sigma(f_k)\right) \geq \sigma(f).$$

Passing to the infimum on the left-hand side gives

$$C\varrho(f) \geq \sigma(f).$$

Now, take $f_1, f_2 \in X$. Let

$$\sum_{k=1}^{N_1} f_k^1 = f_1, \quad \sum_{k=1}^{N_2} f_k^2 = f_2,$$

then

$$\varrho(f_1 + f_2) \leq \sum_{k=1}^{N_1} \sigma(f_k^1) + \sum_{k=1}^{N_2} \sigma(f_k^2).$$

By passing to the infimum on the right-hand side we obtain the triangle inequality for ϱ. □

Proof of Theorem 1: Let us first prove that (i) implies (ii). We shall give an indirect proof. Suppose that (ii) is not true. Then either

$$\int_0^\infty w(s)\mathrm{d}s = \infty \tag{2.2}$$

or

$$\int_0^\infty s^{-p} w(s)\mathrm{d}s = \infty. \tag{2.3}$$

First, note that if $w \in \mathcal{B}_p$ then $\|\cdot\|_X \approx \|\cdot\|_{\Lambda_w^p}$. Since the functional $\|\cdot\|_{\Lambda_w^p}$ is not normable for $p < 1$ (as was shown in [2]), neither is $\|\cdot\|_X$. This allows us to focus on the case when $w \notin \mathcal{B}_p$. Therefore we may suppose that there exists a sequence $\{a_n\}_{n=1}^\infty$ such that

$$a_n^p \int_{a_n}^\infty w(s)s^{-p}\mathrm{d}s \geq 2^n W(a_n). \tag{2.4}$$

Now let us define

$$H(t) := \frac{t^p \int_t^\infty w(s)s^{-p}\mathrm{d}s}{W(t)}. \tag{2.5}$$

Since H is continuous on $(0, \infty)$ and therefore bounded on every $[c, d] \subset (0, \infty)$, we may without loss of generality (by choosing appropriate sub-sequence) assume that either $a_n \downarrow 0$ or $a_n \uparrow \infty$. Now, let us consider three cases:

(1) $a_n \uparrow \infty$;

(2) $a_n \downarrow 0$ and (2.3) holds;
(3) $a_n \downarrow 0$, (2.2) holds and $\sup_{t>1} H(t) < \infty$ (We can assume this otherwise it is in fact Case 1).

Case 1. Now, if $a_n \uparrow \infty$, we may again without loss of generality suppose that

$$\int_{a_{n+1}}^{\infty} w(s)s^{-p}\mathrm{d}s \leq \frac{1}{2}\int_{a_n}^{\infty} w(s)s^{-p}\mathrm{d}s. \tag{2.6}$$

Fix $N \in \mathbb{N}$. Pick $\{f_k\}_{k=1}^N$, such that

(1) $\mathrm{supp}(f_{k+1}) \subset \mathrm{supp}(f_k)$,
(2) $f_k^*(s) = q_k \chi_{(0,a_k)}$, where

$$q_k = \left(a_k^p \int_{a_k}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{-\frac{1}{p}}.$$

Then (2.6) gives

$$\int_{a_n}^{\infty} w(s)s^{-p}\mathrm{d}s \leq 2\int_{a_n}^{a_{n+1}} w(s)s^{-p}\mathrm{d}s. \tag{2.7}$$

Note that

$$f_k^{**}(s) = q_k\left(\chi_{(0,a_k)} + a_k s^{-1}\chi_{(a_k,\infty)}\right).$$

Now, by (2.4) we have

$$\begin{aligned}\|f_k\|_X &= q_k\left(W(a_k) + a_k^p\int_{a_k}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\ &\leq q_k\left(2a_k^p\int_{a_k}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}.\end{aligned} \tag{2.8}$$

Calculate

$$\begin{aligned}\left\|\sum_{k=1}^N f_k\right\|_X &\geq \left\|\sum_{k=1}^N f_k^{**}\chi_{(a_k,a_{k+1})}\right\|_{L_w^p} \\ &= \left(\sum_{k=1}^N q_k^p a_k^p\int_{a_k}^{a_{k+1}} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\ &\geq 2^{-\frac{1}{p}}\left(\sum_{k=1}^N q_k^p a_k^p\int_{a_k}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\ &= 2^{-\frac{1}{p}}\left(\sum_{k=1}^N 1\right)^{\frac{1}{p}} \approx N^{\frac{1}{p}}.\end{aligned}$$

The third inequality follows from (2.7), while the next one from (2.4). Therefore by Lemma 1 we obtain that $\|\cdot\|_X$ cannot be equivalently normed.

Case 2. Suppose (2.3) holds. If $a_n \downarrow 0$, define $a_0 = \infty$. We may without loss of generality suppose that

$$\int_{a_{n+1}}^{\infty} w(s)s^{-p}\mathrm{d}s \geq 2\int_{a_n}^{\infty} w(s)s^{-p}\mathrm{d}s. \tag{2.9}$$

Fix $N \in \mathbb{N}$. Now, let us pick $\{f_k\}_{k=1}^N$ with the following properties

(1) $\operatorname{supp}(f_{k+1}) \subset \operatorname{supp}(f_k)$,
(2) $f_k^* = q_k \chi_{(0,a_k)}$, where

$$q_k = \left(a_k^p \int_{a_k}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{-\frac{1}{p}}.$$

The same calculation as in (2.8) gives

$$\|f_k\| \leq 2^{\frac{1}{p}}.$$

Now, by (2.9), we have

$$\int_{a_{n+1}}^{a_n} w(s)s^{-p}\mathrm{d}s \geq \frac{1}{2}\int_{a_{n+1}}^{\infty} w(s)s^{-p}\mathrm{d}s. \tag{2.10}$$

Calculate

$$\begin{aligned}
\left\|\sum_{k=1}^{N} f_k\right\|_X &\geq \left\|\sum_{k=1}^{N-1} f_{k+1}^{**}\chi_{(a_{k+1},a_k)}\right\|_{L_w^p} \\
&= \left(\sum_{k=1}^{N-1} q_{k+1}^p a_{k+1}^p \int_{a_{k+1}}^{a_k} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\
&\geq 2^{-\frac{1}{p}}\left(\sum_{k=1}^{N-1} q_{k+1}^p a_{k+1}^p \int_{a_{k+1}}^{\infty} w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\
&= 2^{-\frac{1}{p}}\left(\sum_{k=1}^{N-1} 1\right)^{\frac{1}{p}} \approx N^{\frac{1}{p}},
\end{aligned}$$

where the third inequality follows from (2.10). Therefore, by Lemma 1, the functional is not normable.

Case 3. Now, suppose that the condition (2.2) holds. Again, if we can choose $\{a_n\}_{n=1}^{\infty}$ satisfying (2.4) and such that $a_n \uparrow \infty$, we may use the same calculation as in the previous one. Now if there is no such sequence, then the function $H(t)$

(where H is defined in (2.5)) is bounded on $[1,\infty)$. Set

$$C := 1 + \sup_{t>1} H(t).$$

Fix $N \in \mathbb{N}$. Since w is not in L^1, we may choose $\{a_k\}_{k=1}^{\infty}$ such that

$$W(a_{k+1}) \geq 2W(a_k), \tag{2.11}$$

and $a_1 > 1$. Observe that

$$\int_{a_{k-1}}^{a_k} w(s)\mathrm{d}s \geq \frac{1}{2} W(a_k), \tag{2.12}$$

for $k = 1, \dots, N$. Find a sequence $\{f_k\}_{k=1}^{N}$ such that

(1) $\operatorname{supp}(f_k) \subset \operatorname{supp}(f_{k+1})$,
(2) $f_k^*(s) = b_k \chi_{(0,a_k)}$, where $b_k = W^{-\frac{1}{p}}(a_k)$.

For technical reasons, set $a_0 := 0$. We have

$$\begin{aligned}
\|f_k\|_X &= W^{-\frac{1}{p}}(a_k) \left(W(a_k) + a_k \int_{a_k}^{\infty} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq W^{-\frac{1}{p}}(a_k) \left[W(a_k)(1 + \sup_{t>1} H(t)) \right]^{\frac{1}{p}} = C^{\frac{1}{p}}.
\end{aligned}$$

Calculate

$$\begin{aligned}
\left\| \sum_{k=1}^{N} f_k \right\|_X &\geq \left\| \sum_{k=1}^{N} \chi_{(a_{k-1},a_k)} b_k \right\|_{L^p_w} \\
&= \left(\sum_{k=1}^{N} b_k^p \int_{a_k}^{a_{k+1}} w(s) \mathrm{d}s \right)^{\frac{1}{p}} \\
&\geq 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} b_k^p W(a_k) \right)^{\frac{1}{p}} \\
&= 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} 1 \right)^{\frac{1}{p}} = N^{\frac{1}{p}}.
\end{aligned}$$

The third inequality follows from (2.12).

Now, let us prove that (ii) implies (iii). We shall prove that if (ii) is satisfied then

$$B \int_0^1 f^*(s)\mathrm{d}s \leq \|f\|_X \leq A \int_0^1 f^*(s)\mathrm{d}s, \tag{2.13}$$

where

$$A := \left[\int_0^1 w(s)s^{-p}\mathrm{d}s\left(1+\frac{\int_1^\infty w(s)\mathrm{d}s}{\int_1^\infty w(s)\mathrm{d}s}\right)\right]^{\frac{1}{p}}$$

and

$$B := \left(\int_0^1 w(s)\mathrm{d}s\right)^{-\frac{1}{p}}.$$

We have

$$\|f\|_X^p = \int_0^1 f^{**}(s)^p w(s)\mathrm{d}s + \int_1^\infty f^{**}(s)^p w(s)\mathrm{d}s =: \mathrm{I}+\mathrm{II}.$$

Let us first estimate the second term by the first one

$$\begin{aligned}\int_1^\infty f^{**}(s)^p w(s)\mathrm{d}s &\leq f^{**}(1)^p \int_0^1 w(s)\mathrm{d}s\left(\frac{\int_1^\infty w(s)\mathrm{d}s}{\int_0^1 w(s)\mathrm{d}s}\right)\\ &\leq \left(\frac{\int_1^\infty w(s)\mathrm{d}s}{\int_0^1 w(s)\mathrm{d}s}\right)\int_0^1 f^{**}(s)^p w(s)\mathrm{d}s.\end{aligned}$$

Now estimate

$$\begin{aligned}\int_0^1 f^{**}(s)^p w(s)\mathrm{d}s &= \int_0^1 w(s)s^{-p}\left(\int_0^s f^*(z)\mathrm{d}z\right)^p\mathrm{d}s\\ &\leq \int_0^1 w(s)s^{-p}\mathrm{d}s\left(\int_0^1 f^*(z)\mathrm{d}z\right)^p.\end{aligned}$$

Due to this two estimates we have

$$\|f\|_X^p \leq A^p\left(\int_0^1 f^*(s)\mathrm{d}s\right)^p.$$

On the other hand note that

$$\begin{aligned}\left(\int_0^1 f^*(s)\mathrm{d}s\right)^p = f^{**}(1)^p &= \left(\int_0^1 w(s)\mathrm{d}s\right)^{-1} f^{**}(1)^p\int_0^1 w(s)\mathrm{d}s\\ &\leq B^p\int_0^1 f^{**}(s)^p w(s)\mathrm{d}s \leq B^p\,\|f\|_X^p.\end{aligned}$$

Therefore the desired equivalence (2.13) holds. □

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: soudsky@karlin.mff.cuni.cz

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