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Differences of two semiconvex functions on the real line

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Abstract. It is proved that real functions on $\mathbb{R}$ which can be represented as the difference of two semiconvex functions with a general modulus (or of two lower C^1-functions, or of two strongly paraconvex functions) coincide with semismooth functions on $\mathbb{R}$ (i.e. those locally Lipschitz functions on $\mathbb{R}$ for which $f'_+(x) = \lim_{t\to x+} f'_+(t)$ and $f'_-(x) = \lim_{t\to x-} f'_-(t)$ for each x). Further, for each modulus ω, we characterize the class DSC_ω of functions on $\mathbb{R}$ which can be written as $f = g - h$, where g and h are semiconvex with modulus $C\omega$ (for some $C > 0$) using a new notion of $[\omega]$-variation. We prove that $f \in DSC_\omega$ if and only if f is continuous and there exists $D > 0$ such that f'_+ has locally finite $[D\omega]$-variation. This result is proved via a generalization of the classical Jordan decomposition theorem which characterizes the differences of two ω-nondecreasing functions (defined by the inequality $f(y) \geq f(x) - \omega(y - x)$ for $y > x$) on $[a, b]$ as functions with finite $[2\omega]$-variation. The research was motivated by a recent article by J. Duda and L. Zajíček on Gâteaux differentiability of semiconvex functions, in which surfaces described by differences of two semiconvex functions naturally appear.

Keywords: semiconvex function with general modulus; difference of two semiconvex functions; ω-nondecreasing function; $[\omega]$-variation; regulated function

Classification: Primary 26A51; Secondary 26B05, 26A45, 26A48

1. Introduction

In this article we investigate real functions on $\mathbb{R}$ which can be represented as the difference of two semiconvex functions (with a general modulus).

Semiconvex functions (and dual semiconcave functions) with general modulus form an important class of functions, cf. the monograph [4]. Note that (see [5, p. 239]) semiconvex functions on a Banach space X essentially coincide with strongly paraconvex functions of Rolewicz and also with uniformly Fréchet subdifferentiable functions and, if $X = \mathbb{R}^n$, then locally semiconvex functions coincide with Spingarn's lower C^1 functions, with approximately convex functions (in the sense of [12]), and also with weakly convex functions in Nurminskii's sense.

The present research is motivated by article [6] on Gâteaux differentiability of semiconvex functions. Namely, a special case of a result of [6] says that the

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set of all nondifferentiability points of a semiconvex function (resp. of a function semiconvex with modulus ω) on $\mathbb{R}^n$ can be covered by a sequence (H_n) of hypersurfaces described by differences of two semiconvex functions (resp. of two semiconvex functions with modulus $C_n\omega$, $C_n > 0$). So a natural question arises, whether functions which are differences of two functions semiconvex (resp. semiconvex with modulus $C\omega$) on $\mathbb{R}^k$ have a nice "internal" characterization.

The class of such functions for $\omega = 0$ coincides with the important class of DC functions (differences of two continuous convex functions), for which, in the case $k = 1$, a simple internal characterization is well-known (f is DC on (a, b) if and only if f is continuous and f'_+ exists and has locally finite variation on (a, b)).

Our first main result (Theorem 5.1) generalizes this characterization of DC functions. It shows that f on (a, b) belongs to the class DSC_ω (i.e., $f = g - h$, where g and h are semiconvex with modulus $C_1\omega$ for some $C_1 > 0$) if and only if f is continuous, and f'_+ exists and has locally finite $[C_2\omega]$-variation for some $C_2 > 0$.

The new notion of $[\omega]$-variation of a function f on $[a, b]$ is defined as the supremum (over all partitions $a = x_0 < x_1 < \cdots < x_n = b$) of the numbers $\sum_{i=1}^{n}(|f(x_i) - f(x_{i-1})| - \omega(x_i - x_{i-1}))$.

Theorem 5.1 is a consequence of Theorem 3.6 which provides a generalization of the classical Jordan decomposition theorem. It characterizes the differences of two ω-nondecreasing functions (defined by the inequality $f(y) \geq f(x) - \omega(y - x)$ for $y > x$) on $[a, b]$ as functions with finite $[2\omega]$-variation.

Note that there is a little chance to find a useful internal characterization of DSC_ω functions of more variables since no such characterization of DC functions is known.

The second main result (Theorem 5.2) gives a simple characterization of DSC functions on (a, b) (i.e. those, which are differences of two semiconvex functions with some modulus). Namely, the class of DSC functions on (a, b) coincides with the class of semismooth functions on (a, b) (i.e. those locally Lipschitz functions on $\mathbb{R}$ for which $f'_+(x) = \lim_{t\to x+} f'_+(t)$ and $f'_-(x) = \lim_{t\to x-} f'_-(t)$ for each $x \in (a, b)$).

Note that semismooth functions on (a, b) coincide also with differences of two locally semiconvex functions (equivalently, with differences of two lower C^1 functions).

Theorem 5.2 is a consequence of a result on decompositions of regulated functions on $[a, b]$ (i.e. those with finite one-sided limits). This result is a part of Proposition 4.2 which gives several characterizations of regulated functions which are possibly new and of some independent interest (cf. Remark 4.3).

Theorem 5.2 gives a more transparent reformulation (which uses semismooth functions instead of DSC functions, see Remark 6.4) of the result (which follows from [6]) on the set of all nondifferentiability points of a semiconvex function on $\mathbb{R}^2$ (and, more generally, a more transparent reformulation of the result on the singular set $\Sigma_{n-1}(f)$ of a semiconvex function on $\mathbb{R}^n$, see Theorem 6.3 below).

2. Preliminaries

Recall that a real function f on an interval $J \subset \mathbb{R}$ is called *regulated* if $f(x+) := \lim_{t\to x+} f(t) \in \mathbb{R}$ (resp. $f(x-) := \lim_{t\to x-} f(t) \in \mathbb{R}$) exists at each point $x \in J$ which is not a right (resp. left) endpoint of J.

Definition 2.1. We denote by Ω the set of all functions $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0.

Definition 2.2. A real valued continuous function f on a convex subset C of a Banach space X is called *semiconvex with modulus* $\omega \in \Omega$ if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) + \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|, \tag{2.1}$$

whenever $\lambda \in [0,1]$ and $x, y \in C$.

A function is called *semiconvex on* C if it is semiconvex on C with some modulus $\omega \in \Omega$.

Definition 2.3. Let A be a convex subset of a Banach space E and $\omega \in \Omega$.

(i) A real function f on A is called a *DSC* function, if $f = g - h$ for some semiconvex functions g, h.
(ii) A real function f on A is called a DSC_ω function, if there exists $C > 0$ and two functions g, h on A which are semiconvex with modulus $C\omega$ such that $f = g - h$.

Remark 2.4. (1) The above definition of *DSC* and DSC_ω functions slightly differs from that of [6, Definition 2.13], where *Lipschitzness* of functions g, h is demanded.
(2) Obviously, f is a *DSC* function if and only if it is a DSC_ω function for some $\omega \in \Omega$.

Semismooth functions in $\mathbb{R}^n$ were used and investigated in a number of articles. They were originally defined in [11] via a property of the Clarke subdifferential. We will use the following simpler definition which is equivalent to the original one by [3, Theorem 5.1].

Definition 2.5. Let G be an open subset of a Banach space X and f a locally Lipschitz function on G. Then f is said to be semismooth if

$$f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv) - f(x)}{t} \text{ exists for every } x \in G \text{ and } v \in X, \text{ and}$$

$$f'_+(x,v) = \lim_{u\to v, t\to 0+} f'_+(x+tu, v) \text{ for every } x \in G \text{ and } v \in X.$$

For functions on the real line we obtain the following characterizations.

Lemma 2.6. *For a real function on* (a,b)*, the following assertions are equivalent.*

(1) f *is semismooth.*

(2) *f is locally Lipschitz, $f'_+(x) = \lim_{t \to x+} f'_+(t)$ and $f'_-(x) = \lim_{t \to x-} f'_-(t)$ for each $x \in (a,b)$.*
(3) *f is continuous and one (or, equivalently, each) of four Dini derivatives D^+f, D^-f, D_+f, D_-f is a regulated function on (a,b).*
(4) *At each $x \in (a,b)$, both one-sided strict derivatives*

$$\lim_{(u,v)\to(x,x), x \le u < v} \frac{f(v)-f(u)}{v-u} \quad \text{and} \quad \lim_{(u,v)\to(x,x), u < v \le x} \frac{f(v)-f(u)}{v-u}$$

exist and are finite.

Proof: The equivalence of (i) and (ii) immediately follows from our definition of semismooth functions. The equivalence of (ii), (iii) and (iv) follows quite easily from classical Dini's theorem on Dini derivatives ([2, Theorems 1.2, p. 39]. □

It is well-known that if f is a semiconvex function on (a,b), then

$$f \text{ is semismooth, and} \tag{2.2}$$

$$f'_+(x) \ge f'_-(x) \quad \text{for each} \quad x \in (a,b). \tag{2.3}$$

For (2.2) see, e.g., the proof of [6, Lemma 2.5], and for (2.3) see, e.g., [6, Lemma 2.5(i)].

It is well-known that the subdifferential of a continuous convex function on a Banach space is a monotone multivalued mapping, and that semiconvexity of a function is closely connected with a generalized monotonicity of its Clarke subdifferential (cf., e.g., [10, p. 221]). For our purposes, we will need the following notion of generalized monotonicity of a single-valued real function of one variable.

Definition 2.7. Let $\omega \in \Omega$. We say that a real function g on an interval $J \subset \mathbb{R}$ is ω-nondecreasing, if $g(x) - g(y) \le \omega(y-x)$ for every $x, y \in J$ with $x \le y$.

Using this terminology, [6, Proposition 2.8] can be reformulated in the following way.

Proposition 2.8. *Let f be a real function on an open interval $I \subset \mathbb{R}$ and $\omega \in \Omega$. Then the following assertions hold.*

(i) *If f is semiconvex with modulus ω on I, then f'_+ is (2ω)-nondecreasing on I.*
(ii) *If f is continuous, $f'_+(x)$ exists for each $x \in I$ and f'_+ is ω-nondecreasing on I, then f is semiconvex with modulus ω on I.*

Observe that if $\omega \in \Omega$, then

$$\text{each } \omega\text{-nondecreasing function on } [a,b] \text{ is bounded.} \tag{2.4}$$

Indeed, if $a \le x \le b$, then $g(x) \le g(b) + \omega(b-x) \le g(b) + \omega(b-a)$ and $g(x) \ge g(a) - \omega(x-a) \ge g(a) - \omega(b-a)$.

3. Differences of two ω-nondecreasing functions: a generalization of Jordan decomposition theorem

By a partition of an interval $[a,b]$ we mean a finite set $P \subset [a,b]$ with $\{a,b\} \subset P$. As usually, the partition P is identified with the finite sequence $a = x_0 < x_1 < \dots < x_n = b$ such that $P = \{x_0, \dots, x_n\}$.

Definition 3.1. Let $\omega \in \Omega$ and f be a function on $[a,b]$ $(a < b)$. Then we define the $[\omega]$-variation of f on $[a,b]$ as

$$V^\omega(f,[a,b]) := \sup \left\{ \sum_{i=1}^{n} (|f(x_i) - f(x_{i-1})| - \omega(x_i - x_{i-1})) \right\},$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \dots < x_n = b$ of $[a,b]$. We set $V^\omega(f,[a,a]) := 0$, if $f(a)$ is defined.

This definition differs from the well-known notion of ω-variation (see [9]) which is defined as the supremum of sums $\sum \omega(|f(x_i) - f(x_{i-1})|)$. An unusual feature of the $[\omega]$-variation is that it can be sometimes negative. The usefullness of the notion of the $[\omega]$-variation is shown by Theorem 3.6 below (from which our first main result Theorem 5.1 is deduced) which coincides for $\omega = 0$ with the classical Jordan decomposition theorem.

Before a proof of our decomposition theorem we prove several basic properties of the $[\omega]$-variation.

Lemma 3.2. *Let f be a function on $[a,b]$. Then the following assertions hold.*

(i) $V^\omega(f,[a,b]) \geq |f(b) - f(a)| - \omega(b-a) > -\infty$.

(ii) *If $a \leq c \leq b$, then*

$$V^\omega(f,[a,c]) + V^\omega(f,[c,b]) \leq V^\omega(f,[a,b]).$$

(iii) *If $a \leq c \leq b$, then*

$$V^\omega(f,[a,b]) \leq V^\omega(f,[a,c]) + V^\omega(f,[c,b]) + \omega(c-a) + \omega(b-c).$$

(iv) *If $a \leq c \leq b$, then $V^\omega(f,[a,b])$ is finite if and only if both $V^\omega(f,[a,c])$ and $V^\omega(f,[c,b])$ are finite.*

Proof: Since other cases are trivial, we can suppose that $a < b$ in (i) and $a < c < b$ in (ii) and (iii). Considering $P := \{a,b\}$, we immediately obtain (i).

To prove (ii), consider arbitrary reals A, B with $A < V^\omega(f,[a,c])$ and $B < V^\omega(f,[c,b])$. We can clearly choose partitions $P_1 = \{a = x_0 < \dots < x_n = c\}$ and $P_2 = \{c = y_0 < \dots < y_m = b\}$ such that

$$V_1 := \sum_{i=1}^{n} (|f(x_i) - f(x_{i-1})| - \omega(x_i - x_{i-1})) > A \text{ and}$$

$$V_2 := \sum_{i=1}^{m}(|f(y_i) - f(y_{i-1})| - \omega(y_i - y_{i-1})) > B.$$

Let $P := P_1 \cup P_2 = \{a = z_0 < \dots < z_{n+m} = b\}$. Then clearly

$$A + B < V_1 + V_2 = \sum_{i=1}^{n+m}(|f(z_i) - f(z_{i-1})| - \omega(z_i - z_{i-1})) \le V^{\omega}(f, [a, b]).$$

Since A, B are arbitrary, we easily obtain (ii).

To prove (iii), consider a partition $P = \{a = z_0 < \dots < z_n = b\}$. Let $c \in [z_{j-1}, z_j)$. Denote $P_1 := \{z_0, \dots, z_{j-1}, c\} = \{x_0 < x_1 < \dots < x_p\}$, $P_2 := \{c, z_j, \dots, z_n\} = \{y_0 < y_1 < \dots < y_q\}$ and $S := \sum_{i\in\{1,\dots,n\}\setminus\{j\}} (|f(z_i) - f(z_{i-1})| - \omega(z_i - z_{i-1}))$. Then we have (also in the case $c = z_{j-1}$)

$$V := \sum_{i=1}^{n}(|f(z_i) - f(z_{i-1})| - \omega(z_i - z_{i-1})) = S + |f(z_j) - f(z_{j-1})| - \omega(z_j - z_{j-1}) \text{ and}$$

$$\begin{aligned}
&V^{\omega}(f, [a, c]) + V^{\omega}(f, [c, b]) \\
&\ge \sum_{i=1}^{p}(|f(x_i) - f(x_{i-1})| - \omega(x_i - x_{i-1})) \\
&\quad + \sum_{i=1}^{q}(|f(y_i) - f(y_{i-1})| - \omega(y_i - y_{i-1})) \\
&= S + |f(c) - f(z_{j-1})| + |f(z_j) - f(c)| - \omega(c - z_{j-1}) - \omega(z_j - c) \\
&= V - |f(z_j) - f(z_{j-1})| + \omega(z_j - z_{j-1}) + |f(c) - f(z_{j-1})| \\
&\quad + |f(z_j) - f(c)| - \omega(c - z_{j-1}) - \omega(z_j - c) \\
&\ge V - \omega(c - a) - \omega(b - c),
\end{aligned} \tag{3.1}$$

which clearly implies (iii).

Finally, (i), (ii) and (iii) clearly imply (iv). □

Lemma 3.3. *Let $\omega, \tilde{\omega} \in \Omega$ and let $\delta > 0$ be such that $\tilde{\omega}(t) \le \omega(t)$ for each $t \in [0, \delta)$. Then $V^{\tilde{\omega}}(f, [a, b]) < \infty$ implies $V^{\omega}(f, [a, b]) < \infty$.*

Proof: We can suppose $a < b$. Let $a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$. Then

$$S := \sum_{i=1}^{n}(|f(x_i) - f(x_{i-1})| - \tilde{\omega}(x_i - x_{i-1})) \le V^{\tilde{\omega}}(f, [a, b]).$$

Let $B := \{1 \le i \le n : x_i - x_{i-1} \ge \delta\}$. Clearly B has at most $(b - a)/\delta$ elements, and so

$$\sum_{i=1}^{n}(|f(x_i)-f(x_{i-1})|-\omega(x_i-x_{i-1})) = S+\sum_{i=1}^{n}(\tilde{\omega}(x_i-x_{i-1})-\omega(x_i-x_{i-1}))$$
$$\leq S+\sum_{i\in B}\tilde{\omega}(x_i-x_{i-1}) \leq S+\tilde{\omega}(b-a)\frac{b-a}{\delta}.$$

Consequently, $V^{\omega}(f,[a,b]) \leq V^{\tilde{\omega}}(f,[a,b])+\tilde{\omega}(b-a)\frac{b-a}{\delta}$. □

Definition 3.4. Let $\omega \in \Omega$ and f be a real function on (a,b), $-\infty \leq a < b \leq \infty$. We will say that f has locally finite $[\omega]$-variation on (a,b) if for each $x \in (a,b)$ there exists $\delta > 0$ such that $V^{\omega}(f,[x-\delta,x+\delta]) < \infty$.

We will need the following easy fact.

Lemma 3.5. *Let $\omega \in \Omega$ and let f be a real function on (a,b), $-\infty \leq a < b \leq \infty$. Then the following assertions are equivalent.*

(i) *f has locally finite $[\omega]$-variation on (a,b).*
(ii) *f has finite $[\omega]$-variation on each $[c,d] \subset (a,b)$.*

PROOF: Let (i) hold and $[c,d] \subset (a,b)$ be given. Choose $\delta > 0$ such that $V^{\omega}(f,[c-\delta,c+\delta]) < \infty$. Using Lemma 3.2(iv), we obtain $V^{\omega}(f,[c,c+\delta]) < \infty$. If $d \leq c+\delta$, then $V^{\omega}(f,[c,d]) < \infty$ by Lemma 3.2(iv). Otherwise we obtain that $M := \{x \in (c,d] : V^{\omega}(f,[c,x]) < \infty\}$ is nonempty. Set $s := \sup M$. Using (i) and Lemma 3.2(iv), it is easy to obtain first $s \in M$ and then that the case $s < d$ is impossible. Consequently $s = d$ and so (ii) holds. Using Lemma 3.2(iv), we easily obtain the implication (ii)⇒(i). □

Theorem 3.6. *Let $\omega \in \Omega$ and let f be a real function on $[a,b]$, $a < b$. Then the following assertions are equivalent.*

(i) *$f = g - h$ for some functions g, h which are ω-nondecreasing on $[a,b]$.*
(ii) *$V^{2\omega}(f,[a,b]) < \infty$.*

PROOF: a) First suppose that g,h are as in (i). Let $P = \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Denote $I := \{1 \leq i \leq n : f(x_i)-f(x_{i-1}) \geq 0\}$ and $J := \{1 \leq i \leq n : f(x_i)-f(x_{i-1}) < 0\}$. Then, using two times that $g(x_i)-g(x_{i-1})+\omega(x_i-x_{i-1}) \geq 0$ and $h(x_i)-h(x_{i-1})+\omega(x_i-x_{i-1}) \geq 0$ for each $1 \leq i \leq n$, we obtain

$$\begin{aligned}
(3.2)\qquad & \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \\
&= \sum_{i\in I}((g(x_i)-g(x_{i-1}))-(h(x_i)-h(x_{i-1}))) \\
&\quad + \sum_{i\in J}((h(x_i)-h(x_{i-1}))-(g(x_i)-g(x_{i-1}))) \\
&\leq \sum_{i\in I}((g(x_i)-g(x_{i-1}))+\omega(x_i-x_{i-1}))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\in J}((h(x_i)-h(x_{i-1}))+\omega(x_i-x_{i-1}))\\
\le{}&\sum_{i=1}^{n}((g(x_i)-g(x_{i-1}))+\omega(x_i-x_{i-1}))\\
&+\sum_{i=1}^{n}((h(x_i)-h(x_{i-1}))+\omega(x_i-x_{i-1}))\\
={}&(g(b)-g(a))+(h(b)-h(a))+\sum_{i=1}^{n}2\omega(x_i-x_{i-1}).
\end{aligned}
$$

So we easily obtain $V^{2\omega}(f,[a,b])\le(g(b)-g(a))+(h(b)-h(a))$.

b) To prove the second implication, suppose that (ii) holds. For $x\in[a,b]$, set

$$V(x):=V^{2\omega}(f,[a,x]),\quad g(x):=\frac{1}{2}(V(x)+f(x)),\quad h(x):=\frac{1}{2}(V(x)-f(x)).$$

Lemma 3.2(iv) gives that the functions V, g and h are finite. Obviously, $f=g-h$.

To prove that g is ω-nondecreasing, we must prove that, for every $a\le x<y\le b$, we have

$$g(y)-g(x)=\frac{1}{2}(V^{2\omega}(f,[a,y])-V^{2\omega}(f,[a,x])+f(y)-f(x))\ge-\omega(y-x),$$

equivalently,

$$V^{2\omega}(f,[a,y])\ge V^{2\omega}(f,[a,x])+(f(x)-f(y))-2\omega(y-x). \tag{3.3}$$

Using Lemma 3.2(ii) (with $\omega^*:=2\omega$, $a^*:=a$, $c^*:=x$ and $b^*:=y$) and then Lemma 3.2(i) (with $\omega^*:=2\omega$, $a^*:=x$ and $b^*:=y$), we obtain

$$V^{2\omega}(f,[a,y])\ge V^{2\omega}(f,[a,x])+|f(x)-f(y)|-2\omega(y-x),$$

which implies (3.3).

Observing that $V^{2\omega}(-f,[a,x])=V^{2\omega}(f,[a,x])$ and by the fact that h is "the function g^* corresponding to the function $f^*:=-f$", we obtain that also h is ω-nondecreasing. □

Theorem 3.7. *Let $\omega\in\Omega$ and f be a real function on (a,b), $-\infty\le a<b\le\infty$. Then the following assertions are equivalent.*

(i) *$f=g-h$ for some functions g, h which are ω-nondecreasing on (a,b).*
(ii) *f has locally finite $[2\omega]$-variation on (a,b).*

Proof: The implication (i)⇒(ii) clearly follows from Theorem 3.6.

So suppose that (ii) holds. Choose points

$$a<\dots<a_2<a_1<a_0<b_0<b_1<b_2<\dots<b$$

such that $a_n \to a$ and $b_n \to b$. For each $n = 0, 1, \dots$ choose by Theorem 3.6 ω-nondecreasing functions p_n, q_n on $[a_n, b_n]$ such that $f(t) = p_n(t) - q_n(t)$, $t \in [a_n, b_n]$.

Set $g_0 := p_0$ and $h_0 := q_0$. Further we will inductively define functions g_n, h_n for $n \in \mathbb{N}$ such that

(3.4) $\qquad g_n$ and h_n are ω-nondecreasing on $[a_n, b_n]$,

(3.5) $\qquad f(t) = g_n(t) - h_n(t)$, $t \in [a_n, b_n]$, and

$\qquad g_n$ and h_n extend g_{n-1} and h_{n-1}, respectively.

So suppose that $n \in \mathbb{N}$ and the functions g_{n-1}, h_{n-1} are defined. For any $K > 0$ set

(i) $g_n^K(t) := g_{n-1}(t)$ and $h_n^K(t) := h_{n-1}(t)$ for $t \in [a_{n-1}, b_{n-1}]$;
(ii) $g_n^K(t) := p_n(t) + K$ and $h_n^K(t) := q_n(t) + K$ for $t \in (b_{n-1}, b_n]$;
(iii) $g_n^K(t) := p_n(t) - K$ and $h_n^K(t) := q_n(t) - K$ for $t \in [a_n, b_{n-1})$.

Since the functions g_{n-1}, h_{n-1}, p_n, q_n are bounded by (2.4), it is easy to see that we can choose K so large, that

$$g_n^K(x_1) \leq g_n^K(x_2) \text{ if } x_1 \in [a_n, a_{n-1}],\ x_2 \in [a_{n-1}, b_{n-1}] \text{ or } x_1 \in [a_{n-1}, b_{n-1}],\ x_2 \in [b_{n-1}, b_n]. \tag{3.6}$$

Set $g_n := g_n^K$ and $h_n := h_n^K$. Obviously, (3.5) holds. Using (3.6) and the fact that both g_n and h_n are ω-nondecreasing on intervals $[a_n, b_{n-1})$, $[a_{n-1}, b_{n-1}]$, $(b_{n-1}, b_n]$, we easily obtain that also (3.4) holds.

Now let g (resp. h) be the unique common extension of all g_n's (resp. h_n's). Then clearly g, h are ω-nondecreasing on (a, b) and $f = g - h$. □

4. Decompositions of regulated functions

We start with a characterization of functions which are ω-nondecreasing for some $\omega \in \Omega$.

Proposition 4.1. *Let f be a function on $[a, b]$, $a < b$. Then the following assertions are eqivalent.*

(i) *There exists $\omega \in \Omega$ such that f is ω-nondecreasing.*
(ii) *f is regulated, $f(a) \leq f(a+)$, $f(b-) \leq f(b)$ and $f(x_-) \leq f(x) \leq f(x+)$ for each $x \in (a, b)$.*

Proof: To prove (i)⇒(ii), choose $\omega \in \Omega$ such that f is ω-nondecreasing. We will first show that

$$f(x) \leq f(x+) < \infty \quad \text{whenever} \quad x \in [a, b). \tag{4.1}$$

So choose $x \in [a,b)$. Since f is ω-nondecreasing, we have

$$\begin{aligned} f(x) &\leq f(t) + \omega(t-x) \leq f(v) + \omega(v-t) + \omega(t-x) \\ &\leq f(v) + 2\omega(v-x) \text{ if } x < t < v < b. \end{aligned} \tag{4.2}$$

So $f(x) \leq \limsup_{t\to x+} f(t) \leq f(v) + 2\omega(v-x)$ for each $x < v \leq b$. Consequently we obtain $\limsup_{t\to x+} f(t) < \infty$ and $f(x) \leq \limsup_{t\to x+} f(t) \leq \liminf_{v\to x+} f(v)$, and (4.1) follows.

Quite analogously we obtain

$$-\infty < f(x-) \leq f(x) \quad \text{whenever} \quad x \in (a,b], \tag{4.3}$$

using that

$$f(v) - 2\omega(x-v) \leq f(t) - \omega(x-t) \leq f(x) \quad \text{whenever} \quad a < v < t < x.$$

Obviously, (4.1) and (4.3) imply (ii).

To prove (ii)⇒(i), suppose that (ii) holds. Then f is a regulated function and so it is bounded, see e.g. [8]. Set

$$\omega(t) := \sup\{f(r) - f(s) : a \leq r \leq s \leq b,\ s - r \leq t\}, \qquad t \in [0,\infty). \tag{4.4}$$

Since f is bounded, ω is finite. Clearly ω is nondecreasing and $\omega(0) = 0$. So, to prove $\omega \in \Omega$, it is sufficient to prove $\omega(0+) = 0$. Suppose the opposite. Then it is easy to see that there exists $\varepsilon > 0$ and for each $n \in \mathbb{N}$ points $r_n < s_n$ from $[a,b]$ such that $s_n - r_n < 1/n$ and $f(r_n) - f(s_n) > \varepsilon$. Considering suitable subsequences of (r_n) and (s_n), if necessary, we can suppose that there exists $c \in [a,b]$ such that $r_n \to c$ and $s_n \to c$. For formal reasons, define $f(x) := f(a)$ for $x < a$ and $f(x) := f(b)$ for $x > b$. Then we can clearly find $\delta > 0$ such that $|f(x) - f(c_-)| < \varepsilon/2$ for $x \in (c-\delta, c)$ and $|f(x) - f(c_+)| < \varepsilon/2$ if $x \in (c, c+\delta)$. We will show that $f(r) - f(s) < \varepsilon$ whenever $r < s$ are points from $(c-\delta, c+\delta)$.

This inequality is obvious if $\{r,s\} \subset (c-\delta, c)$ or $\{r,s\} \subset (c, c+\delta)$.

If $r = c$, then $f(r) - f(s) \leq f(c_+) - f(s) < \varepsilon/2$. Similarly if $s = c$, then $f(r) - f(s) \leq f(r) - f(c_-) < \varepsilon/2$.

Finally, if $r < c < s$, we obtain

$$\begin{aligned} f(r) - f(s) &= (f(r) - f(c)) + (f(c) - f(s)) \\ &\leq (f(r) - f(c_-)) + (f(c_+) - f(s)) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So, choosing n such that $\{r_n, s_n\} \subset (c-\delta, c+\delta)$, we obtain $f(r_n) - f(s_n) < \varepsilon$, which is a contradiction. So $\omega \in \Omega$. Since f is ω-monotone by (4.4), we are done. □

Proposition 4.2. *Let f be a function on $[a,b]$, $a < b$. Then the following conditions are equivalent.*

(i) *f is regulated.*

(ii) *f can be decomposed as $f = u_1 - u_2$, where the functions u_i (i=1,2) are regulated, $u_i(a) \leq u_i(a+)$, $u_i(b-) \leq u_i(b)$ and $u_i(x_-) \leq u_i(x) \leq u_i(x+)$ for each $x \in (a,b)$.*

(iii) *There exists $\omega \in \Omega$ such that f can be decomposed as $f = u_1 - u_2$, where the functions u_1, u_2 are ω-nondecreasing on $[a,b]$.*

(iv) *There exists $\omega \in \Omega$ such that $V^\omega(f,[a,b]) < \infty$.*

(v) *f can be decomposed as $f = p - q$, where the functions p, q are regulated and lower semicontinuous on $[a,b]$.*

Proof: First we will simultaneously prove that (i)⇒(ii) and (i)⇒(v). So suppose that (i) holds and denote by $\rho(x)$ the oscillation of f at x for $x \in [a,b]$. (Thus $\rho(a) = |f(a+) - f(a)|$, $\rho(b) = |f(b-) - f(b)|$ and $\rho(x) = \operatorname{diam}\{f(x-), f(x), f(x+)\}$ for $x \in (a,b)$.) For an arbitrary interval $I \subset \mathbb{R}$ denote by $c_0(I)$ the set of all $\varphi : I \to \mathbb{R}$ such that the set $\{x : |\varphi(x)| > \varepsilon\}$ is finite for each $\varepsilon > 0$. Observe that [8, Proposition 1.9] immediately implies the (easy) fact that $\rho \in c_0([a,b])$.

Denote by D the set of all regulated functions on $[a,b]$ which are left continuous on $(a,b]$. For each $g \in D$, define the function φ_g by

$$\varphi_g(x) = g(x_+) - g(x), \quad x \in [a,b).$$

It is well-known (see the proof of [7, Theorem 2.3.1]) that

$$T : g \mapsto \varphi_g \text{ is a surjection of } D \text{ onto } c_0([a,b)). \tag{4.5}$$

So we can find a regulated function g on $[a,b]$ which is left continuous on $(a,b]$ and $g(x_+) - g(x) = \rho(x)$ for each $x \in [a,b)$. By a quite symmetric way we can obtain a regulated function h on $[a,b]$ which is right continuous on $[a,b)$ and $h(x) - h(x_-) = \rho(x)$ for each $x \in (a,b]$.

Now it is easy to check that (ii) holds with

$$u_1 := \frac{1}{2}(g + h + f) \text{ and } u_2 := \frac{1}{2}(g + h - f).$$

Indeed, clearly $f = u_1 - u_2$. Further, for each $x \in [a,b)$, we have

$$u_1(x_+) - u_1(x) = \frac{1}{2}(g(x_+) - g(x) + f(x_+) - f(x)) \geq 0,$$

since $g(x_+) - g(x) = \rho(x) \geq |f(x_+) - f(x)|$. The other three inequalities for u_1 and u_2 follow quite similarly.

Setting $p := \frac{1}{2}(g - h + f)$ and $q := \frac{1}{2}(g - h - f)$ and proceeding quite similarly, we obtain the decomposition from (v).

To prove (ii)⇒(iii), suppose that u_1 and u_2 are as in (ii). By Proposition 4.1 there exist $\omega_i \in \Omega$ such that u_i is ω_i-nondecreasing, $i = 1,2$. Setting $\omega := \max(\omega_1, \omega_2)$, we obtain that u_1, u_2 are ω-nondecreasing.

By Theorem 3.6 we have (iii)⇔(iv). Since (iii)⇒(i) follows from Proposition 4.1 and (v)⇒(i) is obvious, we are done. □

Remark 4.3. As noted after Definition 3.1, ω-variation used in [9] is quite different from our $[\omega]$-variation. So it is interesting that for a function f on $[a,b]$ the following assertions are equivalent:

(i) f is regulated;
(ii) f has finite ω-variation for some $\omega \in \Omega$ with $\omega(t) > 0$ for $t > 0$;
(iii) f has finite $[\omega]$-variation for some $\omega \in \Omega$.

(Proposition 4.2 gives (i)$\Leftrightarrow$(iii) and (i)$\Leftrightarrow$(ii) follows from [9].)

Proposition 4.4. *Let f be a function on (a,b), $-\infty \le a < b \le \infty$. Then the following conditions are equivalent.*

(i) *f is regulated.*
(ii) *There exists $\omega \in \Omega$ such that f has locally finite $[\omega]$-variation.*
(iii) *There exists $\omega \in \Omega$ such that f can be decomposed as $f = g - h$, where the functions g, h are ω-nondecreasing on (a,b).*
(iv) *f can be decomposed as $f = u_1 - u_2$, where the functions u_i $(i = 1,2)$ are regulated and $u_i(x_-) \le u_i(x) \le u_i(x+)$ for each $x \in (a,b)$.*

Proof: To prove (i)$\Rightarrow$(ii), choose points $a < \dots < a_2 < a_1 < b_1 < b_2 < \dots < b$ such that $a_n \to a$ and $b_n \to b$. By Proposition 4.2 for each $n \in \mathbb{N}$ there exists $\omega_n \in \Omega$ such that $V^{\omega_n}(f,[a_n,b_n]) < \infty$. We can suppose that $\omega_n(x) > 0$ for $x > 0$ (otherwise we can work with $\widetilde{\omega}_n(x) = \omega(x) + x$ instead of ω_n). Considering $\max(\omega_1,\dots,\omega_n)$ instead of ω_n, if necessary, we can suppose $\omega_1 \le \omega_2 \le \dots$. Let $\delta_1 > \delta_2 > \dots > 0$ be a decreasing sequence such that $\delta_n \to 0$ and

$$\omega_{n+1}(\delta_{n+1}) < \min(\omega_n(\delta_n), \frac{1}{n}), \quad n \in \mathbb{N}.$$

Set $\omega(0) := 0$, $\omega(x) = \omega_1(\delta_1)$ for $x \in [\delta_2,\infty)$, and $\omega(x) = \omega_n(\delta_n)$ for $x \in [\delta_{n+1},\delta_n)$, $n \ge 2$. It is easy to see that $\omega \in \Omega$ and $\omega_n(x) \le \omega(x)$ for $x \in (0,\delta_n)$. Thus Lemma 3.3 clearly implies that f has finite $[\omega]$-variation on each $[a_n,b_n]$, and so f has locally finite $[\omega]$-variation on (a,b).

Theorem 3.7 clearly gives (ii)$\Rightarrow$(iii) and Proposition 4.1 gives (iii)$\Rightarrow$(iv). Finally, (iv)$\Rightarrow$(i) holds, since the difference of two regulated functions is clearly regulated. $\square$

5. DSC_ω and DSC functions on $\mathbb{R}$

Theorem 5.1. *Let F be a function on an open interval $I \subset \mathbb{R}$ and $\omega \in \Omega$. Then the following assertions are equivalent.*

(i) *F is DSC_ω on I.*
(ii) *F is continuous on I, has finite right-hand derivative at each point of I and there exists $D > 0$ such that F'_+ has locally finite $[D\omega]$-variation on I.*

Proof: Suppose that (i) holds. Choose $C > 0$ and functions G and H on I which are semiconvex with modulus $C\omega$ and $F = G - H$. Since G and H are locally Lipschitz and have finite right-hand derivative at each point of I by (2.2), the same also holds for F and $F'_+ = G'_+ - H'_+$. By Proposition 2.8(i) functions

G'_+ and H'_+ are $(2C\omega)$-nondecreasing on I. So Theorem 3.7 implies (ii) (with $D = 4C$).

Now suppose that (ii) holds and set $C := D/2$. By Theorem 3.7 there exist functions f_1 and f_2 which are $(C\omega)$-nondecreasing on I and $F'_+ = f_1 - f_2$. By Proposition 4.4 we obtain that the functions F'_+, f_1, f_2 are regulated, and thus they are locally Lebesgue integrable. Indeed, each regulated function on I is locally bounded (see, e.g., [8]) and continuous except for a countable set. So we can define, for a fixed $x_0 \in I$, $F_i(x) := \int_{x_0}^x f_i$, $x \in I$, for $i \in \{1, 2\}$. Further, since F'_+ is regulated, Lemma 2.3 ((iii)$\Rightarrow$(i)) gives that F is semismooth. Thus F is locally Lipschitz by (2.2) and therefore

$$F(x) = F(x_0) + \int_{x_0}^x F' = F(x_0) + \int_{x_0}^x F'_+ = F(x_0) + F_1(x) - F_2(x), \quad x \in I.$$

A classical argument on differentiation of an indefinite integral gives that $(F_i)'_+(x) = f_i(x+)$ for each $x \in I$ and $i \in \{1, 2\}$.

Note that F_1 and F_2 are clearly continuous. Since f_1 and f_2 are $(C\omega)$-nondecreasing on I,

$$\begin{aligned}(F_i)'_+(x_1) &= \lim_{h \Rightarrow 0+} f_i(x_1 + h) \le \lim_{h \Rightarrow 0+} (f_i(x_2 + h) + C\omega(x_2 - x_1)) \\ &= (F_i)'_+(x_2) + C\omega(x_2 - x_1)\end{aligned}$$

holds for each $x_1, x_2 \in I$, $x_1 < x_2$, and each $i \in \{1, 2\}$. So Proposition 2.8(ii) implies that $F(x_0) + F_1$ and F_2 are on I semiconvex with modulus $C\omega$, and so (i) holds. □

Theorem 5.2. *Let F be a function on an open interval $I \subset \mathbb{R}$. Then the following assertions are equivalent.*

(i) *F is semismooth.*
(ii) *F is a difference of two semiconvex functions.*
(iii) *F is a difference of two locally semiconvex functions* (*equivalently: of two lower C^1-functions*).

Proof: Suppose (i). Then F'_+ is regulated by Lemma 2.6. By Proposition 4.4 there exists $\omega \in \Omega$ such that F'_+ has locally finite $[\omega]$-variation on I. So F is DSC_ω on I by Theorem 5.1 and (ii) follows. The implication (ii)$\Rightarrow$(iii) is trivial.

Suppose (iii). Since each semiconvex function is semismooth (see (2.2)) and the difference of two semismooth functions is clearly semismooth, we obtain that F is locally semismooth. Thus (i) follows. □

Below (Proposition 5.4) we present an application of Theorem 5.1 which concerns a natural question on comparison of classes DSC_{ω_1} and DSC_{ω_2} with different ω_1 and ω_2. Note that the existence of g from Proposition 5.4 which is not C^1 follows from [6, Lemma 4.6]. However, the method of [6] (which does not use the notion of $[\omega]$-variation) cannot yield a smooth g. Our modification of construction from [6] uses also the following lemma ([6, Lemma 4.5], in which proof the assumption that $\omega_1(t) > 0$ for $t > 0$ was used but not assumed).

Lemma 5.3. *Let $\omega_1 \in \Omega$ and $\omega_2 \in \Omega$ be concave functions such that $\omega_1(t) > 0$ for $t > 0$ and $\lim_{t\to 0+} \frac{\omega_1(t)}{\omega_2(t)} = 0$. Then there exists a sequence $(\Delta_i)_{i=1}^{\infty}$ of positive numbers such that*

$$\sum_{i=1}^{\infty} \omega_1(\Delta_i) < \infty, \sum_{i=1}^{\infty} \Delta_i < \infty \text{ and } \sum_{i=1}^{\infty} \omega_2(\Delta_i) = \infty. \tag{5.1}$$

Proposition 5.4. *Let $\omega_1 \in \Omega$ and $\omega_2 \in \Omega$ be concave functions such that $\omega_1(t) > 0$ for $t > 0$ and $\lim_{t\to 0+} \frac{\omega_1(t)}{\omega_2(t)} = 0$. Let $I \subset \mathbb{R}$ be an open interval.*

Then there exists a C^1 function g on I which is semiconvex with modulus ω_2 and is not a DSC_{ω_1} function on I.

Proof: Choose by Lemma 5.3 a sequence $(\Delta_i)_{i=1}^{\infty}$ which fulfils (5.1). Considering, if necessary, $\widetilde{\Delta_i} := \Delta_{i+p}$ instead of Δ_i, we can suppose that the length of I is greater than $2\sum_{i=1}^{\infty} \Delta_i$. So we can choose points

$$a_1 < b_1 < a_2 < b_2 < \dots$$

such that $a_1 \in I$, $c := \lim a_i \in I$, and $b_i - a_i = a_{i+1} - b_i = \Delta_i$, $i \in \mathbb{N}$.

Let g be the function on I which is affine on each $[a_n, b_n]$ and $[b_n, a_{n+1}]$, $g(a_n) = 0$, $g(b_n) = \omega_2(\Delta_n)$, $n \in \mathbb{N}$, and $g(x) = 0$ for $x \in I \cap ((-\infty, a_1) \cup [c, \infty))$.

Now we will show that g is ω_2-nondecreasing, i.e.

$$g(x_1) \le g(x_2) + \omega_2(x_2 - x_1), \quad \text{whenever } x_1, x_2 \in I,\ x_1 < x_2. \tag{5.2}$$

Since $g \ge 0$, (5.2) is obvious if $g(x_1) = 0$. So we can suppose that $x_1 \in [a_n, a_{n+1}]$ for some n. To prove the inequality of (5.2), we will distinguish several possibilities.

a) If $x_1, x_2 \in [b_n, a_{n+1}]$, then observe that g is affine on $[b_n, a_{n+1}]$ with the slope $-\omega_2(\Delta_n)(\Delta_n)^{-1}$, and so

$$g(x_1) = g(x_2) + (x_2 - x_1) \cdot \omega_2(\Delta_n) \cdot (\Delta_n)^{-1} \le g(x_2) + \omega_2(x_2 - x_1).$$

The last inequality holds, since $\omega_2 \in \Omega$ is concave, and so $\omega_2(x_2-x_1)\cdot(x_2-x_1)^{-1} \ge \omega_2(\Delta_n)\cdot(\Delta_n)^{-1}$.

b) If $x_1 \in [b_n, a_{n+1}]$ and $x_2 > a_{n+1}$, then (using case a)) we obtain

$$g(x_1) \le g(a_{n+1}) + \omega_2(a_{n+1} - x_1) \le g(x_2) + \omega_2(x_2 - x_1).$$

c) The case $x_1, x_2 \in [a_n, b_n]$ is obvious.

d) If $x_1 \in [a_n, b_n]$ and $x_2 > b_n$, then (using cases a) and b)) we obtain

$$g(x_1) \le g(b_n) \le g(x_2) + \omega_2(x_2 - b_n) \le g(x_2) + \omega_2(x_2 - x_1).$$

Since $\lim g(b_n) = \lim \omega_2(\Delta_n) = 0$, we see from the definition of g that g is continuous. So there exists a function G defined on I which is primitive to g on I. Clearly G is a C^1 function and G is semiconvex with modulus ω_2 by (5.2) and Proposition 2.8.

Now we will prove that G is not DSC_{ω_1} on I. If the opposite holds, then we can by Theorem 5.1 choose $C > 0$ such that $V^{C\omega_1}(g, [a_1, c]) < \infty$. For each $N \in \mathbb{N}$ let

$$\begin{aligned} P_N := &\sum_{n=1}^{N} (|g(b_n) - g(a_n)| - C\omega_1(b_n - a_n)) \\ &+ \sum_{n=1}^{N} (|g(a_{n+1}) - g(b_n)| - C\omega_1(a_{n+1} - b_n)) \\ &+ |g(c) - g(a_{N+1})| - C\omega_1(c - a_{N+1}) \end{aligned}$$

be the sum corresponding to the partion $a_1 < b_1 < \cdots < a_{N+1} < c$ of the interval $[a_1, c]$ and the modulus $C\omega_1$ according to Definition 3.1. Then, using (5.1), we obtain

$$\begin{aligned} V^{C\omega_1}(g, [a_1, c]) &\geq \lim_{N \Rightarrow \infty} P_N \\ &= \lim_{N \Rightarrow \infty} \Big(\sum_{n=1}^{N} (\omega_2(\Delta_n) - C\omega_1(\Delta_n)) \\ &\quad + \sum_{n=1}^{N} (\omega_2(\Delta_n) - C\omega_1(\Delta_n)) - C\omega_1(c - a_{N+1}) \Big) \\ &= \lim_{N \Rightarrow \infty} \Big(2 \sum_{n=1}^{N} \omega_2(\Delta_n) - 2C \sum_{n=1}^{N} \omega_1(\Delta_n) - C\omega_1(c - a_{N+1}) \Big) = \infty, \end{aligned}$$

which is a contradiction. □

6. Consequences for singular sets of semiconvex functions in Euclidean spaces

A result of [6] implies that a set of singular points of a semiconvex function in $\mathbb{R}^n$ can be covered by countably many of 1-dimensional surfaces, which are described by DSC functions (equivalently by Theorem 5.2: “described by semismooth functions”). Since the definition of semismooth functions is much more transparent than that of a DSC function, we obtain much more transparent characterization of smallness of this singular set.

For precise formulation of this observation, we need some definitions and simple arguments.

First recall that if f is a semiconvex function on a Banach space and $k \in \mathbb{N}$, then we consider the set $\Sigma_k(f)$ of *singular points of order* k (or of magnitude k

by [4]) at which the Clarke subdifferential $\partial f(x)$ is at least k-dimensional. Note that $\Sigma_1(f)$ is the set of all points at which f is not Gâteaux differentiable.

Definition 6.1. Let E be a Banach space and F a finite-dimensional Banach space. Then a mapping $\varphi : E \to F$ is called a *DSC* mapping (or semismooth mapping), if $f^* \circ \varphi$ is a *DSC* (or semismooth) function for each functional $f^* \in F^*$.

Definition 6.2. We say that $A \subset \mathbb{R}^n$ $(n \geq 2)$ is a *DSC* surface (resp. a semismooth surface) of dimension $1 \leq k < n$ if there exist a k-dimensional space $E \subset \mathbb{R}^n$ and a *DSC* mapping (resp. semismooth mapping) $\varphi : E \to F := E^\perp$ such that

$$A = \{x + \varphi(x) : \ x \in E\}.$$

Note that the above definition of *DSC* surfaces formally differs from this of [6, Definition 2.15] which

a) works in an arbitrary Banach space X and so F is a topological (instead of orthogonal) complement of E;

b) works with slightly different definition of *DSC* mappings (see Remark 2.4) and so it introduces *DSC*-surfaces, which are apriori Lipschitz surfaces;

c) works with codimension of a surface instead of dimension.

However, for our application, these differences are not essential.

First, using [6, Lemma 2.16] (which shows that in $\mathbb{R}^n$ "we can consider $F := E^\perp$" only) we see that if $A \subset \mathbb{R}^n$ is a DSC surface of codimension $n-k$ in the sense of [6], then it is a DSC surface of dimension k in the sense of our definition.

Second, if $A \subset \mathbb{R}^n$ is a DSC surface of dimension k in the sense of our definition, then there exist DSC surfaces A_i $(i = 1, 2, \dots)$ of codimension $n-k$ in the sense of [6], such that $A \subset \bigcup A_i$. This statement easily follows from the fact that semiconvex functions are locally Lipschitz and [5, Theorem 5.7] which asserts that each Lipschitz semiconvex function on a bounded convex subset of a superreflexive space E has a Lipschitz semiconvex extension to the whole space E.

Further note that Theorem 5.2 easily implies that if E is a 1-dimensional and F a k-dimensional subspace of $\mathbb{R}^n$, then $\varphi : E \to F$ is *DSC* if and only if it is semismooth.

Using the above observations, we easily see that [6, Theorem 3.8(ii)] implies the following result.

Theorem 6.3. *Let $n \geq 2$. Then we have:*

(i) *if f is a semiconvex function on $\mathbb{R}^n$, then $\Sigma_{n-1}(f)$ can be covered by countably many semismooth surfaces of dimension* 1*;*

(ii) *if $A \subset \mathbb{R}^n$ is a countable union of semismooth surfaces of dimension* 1*, then there exists a semiconvex function f on $\mathbb{R}^n$ such that $A \subset \Sigma_{n-1}(f)$.*

Remark 6.4. (i) Using the extension result [5, Theorem 5.7] which was recalled above, we easily see that (i) holds also for a locally semiconvex (equivalently lower C^1 or approximately convex) function f on an open subset of $\mathbb{R}^n$.

(ii) If $n = 2$, then $\Sigma_1(f)$ coincides with the set $N(f)$ of all nondifferentiability points of f. So we obtain a simple complete characterization of smallness of sets $N(f)$ for semiconvex (locally semiconvex; lower C^1; approximately convex) functions on $\mathbb{R}^2$.

(1) Using Theorem 6.3 and known properties of semismooth functions, it is possible to show that, for each semiconvex function f on $\mathbb{R}^2$, the set $N(f)$ can be covered by 1-dimensional semismooth surfaces of the form $\{(x, y) : y = \varphi_j(x)\}$ and $\{(x, y) : x = \psi_j(y)\}$, $j = 1, 2, \ldots$, where φ_j and ψ_j are semismooth functions on $\mathbb{R}$.

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