Some results on spaces with $\aleph_1$-calibre

Wei-Feng Xuan, Wei-Xue Shi

Abstract. We prove that, assuming *CH*, if X is a space with $\aleph_1$-calibre and a zeroset diagonal, then X is submetrizable. This gives a consistent positive answer to the question of Buzyakova in *Observations on spaces with zeroset or regular G_δ-diagonals*, Comment. Math. Univ. Carolin. **46** (2005), no. 3, 469–473. We also make some observations on spaces with $\aleph_1$-calibre.

Keywords: $\aleph_1$-calibre; star countable; zeroset diagonal

Classification: Primary 54D20; Secondary 54E35

1. Introduction

H. Martin in [6] proved that a separable space with a zeroset diagonal is submetrizability. However, having a zeroset diagonal does not guarantee submetrizable. Recall that a space has $\aleph_1$-calibre if every uncountable family of open sets contains an uncountable subfamily with non-emptyset intersection. It is clear that every separable space has $\aleph_1$-calibre. Naturally, Buzyakova in [1] posted the following question.

Question 1.1. *Let X have $\aleph_1$-calibre and a zeroset diagonal. Is X submetrizable?*

In this paper, we prove that, assuming *CH*, if X is a space with $\aleph_1$-calibre and a zeroset diagonal, then X is submetrizable. This gives a consistent positive answer to the Question 1.1. We also make some observations on spaces with $\aleph_1$-calibre.

2. Notation and terminology

All the spaces are assumed to be Hausdorff unless otherwise stated.

Definition 2.1. A space X has a zeroset diagonal if there is a continuous mapping $f : X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x,x) : x \in X\}$.

Definition 2.2. A space X is called submetrizable if there exists a continuous injection of X into a metrizable space.

Clearly, every submetrizable space has a zeroset diagonal. Note that there is a space which has a zeroset diagonal but not submetrizable [7].

DOI 10.14712/1213-7243.2015.154

The authors are supported by NSFC project 11271178.

Definition 2.3. A space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset A of X such that $\mathrm{St}(A,\mathcal{U}) = X$, where $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

This notation was first introduced and studied by S. Ikenaga in [5]. Sometimes a star countable space is also called that it has countable weak extent.

Lemma 2.4. *Δ-system Lemma states that every uncountable collection of finite sets contains an uncountable Δ-system, i.e., a collection of sets whose pairwise intersection is constant.*

All notation and terminology not explained here is given in [4].

3. Results

Theorem 3.1. *Assume* CH. *If X is a space with $\aleph_1$-calibre and a zeroset diagonal, then X is submetrizable.*

Proof: In [3], it has been proved that if X has a zeroset diagonal and X^2 is star countable, then X is submetrizable. So, it is sufficient to prove that X^2 is star countable. Notice that a space with $\aleph_1$-calibre has countable Souslin number and a zeroset diagonal implies a regular G_δ-diagonal. We can apply a known result from [2] that the cardinality of a space with a regular G_δ-diagonal and countable Souslin number is at most $\mathfrak{c}$ to conclude that $|X| \leq \mathfrak{c}$. Clearly, $|X^2| \leq \mathfrak{c}$, and hence $|X^2| \leq \aleph_1$ since CH. Assume $|X^2| = \aleph_1$. Enumerate X^2 as $\{x_\alpha : \alpha < \aleph_1\}$.

Suppose that X^2 is not star countable. Then there exists an open cover $\mathcal{U}$ of X^2 such that for any countable subset A of X^2, $X^2 \setminus \mathrm{St}(A,\mathcal{U}) \neq \emptyset$. It is clear that $\overline{A} \subset \mathrm{St}(A,\mathcal{U})$. In fact, for any $x \in \overline{A}$, there exists an open set $U \in \mathcal{U}$ which contains x, satisfying that $U \cap A \neq \emptyset$, and hence $x \in U \subset \mathrm{St}(A,\mathcal{U})$. So, $X^2 \setminus \overline{A} \neq \emptyset$. For each $\alpha < \aleph_1$, let $U_\alpha = X^2 \setminus \overline{\{x_\beta : \beta < \alpha\}}$. Then $\{U_\alpha : \alpha < \aleph_1\}$ is an uncountable decreasing family of non-empty open sets of X^2 and $\bigcap\{U_\alpha : \alpha < \aleph_1\} = \emptyset$. However, since X has $\aleph_1$-calibre hence X^2 also has $\aleph_1$-calibre [4, p. 116], which implies that $\bigcap\{U_\alpha : \alpha < \aleph_1\} \neq \emptyset$. This is a contradiction! $\square$

Theorem 3.1 gives a consistent positive answer to the Question 1.1. A natural question then arises: Assume $\neg$ CH. Let X have $\aleph_1$-calibre and $|X| \leq \mathfrak{c}$. Is X^2 star countable? The answer to this question is negative. The following examples will show that we cannot drop the assumption of CH.

Example 3.2. Assume $2^{\aleph_1} = \mathfrak{c}$. There is a space X having $\aleph_1$-calibre and $|X| = \mathfrak{c}$, however, X^2 is not star countable.

Proof: Let $X = \{x \in D^{\mathfrak{c}} : 0 < |\{\alpha < \mathfrak{c} : x(\alpha) = 1\}| \leq \aleph_1\}$, where $D = \{0, 1\}$. Clearly, since $2^{\aleph_1} = \mathfrak{c}$, then $|X| = \mathfrak{c}^{\aleph_1} = (2^{\aleph_1})^{\aleph_1} = 2^{\aleph_1} = \mathfrak{c}$.

We firstly prove that X has $\aleph_1$-calibre. For any finite partial function $\varphi : \mathfrak{c} \to D$, let $B(\varphi) = \{x \in X : x|_{\operatorname{dom}\varphi} = \varphi\}$; then the sets $B(\varphi)$ are a base of X. Let $\mathcal{U} = \{U_\alpha : \alpha < \aleph_1\}$ be a family of open sets in X. For $\alpha < \aleph_1$ let φ_α be a finite partial function from $\mathfrak{c}$ to D such that $B(\varphi_\alpha) \subset U_\alpha$, and let $S_\alpha = \operatorname{dom}\varphi_\alpha$. By the Δ-system Lemma, there is an uncountable subset $\Lambda \subset \aleph_1$ and a finite $S \subset \mathfrak{c}$ such that $S_\xi \cap S_\eta = S$ whenever $\xi, \eta \in \Lambda$ and $\xi \neq \eta$. Since S is finite, there is an uncountable $\Lambda_0 \subset \Lambda$ such that $\varphi_\xi|_S = \varphi_\eta|_S$ whenever $\xi, \eta \in \Lambda_0$, and hence $\bigcap_{\alpha\in\Lambda_0} U_\alpha \supset \bigcap_{\alpha\in\Lambda_0} B(\varphi_\alpha) \neq \emptyset$. Thus, X has $\aleph_1$-calibre.

To show that X^2 is not star countable, we only need to prove that X is not star countable. For $\alpha < \mathfrak{c}$ let $\varphi_\alpha = \langle \alpha, 1\rangle$ and $U_\alpha = B(\varphi_\alpha)$; clearly $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$ is an open cover of X. Let A be any countable subset of X, and let $S = \bigcup_{x\in A}\{\alpha < \mathfrak{c} : x(\alpha) = 1\}$. It is easy to see that $|S| \leq \aleph_1 < 2^{\aleph_1} = \mathfrak{c}$, so there is some $\gamma \in \mathfrak{c} \setminus S$. Let x be the unique point of X such that $x(\gamma) = 1$ and $x(\alpha) = 0$ for any other $\alpha < \mathfrak{c}$. Suppose that there exists U_α of $\mathcal{U}$ such that $U_\alpha \cap A \neq \emptyset$ and $x \in U_\alpha$. Then $x(\alpha) = 1$ and hence $\alpha = \gamma \notin S$. However, let $y \in U_\alpha \cap A$; clearly, $y(\alpha) = 1$, and hence $\alpha \in S$. This is a contradiction. Thus $\mathrm{St}(A, \mathcal{U}) \neq X$. This shows X is not star countable. □

Example 3.3. Assume MA+ $\neg$ CH. There is a first countable space X with $\aleph_1$-calibre, however, X^2 is not star countable.

Proof: Let X be the space of all nonempty compact nowhere dense subsets of $\mathbb{R}$ with the Pixley-Roy topology. A neighbourhood for $x \in X$ is obtained by taking a neighbourhood U of x on the real line and letting $[x, U] = \{y \in X : x \subset y \subset U\}$. Clearly, $|X| = \mathfrak{c}$. It is shown in [8] that X is a first countable space with $\aleph_1$-calibre.

To show that X^2 is not star countable, we only need to prove that X is not star countable. Let $\mathcal{U} = \{[r, \mathbb{R}] : r \in \mathbb{R}\}$ be an open cover of X. Let A be any countable subset of X. It was established in Baire category theorem that a non-empty complete metric space is not the countable union of nowhere-dense closed sets so $\mathbb{R} \setminus \bigcup A \neq \emptyset$. We pick some $r_0 \in \mathbb{R} \setminus \bigcup A$. Hence, $r_0 \notin \mathrm{St}(A, \mathcal{U})$, since $[r_0, \mathbb{R}]$ is the only element of $\mathcal{U}$ containing r_0 and $[r_0, \mathbb{R}] \cap A = \emptyset$. This shows X is not star countable. □

We say that X has countable tightness if $x \in \overline{A}$ for any A of X, then there exists a countable subset A_0 of A such that $x \in \overline{A_0}$; it is denoted by $t(X) = \aleph_0$.

Proposition 3.4. *Let X be a space with $\aleph_1$-calibre and $t(X) = \aleph_0$. If $d(X) \leq \aleph_1$, then X is separable.*

Proof: Since $d(X) \leq \aleph_1$, there exists a dense subset A of X with $|A| \leq \aleph_1$. If $|A| < \aleph_1$, it is obvious that X is separable. We assume that $|A| = \aleph_1$. Enumerate A as $\{x_\alpha : \alpha < \aleph_1\}$ and let $F_\alpha = \overline{\{x_\beta \in A : \beta < \alpha\}}$ for each $\alpha < \aleph_1$. Then we have an $\aleph_1$-sequence $\mathcal{F} = \{F_\alpha : \alpha < \aleph_1\}$ of increasing closed separable subsets of X. For any point $x \in X$, $x \in \overline{A}$. Since $t(X) = \aleph_0$, there exists a countable subset A_0 of A such that $x \in \overline{A_0}$, and hence there exists some F_α such that $x \in A_0 \subset F_\alpha$. Thus $\bigcup \mathcal{F} = X$. We prove that there exists some $F_\alpha = X$. If $F_\alpha \neq X$ for any $\alpha < \aleph_1$ then the family $\{X \setminus F_\alpha : \alpha < \aleph_1\}$ is point-countable and uncountable which is a contradiction. Therefore $F_\alpha = X$ for some $\alpha < \aleph_1$, and hence X is separable. □

References

[1] Buzyakova R.Z., *Observations on spaces with zeroset or regular G_δ-diagonals*, Comment. Math. Univ. Carolin. **46** (2005), no. 3, 469–473.

[2] Buzyakova R.Z., *Cardinalities of ccc-spaces with regular G_δ-diagonals*, Topology Appl. **153** (2006), no. 11, 1696–1698.

[3] Basile D., Bella A., Ridderbos G.J., *Weak extent, submetrizability and diagonal degrees*, Houston J. Math. **40** (2014), no. 1, 255–266.

[4] Engelking R., *General Topology*, Heldermann, Berlin, 1989.

[5] Ikenaga S., *A class which contains Lindelöf spaces, separable spaces and countably compact spaces*, Memoirs of Numazu College of Technology **18** (1983), 105–108.

[6] Martin H.W., *Contractibility of topological spaces onto metric spaces*, Pacific J. Math. **61** (1975), no. 1, 209–217.

[7] Wage M.L., Fleissner W.G., Reed G.M., *Normality versus countable paracompactness in perfect spaces*, Bull. Amer. Math. Soc. **82** (1976), no. 4, 635–639.

[8] Tall F.D., *First Countable Space with $\aleph_1$-Calibre May or May not be Separable*, Set-theoretic Topology, Academic Press, New York, 1977.

College of Science, Nanjing Audit University, Nanjing, China, 210093

E-mail: wfxuan@nau.edu.cn

Department of Mathematics, Nanjing University, Nanjing, China, 210093

E-mail: wxshi@nju.edu.cn

(*Received* May 2, 2015*, revised* July 10, 2015)