Summation equations with sign changing kernels and applications to discrete fractional boundary value problems

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Abstract. We consider the summation equation, for $t \in [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$,

$$y(t) = \gamma_1(t)H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \gamma_2(t)H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right) + \lambda\sum_{s=0}^{b} G(t,s)f(s+\mu-1,y(s+\mu-1))$$

in the case where the map $(t,s) \mapsto G(t,s)$ may change sign; here $\mu \in (1,2]$ is a parameter, which may be understood as the order of an associated discrete fractional boundary value problem. In spite of the fact that G is allowed to change sign, by introducing a new cone we are able to establish the existence of at least one positive solution to this problem by imposing some growth conditions on the functions H_1 and H_2. Finally, as an application of the abstract existence result, we demonstrate that by choosing the maps $t \mapsto \gamma_1(t)$, $\gamma_2(t)$ in particular ways, we can recover the existence of at least one positive solution to various discrete fractional- or integer-order boundary value problems possessing Green's functions that change sign.

Keywords: summation equation; sign-changing kernel; discrete fractional calculus; positive solution; nonlocal boundary condition

Classification: Primary 39A05, 39A12, 39A99; Secondary 26A33, 47H07

1. Introduction

Let $\lambda > 0$ and $1 < \mu \leq 2$ be parameters, and let $\{\xi_i\}_{i=1}^{n}$, $\{\zeta_i\}_{i=1}^{m} \subseteq [\mu-1,\mu+b-1]_{\mathbb{N}_{\mu-2}}$ and $\{a_i\}_{i=1}^{n}$, $\{b_i\}_{i=1}^{m} \subseteq \mathbb{R}$ be given sequences of numbers; here we use the standard notation $\mathbb{N}_r := \{r, r+1, r+2, \dots\}$ for a fixed number $r \in \mathbb{R}$. We then consider the summation equation, for $t \in [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$,

$$y(t) = \gamma_1(t)H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \gamma_2(t)H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right) + \lambda\sum_{s=0}^{b} G(t,s)f(s+\mu-1,y(s+\mu-1)), \tag{1.1}$$

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where we note that the coefficients in the linear functionals $y \mapsto \sum_{i=1}^{n} a_i y(\xi_i)$ and $y \mapsto \sum_{i=1}^{m} b_i y(\zeta_i)$ need not be nonnegative. The maps $H_1, H_2 : [0,+\infty) \to [0,+\infty)$ and $\gamma_1, \gamma_2 : [0,+\infty) \to [0,+\infty)$ are continuous maps. Later we shall impose some growth conditions on the maps H_1 and H_2. One of the principal contributions of this work is to demonstrate that problem (1.1) may possess a positive solution even in case the kernel G is *sign changing*.

As an application of this abstract existence result, we shall demonstrate that by choosing the maps γ_1 and γ_2 in particular ways we can obtain existence of at least one positive solution to fractional boundary value problems (FBVPs) in the setting of the forward (or delta) fractional calculus. For example, letting $\mu \in (1,2)$, $\alpha < 0$, and $K \in [-1, b-1]_{\mathbb{N}_{-1}}$, if we select

$$\gamma_1(t) = \frac{1}{\Gamma(\mu-1)} t^{\underline{\mu-2}} - \frac{(\mu+b)^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-1}}\Gamma(\mu-1)} t^{\underline{\mu-1}}$$

and

$$\gamma_2(t) = \frac{1}{(\mu+b)^{\underline{\mu-1}}} t^{\underline{\mu-1}},$$

and define G by (3.9), then we obtain that a solution of (1.1) is likewise a solution of the FBVP

$$\begin{aligned} -\Delta_{\mu-2}^{\mu} y(t) &= f(t+\mu-1, y(t+\mu-1)),\ t \in [0,10]_{\mathbb{N}_0} \\ y(\mu-2) &= H_1\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right) \\ y(\mu+b) - \alpha y(\mu+K) &= -\alpha\gamma_1(\mu+K) H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) \\ &\quad + \left(1-\alpha\gamma_2(\mu+K)\right) H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right). \end{aligned} \tag{1.2}$$

If we then specialize to the case where $\mu = 2$ and put $\gamma_1(t) := 1 - \frac{t}{12}$, $\gamma_2(t) := \frac{t}{12}$, $b := 10$, $K := 3$, and $\alpha := -1$, then problem (1.2) becomes

$$\begin{aligned} -\Delta^2 y(t) &= f(t+1, y(t+1)),\ t \in [0,10]_{\mathbb{N}_0} \\ y(0) &= H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right) \\ y(12) + y(5) &= \frac{7}{12} H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \frac{17}{12} H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right). \end{aligned} \tag{1.3}$$

The point of the specific examples such as (1.2)–(1.3) is to show that by choosing γ_1 and γ_2 in various ways, we obtain solutions to a variety of BVPs with nonlocal boundary conditions. Since it is possible that the maps H_1 and H_2 are nonlinear, the boundary conditions allowable in our theory can be nonlinear in addition to nonlocal. However, it should be emphasized, as will be clarified later, that the maps H_1 and H_2 and hence the boundary conditions in our FBVPs need not be nonlinear, for, in fact, H_1 and H_2 can be either affine or linear in ways described later. Consequently, the results described herein are fairly generally applicable. Moreover, even in the case where $\mu = 2$, e.g., in (1.3), our results are new, to the best of our knowledge, and so, we not only provide new results for discrete BVPs of fractional order, but also provide new results in the integer-order setting — see, for example, Example 3.7.

In our treatment of problem (1.1) we should like to impose growth conditions on the maps H_1 and H_2 that are relatively flexible. In particular, we wish to be able to include nonlinear, affine, and linear maps as part of our theory. As part of this program, a typical condition we might impose is one on the ratio $\frac{H_i(z)}{z}$ as $z \to +\infty$. However, this runs into an immediate difficulty since, for example, H_1 is composed with the nonlocal element $\sum_{i=1}^{n} a_i y(\xi_i)$, where the a_i's are not necessarily of one sign. Consequently, in order to invoke an asymptotic-type growth condition, we shall require some control (specifically a lower bound) over the quantity $\sum_{i=1}^{n} a_i y(\xi_i)$ as $\|y\| \to +\infty$.

In previous work (for a simple example in the discrete setting, see [35]) we have achieved this by utilizing a Harnack inequality approach. That is to say, denoting by $\mathcal{B}$ the collection of all maps

$$\mathcal{B} := \left\{ y \ : \ [\mu-2, \mu+b]_{\mathbb{N}_{\mu-2}} \to \mathbb{R} \right\},$$

where we equip $\mathcal{B}$ with the usual maximum norm, henceforth denoted by $\|\cdot\|$, we have utilized, roughly speaking, the cone

$$\mathcal{K}_0 := \left\{ y \in \mathcal{B} \ : \ y \geq 0, \min_{t \in [m_1, m_2]_{\mathbb{N}_{\mu-2}}} y(t) \geq \gamma \|y\| \right\},$$

for some m_1, $m_2 \in \mathbb{N}_{\mu-2}$ satisfying $\mu - 2 < m_1 \leq m_2 < b + \mu$, where $b \geq 2$ is some number and $\gamma := \gamma(m_1, m_2, b, G) > 0$ is an explicitly computable constant in terms of initial data — see, for example, Erbe and Peterson [17, Lemma 6] and [18] in the case where $\mu = 2$. Then by means of the Harnack-like inequality

$$\min_{t \in [m_1, m_2]_{\mathbb{N}_{\mu-2}}} y(t) \geq \gamma \|y\|$$

we can obtain a coercivity-type bound that is sufficient for the purposes of utilizing asymptotic conditions. For example, if $\mu = 2$ (i.e., the integer-order setting) and the argument of H_1 is, say,

$$\frac{1}{2} y(1) - \frac{1}{5} y(3) + y(10),$$

then we might write

$$\varphi(y) := \frac{1}{2}y(1) - \frac{1}{5}y(3) + y(10) = \underbrace{\left(\frac{1}{2}y(1) - \frac{1}{5}y(3) + \frac{1}{2}y(10)\right)}_{=:\varphi_1(y)} + \underbrace{\frac{1}{2}y(10)}_{=:\varphi_2(y)}.$$

Now suppose that we add the condition $\varphi_1(y) \geq 0$ to the cone $\mathcal{K}_0$ and take $m_1 \leq m_2$ such that $[m_1, m_2]_{\mathbb{N}_0} \supseteq \{10\}$. Then it would follow that

$$\frac{1}{2}y(10) \geq \frac{1}{2}\gamma\|y\|,$$

and so, using that $\varphi_1(y) \geq 0$ we may thus deduce that

$$\varphi(y) = \frac{1}{2}y(1) - \frac{1}{5}y(3) + y(10) = \underbrace{\left(\frac{1}{2}y(1) - \frac{1}{5}y(3) + \frac{1}{2}y(10)\right)}_{=:\varphi_1(y)} + \underbrace{\frac{1}{2}y(10)}_{=:\varphi_2(y)} \geq \frac{1}{2}\gamma\|y\|,$$

whenever $y \in \mathcal{K}_0$. Thus, we obtain a coercivity condition for the functional $\varphi(y)$, and this then allows the use of asymptotic growth conditions on H_1 and H_2 — as an example of this procedure in the setting of ordinary differential equations, see, for instance, [33], [34], [36].

A significant problem that we encounter in trying to transplant this previously developed idea into the present setting of a sign-changing Green's function is that an approach that relied on a Harnack-like inequality would be severely restricted in applications. Indeed, we would have to insist on the existence of numbers $m_1', m_2' \in \mathbb{N}_{\mu-2}$ such that the kernel G satisfied

$$\min_{t\in\left[m_1',m_2'\right]_{\mathbb{N}_{\mu-2}}} G(t,s) > 0$$

for each $s \in [0, b]_{\mathbb{N}_0}$, which would be very restrictive and unnatural; indeed, the necessity of this inequality would restrict greatly the sorts of nonlocal elements that could be utilized in the application of (1.1) to boundary value problems. And, in addition, evidently there would be sign-changing and vanishing kernels that would not even satisfy this condition. So, all in all, it seems strongly preferable to develop a new approach, which can more naturally accommodate a sign-changing kernel and which relies neither on any Harnack-like inequality nor on any explicit decomposition of the linear functionals.

To this end, in this work we introduce the cone $\mathcal{K} \subseteq \mathcal{B}$ defined by

$$\begin{aligned}\mathcal{K} := \Bigg\{ y \in \mathcal{B} \ :\ y(t) \geq 0, \sum_{i=1}^{n} a_i y\left(\xi_i\right) \geq \left(\min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{n} a_i G\left(\xi_i, s\right)\right) \|y\|, \\ \sum_{i=1}^{m} b_i y\left(\zeta_i\right) \geq \left(\min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{m} b_i G\left(\zeta_i, s\right)\right) \|y\| \Bigg\},\end{aligned}$$

where $\mathcal{G}(s) := \max_{s\in[0,b]_{\mathbb{N}_0}} |G(t,s)|$, and it is assumed that the coercivity constants

$$\min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{n} a_i G(\xi_i, s) \quad \text{and} \quad \min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{m} b_i G(\zeta_i, s)$$

exist as positive, finite numbers. Here the set S_0 is defined by

$$S_0 := \left\{ s \in [0,b]_{\mathbb{N}_0} \ : \ \mathcal{G}(s) \neq 0 \right\}.$$

We identify two mathematical advantages of incorporating the coercivity conditions directly into the cone; there are additional advantages that for the sake of length we intend to detail in other works — for additional details, see [37].

- First of all and most evidently, the development of the cone $\mathcal{K}$ allows us to treat the case of sign-changing and vanishing kernels, as explained earlier.
- Second of all, it should also be mentioned that by utilizing $\mathcal{K}$ as above, we dispense with having to identify and check a particular decomposition of the nonlocal elements, as above, for example. Rather, we have identified a constant, in some sense internal to the problem (1.1) under study, that provides the control needed over the nonlocal elements. Thus, in particular, we do not need to check to see if a Harnack-like inequality holds, for it is ***never*** utilized in the proofs. Moreover, we do not need to spend time constructing the interval over which the Harnack inequality is taken so that its application is permitted with respect to the nonlocal element. All of this is obviated by use of the new cone. Therefore, overall, we feel this approach to be not only of pragmatic value in the particular setting of problem (1.1), but also simply a more elegant and aesthetically pleasing way to treat nonlocal boundary value problems in the difference equations setting.

Having described the particular contributions of this work, we conclude the introduction with a brief mention of the existing literature and its relationship to this paper. In particular, insofar as discrete fractional calculus is concerned there has been relatively intense research interest over the past several years. Beginning with some initial work of Atici and Eloe [3], [4], [5], [6], [7], which established some of the basic theoretical results in the field, many additional works have appeared. These have included works on operational properties of fractional differences, such as those by Atici and Acar [2], Atici and Eloe [8], Atici and Uyanik [10], Bastos, et al. [13], Erbe, et al. [46], [47], [48], Dahal and Goodrich [15], [16], Ferreira [21], Ferreira and Torres [24], Goodrich [31], Holm [41]; on boundary value problems, such as Atici and Eloe [6], Dahal, et al. [14], Ferreira [19], [20], [22], Ferreira and Goodrich [23], Goodrich [26], [27], [29], [30], Graef and Kong [40]; on modeling, which was explored in a paper by Atici and Sengül [9]; and on chaos in discrete fractional calculus, which was introduced by Wu and Baleanu [55]. At the same time, the study of nonlocal boundary value problems, whether equipped with linear, affine, or nonlinear boundary conditions, has seen much interest lately.

Recent studies have included those by Anderson [1], Goodrich [33], Infante, et al. [42], [43], [44], Jankowski [45], Karakostas [49], and Yang [56], [57]; furthermore, some classical works of interest are those by Picone [51] and Whyburn [54].

As concerns the study of boundary value problems with sign-changing Green's functions in the discrete setting, or, more generally as we treat here, summation equations with sign-changing kernels, the results seem very scarce. While there are a few such studies in the differential equations and Hammerstein integral equation settings, see, for example, [25], [39], [50], [52], [59], the only work of which the author is aware in the discrete setting is the relatively recent paper by Wang and Gao [53], which treats a third-order problem.

Thus, there seems to be a gap in the difference equations literature in this regard. Part of this may be attributed to the fact that deducing existence of *positive* solutions in the setting of a sign-changing kernel (or Green's function) is not as easy in the case where these maps are nonnegative, and so, extra difficulties are encountered, just as we encounter here — see, for example, the proof of Lemma 2.6. So, in this sense, this paper fills a gap in the literature and makes a connection back to the differential equations literature, whose development seems more complete. Moreover, since at the same time we treat the nonlocal boundary conditions setting, we are able to demonstrate ways in which those elements can be used profitably in the sign-changing setting — again, see, for example, the proof of Lemma 2.6. And since we must use a novel cone to achieve these ends, we believe this to be one of the contributions of this work.

2. Preliminaries

We begin by recalling fundamental definitions in discrete fractional calculus; the textbook by Goodrich and Peterson [38] is an excellent source for the basic theory of both the delta and nabla discrete fractional calculus as well as the classical difference calculus. After the preliminary definitions, we also introduce some notational conventions that shall be utilized in this work.

Definition 2.1. We define the ***falling factorial function***, denoted $t \mapsto t^{\underline{\nu}}$, by

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} := 0$.

Definition 2.2. The ***ν-th fractional sum***, $\nu > 0$, of a function $f : \mathbb{N}_a \to \mathbb{R}$, where $a \in \mathbb{R}$ is given, is

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. We also define the ***ν-th fractional difference*** of f, for $\nu > 0$, by

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{\nu-N} f(t),$$

where $t \in \mathbb{N}_{a-\nu+N}$ and $N \in \mathbb{N}_1$ is the unique number satisfying $N-1 < \nu \leq N$.

Notation 2.3. To facilitate the discussion in Section 3, we introduce the following notation. This shall be used throughout the remainder of the paper without further mention.

- Given a function $a : X \subset \mathbb{R} \to \mathbb{R}$, with $|X| < +\infty$, we denote by $\widetilde{a}_X^m$ and $\widetilde{a}_X^M$, respectively, the quantities
$$\widetilde{a}_X^m := \min_{s \in X} a(s) \quad \text{and} \quad \widetilde{a}_X^M := \max_{s \in X} a(s).$$
- Given a continuous function $f : X \times [0, +\infty) \to [0, +\infty)$, for some set $X \subseteq \mathbb{R}$ with $|X| < +\infty$, for real numbers $0 \leq a < b \leq +\infty$ we denote by $\widetilde{f}_{[a,b]}^m$ and $\widetilde{f}_{[a,b]}^M$, respectively, the quantities
$$\widetilde{f}_{[a,b]}^m := \min_{(t,y) \in X \times [a,b]} f(t,y) \quad \text{and} \quad \widetilde{f}_{[a,b]}^M := \max_{(t,y) \in X \times [a,b]} f(t,y).$$
- Denote by Γ_0 the quantity
$$\Gamma_0 := \min_{t \in [\mu-2, \mu+b]_{\mathbb{N}_{\mu-2}}} \left(\gamma_1(t) + \gamma_2(t)\right).$$
- As already suggested in Section 1, given a real number r_0, we denote by $\mathbb{N}_{r_0}$ the set
$$\mathbb{N}_{r_0} := \{r_0, r_0+1, r_0+2, \ldots\}.$$
Furthermore, given an interval $[a,b]$ with $-\infty \leq a < b \leq +\infty$ we denote by $[a,b]_{\mathbb{N}_{r_0}}$ the set
$$[a,b]_{\mathbb{N}_{r_0}} := [a,b] \cap \mathbb{N}_{r_0}.$$
- Given a map $G : X \times Y \to \mathbb{R}$, we denote by $G^+ : X \times Y \to [0, +\infty)$ and $G^- : X \times Y \to [0, +\infty)$ the maps
$$G^+(t,s) := \max\{0, G(t,s)\},$$
$$G^-(t,s) := \max\{0, -G(t,s)\}.$$
Thus, in particular G^+ and G^- are, respectively, the positive and negative parts of G.
- For $\rho > 0$, we denote by $\Omega_\rho \subseteq \mathcal{K}$ the open set $\Omega_\rho := \{y \in \mathcal{K} : \|y\| < \rho\}$.

We next list the growth and regularity assumptions imposed on the various maps involved in the summation equation (1.1). In particular, condition (A1) places a natural restriction on the kernel G. Condition (A2) provides some growth conditions on the maps H_1 and H_2. Conditions (A3)–(A4) are assumptions about the existence of the coercivity constants C_0 and D_0. Finally, condition (A5) facilitates the computation of the admissible range of the parameter λ appearing in (1.1) so that the existence result is not merely abstract, citing some uncomputable "sufficiently small" λ.

A1: Denote by $\mathcal{G} : [0,b]_{\mathbb{N}_0} \to [0,+\infty)$ the quantity

$$\mathcal{G}(s) := \max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}} |G(t,s)|,$$

and assume that $\mathcal{G}(s) < +\infty$ for each $s \in [0,b]_{\mathbb{N}_0}$.

A2: There exist numbers A_0, $B_0 \in (0,+\infty)$ such that

$$+\infty > \lim_{z\to+\infty} \frac{H_1(z)}{z} > A_0 \quad \text{and} \quad +\infty > \lim_{z\to+\infty} \frac{H_2(z)}{z} > B_0.$$

A3: Define the set $S_0 \subseteq [0,b]_{\mathbb{N}_0}$ by

$$S_0 := \left\{s \in [0,b]_{\mathbb{N}_0} \ : \ \mathcal{G}(s) \neq 0\right\}.$$

Then assume that the quantities

$$C_0 := \min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{n} a_i G\left(\xi_i, s\right)$$

and

$$D_0 := \min_{s\in S_0} \frac{1}{\mathcal{G}(s)} \sum_{i=1}^{m} b_i G\left(\zeta_i, s\right)$$

satisfy C_0, $D_0 \in (0,+\infty)$.

A4: For each $j = 1,2$ it holds that

$$\sum_{i=1}^{n} a_i \gamma_j\left(\xi_i\right) \geq C_0\|\gamma_j\| \quad \text{and} \quad \sum_{i=1}^{m} b_i \gamma_j\left(\zeta_i\right) \geq D_0\|\gamma_j\|.$$

A5: Let ρ_1 be defined by

$$\rho_1 := \inf\left\{\widetilde{\rho} \in [0,+\infty) \ : \ \frac{H_1(z)}{z} > A_0,\ \frac{H_2(z)}{z} > B_0,\ \frac{f(t,y)}{y} < 1 \right.$$
$$\left. \text{for all } \ t \in [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}, \ \text{whenever} \ \ y,z \in [\widetilde{\rho},+\infty)\right\},$$

and assume that ρ_1 is selected so that $\rho_1 \geq 1$. Put

$$\rho_1^* := \max\left\{1, \frac{\rho_1}{\min\left\{C_0, D_0\right\}}\right\}$$

and define the number $\lambda_0 > 0$ as the minimum of

$$\left\{\frac{1}{2}\Gamma_0 \min\left\{A_0C_0, B_0D_0\right\}\left(\max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}}\sum_{s=0}^{b}G^{-}(t,s)\right)^{-1}\min\left\{1, \frac{1}{\widetilde{f}^{M}_{[0,\rho_1]}}\right\},\right.$$
$$\left.\frac{\Gamma_0\left[\widetilde{H}^{m}_{1,\left[0,C_1\rho_1^*\right]}+\widetilde{H}^{m}_{2,\left[0,D_1\rho_1^*\right]}\right]}{\widetilde{f}^{M}_{\left[0,\rho_1^*\right]}}\left(\max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}}\sum_{s=0}^{b}G^{-}(t,s)\right)^{-1}\right\},$$

where we assume that
(1) $\Gamma_0 \min\left\{A_0C_0, B_0D_0\right\} > 0$;
(2) $\min\{\widetilde{f}^{M}_{[0,\rho_1]}, \widetilde{f}^{M}_{[0,\rho_1^*]}\} > 0$;
(3) $\widetilde{H}^{m}_{1,[0,C_1\rho_1^*]} + \widetilde{H}^{m}_{2,[0,D_1\rho_1^*]} > 0$; and
(4) $\max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}} \sum_{s=0}^{b} G^{-}(t,s) > 0$.

Remark 2.4. We emphasize that, as the example of Section 3 shall demonstrate, all of the above constants and conditions can be computed and checked.

Remark 2.5. Note that assumption (4) in condition (A5) essentially asserts that there is at least one point (t, s) such that $G(t, s) < 0$. That is, G is actually a sign-changing kernel.

We next describe the cone utilized in this work. As explained in detail in Section 1 we use the cone defined by

$$\mathcal{K} := \left\{y \in \mathcal{B}\ :\ y(t) \geq 0, \sum_{i=1}^{n} a_i y\left(\xi_i\right) \geq C_0\|y\|, \sum_{i=1}^{m} b_i y\left(\zeta_i\right) \geq D_0\|y\|\right\},$$

where C_0, $D_0 \in (0, +\infty)$ are as defined above as in condition (A3); note that $\mathcal{K}$ is neither empty nor trivial due to the fact that γ_1, $\gamma_2 \in \mathcal{K}$ with at least one of them not zero identically, which follows from the assumption above that $\Gamma_0 > 0$. We will also make use of the operator $T\ :\ \mathcal{B} \to \mathcal{B}$ defined by

$$\begin{aligned}(Ty)(t) = \gamma_1(t)H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \gamma_2(t)H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\\ +\lambda\sum_{s=0}^{b} G(t,s)f(s+\mu-1, y(s+\mu-1)).\end{aligned} \tag{2.1}$$

Obviously a fixed point of the operator T will be a solution of the summation equation (1.1). Thus, in Section 3 we will search for nonnegative, nontrivial fixed points of T in order to identify positive solutions of (1.1). The next lemma will be essential in this endeavor. While in many problems it is trivial to argue that $T(\mathcal{K}) \subseteq \mathcal{K}$, here we provide a thorough proof of this fact since the use of the new cone as well as the fact that G is allowed to change sign jointly cause the proof to be more technical.

Lemma 2.6. *Let T be the operator defined by* (2.1) *and assume that conditions* (A1)–(A5) *hold. Define the numbers C_1, $D_1 > 0$ by the following.*

$$\begin{aligned} C_1 &:= \sum_{i=1}^{n} |a_i| \\ D_1 &:= \sum_{i=1}^{m} |b_i| . \end{aligned} \tag{2.2}$$

Then whenever $\lambda \in (0, \lambda_0)$, it holds that $T(\mathcal{K}) \subseteq \mathcal{K}$.

Proof: We begin by demonstrating that for each $y \in \mathcal{K}$ it holds both that

$$\sum_{i=1}^{n} a_i (Ty)(\xi_i) \geq C_0 \|Ty\| \tag{2.3}$$

and that

$$\sum_{i=1}^{m} b_i (Ty)(\zeta_i) \geq D_0 \|Ty\|, \tag{2.4}$$

as these are the easier verifications. Then we shall show that $(Ty)(t) \geq 0$, for each $t \in [\mu - 2, \mu + b]_{\mathbb{N}_{\mu-2}}$, which is the more technical verification.

So, to demonstrate that (2.3) holds, let $y \in \mathcal{K}$ be fixed but otherwise arbitrary. First observe that

$$\begin{aligned} \|Ty\| &\leq \|\gamma_1\| H_1 \left(\sum_{i=1}^{n} a_i y(\xi_i) \right) + \|\gamma_2\| H_2 \left(\sum_{i=1}^{m} b_i y(\zeta_i) \right) \\ &\quad + \max_{t \in [\mu-2, \mu+b]_{\mathbb{N}_{\mu-2}}} \lambda \sum_{s=0}^{b} |G(t,s)| f(s+\mu-1, y(s+\mu-1)) \\ &\leq \|\gamma_1\| H_1 \left(\sum_{i=1}^{n} a_i y(\xi_i) \right) + \|\gamma_2\| H_2 \left(\sum_{i=1}^{m} b_i y(\zeta_i) \right) \\ &\quad + \lambda \sum_{s=0}^{b} \mathcal{G}(s) f(s+\mu-1, y(s+\mu-1)). \end{aligned} \tag{2.5}$$

At the same time we calculate

$$\sum_{i=1}^{n} a_i (Ty)(\xi_i) = H_1 \left(\sum_{i=1}^{n} a_i y(\xi_i) \right) \left(\sum_{i=1}^{n} a_i \gamma_1(\xi_i) \right) \tag{2.6}$$

$$
\begin{aligned}
&+ H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\left(\sum_{i=1}^{n} a_i \gamma_2\left(\xi_i\right)\right)\\
&+ \lambda \sum_{i=1}^{n}\sum_{s=0}^{b} a_i G\left(\xi_i, s\right) f(s+\mu-1, y(s+\mu-1))\\
\geq{}& C_0\|\gamma_1\| H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + C_0\|\gamma_2\| H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\\
&+ \lambda \sum_{s\in S_0} \underbrace{\left[\min_{s\in S_0}\frac{1}{\mathcal{G}(s)}\sum_{i=1}^{n} a_i G\left(\xi_i, s\right)\right]}_{:=C_0} \mathcal{G}(s) f(s+\mu-1, y(s+\mu-1)),
\end{aligned}
$$

where we have used both the definition of the set S_0 as well as the fact that $\gamma_1, \gamma_2 \in \mathcal{K}$ by assumption. Then putting (2.5)–(2.6) together we deduce that

$$
\begin{aligned}
\sum_{i=1}^{n} a_i (Ty)\left(\xi_i\right) \geq C_0 \Bigg[&\|\gamma_1\| H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + \|\gamma_2\| H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\\
&+ \lambda \sum_{s=0}^{b} \mathcal{G}(s) f(s+\mu-1, y(s+\mu-1))\Bigg] \geq C_0\|Ty\|,
\end{aligned}
\tag{2.7}
$$

whence (2.3) holds, as claimed. A nearly identical calculation reveals that (2.4) holds, for we simply write, as in (2.5)–(2.7),

$$
\begin{aligned}
\sum_{i=1}^{m} b_i (Ty)\left(\zeta_i\right) ={}& H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right)\left(\sum_{i=1}^{m} b_i \gamma_1\left(\zeta_i\right)\right)\\
&+ H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\left(\sum_{i=1}^{m} b_i \gamma_2\left(\zeta_i\right)\right)\\
&+ \lambda \sum_{i=1}^{m}\sum_{s=0}^{b} b_i G\left(\zeta_i, s\right) f(s+\mu-1, y(s+\mu-1))\\
\geq{}& D_0\|\gamma_1\| H_1\left(\sum_{i=1}^{n} a_i y\left(\xi_i\right)\right) + D_0\|\gamma_2\| H_2\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right)\\
&+ \lambda \sum_{s\in S_0} \underbrace{\left[\min_{s\in S_0}\frac{1}{\mathcal{G}(s)}\sum_{i=1}^{m} b_i G\left(\zeta_i, s\right)\right]}_{:=D_0} \mathcal{G}(s) f(s+\mu-1, y(s+\mu-1))\\
\geq{}& D_0\|Ty\|,
\end{aligned}
$$

as desired.

So, it remains to demonstrate that $(Ty)(t) \geq 0$ for each $t \in [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$. To accomplish this, we shall consider cases. Therefore, first suppose that $y \in \mathcal{K}$ satisfying $\|y\| \geq \rho_1^*$ is otherwise arbitrary and fixed; recall that $\rho_1^* := \max\{1, \frac{\rho_1}{\min\{C_0,D_0\}}\}$, where ρ_1 is defined as in condition (A5) earlier. Let us first observe that by using the coercivity of the linear functionals we obtain

$$\sum_{i=1}^{n} a_i y(\xi_i) \geq C_0\|y\| \geq \rho_1 \quad \text{and} \quad \sum_{i=1}^{m} b_i y(\zeta_i) \geq D_0\|y\| \geq \rho_1.$$

Now, for notational convenience in the sequel define the set $E_1 \subseteq [0,b]_{\mathbb{N}_0}$ by

$$E_1 := \{s \in [0,b]_{\mathbb{N}_0} \ : \ y(s+\mu-1) \geq \rho_1\}.$$

Then we may write, recalling that $\Gamma_0 > 0$,

(2.8)

$$\begin{aligned}
(Ty)(t) &\geq \gamma_1(t)A_0\sum_{i=1}^{n} a_i y(\xi_i) + \gamma_2(t)B_0\sum_{i=1}^{m} b_i y(\zeta_i) \\
&\quad + \lambda\sum_{s=0}^{b} G(t,s)f(s+\mu-1, y(s+\mu-1)) \\
&\geq \Gamma_0 \min\{A_0C_0, B_0D_0\}\|y\| - \lambda\sum_{s\in E_1} G^-(t,s)f(s+\mu-1, y(s+\mu-1)) \\
&\quad - \lambda\sum_{s\in[0,b]_{\mathbb{N}_0}\setminus E_1} G^-(t,s)f(s+\mu-1, y(s+\mu-1)) \\
&\geq \Gamma_0 \min\{A_0C_0, B_0D_0\}\|y\| - \lambda\sum_{s\in E_1} G^-(t,s)y(s+\mu-1) \\
&\quad - \lambda\sum_{s\in[0,b]_{\mathbb{N}_0}\setminus E_1} G^-(t,s)\widetilde{f}^M_{[0,\rho_1]} \\
&\geq \underbrace{\left[\Gamma_0\min\{A_0C_0, B_0D_0\} - \lambda \max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}}\sum_{s=0}^{b} G^-(t,s)\right]}_{\geq\frac{1}{2}\Gamma_0(A_0C_0, B_0D_0)}\|y\| \\
&\quad - \frac{1}{2}\Gamma_0(A_0C_0 + B_0D_0) \\
&\geq 0,
\end{aligned}$$

for each $t \in [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$, using both the fact that

$$\lambda \leq \frac{1}{2}\Gamma_0\min\{A_0C_0, B_0D_0\}\left(\max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}}\sum_{s=0}^{b} G^-(t,s)\right)^{-1}\min\left\{1, \frac{1}{\widetilde{f}^M_{[0,\rho_1]}}\right\}$$

and the fact that

$$\rho_1^* = \|y\| \geq 1.$$

On the other hand, we next consider the case in which $\|y\| \leq \rho_1^*$. Recall the definition of C_1 and D_1 from (2.2). Then using the fact that

$$0 \leq \sum_{i=1}^{n} a_i y(\xi_i) \leq \|y\| \sum_{i=1}^{n} |a_i| \leq \rho_1^* \sum_{i=1}^{n} |a_i| = C_1 \rho_1^*$$

and similarly with respect to the argument of H_2, we compute

(2.9)

$$\begin{aligned}
(Ty)(t) &= \gamma_1(t) H_1\left(\sum_{i=1}^{n} a_i y(\xi_i)\right) + \gamma_2(t) H_2\left(\sum_{i=1}^{m} b_i y(\zeta_i)\right) \\
&\quad + \lambda \sum_{s=0}^{b} G(t,s) f(s+\mu-1, y(s+\mu-1)) \\
&\geq \Gamma_0\left[\widetilde{H}^m_{1,[0,C_1\rho_1^*]} + \widetilde{H}^m_{2,[0,D_1\rho_1^*]}\right] + \lambda \sum_{s=0}^{b} G(t,s) f(s+\mu-1, y(s+\mu-1)) \\
&\geq \Gamma_0\left[\widetilde{H}^m_{1,[0,C_1\rho_1^*]} + \widetilde{H}^m_{2,[0,D_1\rho_1^*]}\right] - \lambda \sum_{s=0}^{b} G^-(t,s) f(s+\mu-1, y(s+\mu-1)) \\
&\geq \Gamma_0\left[\widetilde{H}^m_{1,[0,C_1\rho_1^*]} + \widetilde{H}^m_{2,[0,D_1\rho_1^*]}\right] - \lambda \max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}} \sum_{s=0}^{b} G^-(t,s) \widetilde{f}^M_{[0,\rho_1^*]} \\
&\geq 0,
\end{aligned}$$

using that

$$\lambda < \frac{\Gamma_0\left[\widetilde{H}^m_{1,[0,C_1\rho_1^*]} + \widetilde{H}^m_{2,[0,D_1\rho_1^*]}\right]}{\widetilde{f}^M_{[0,\rho_1^*]}} \left(\max_{t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}} \sum_{s=0}^{b} G^-(t,s)\right)^{-1}.$$

All in all, then, putting (2.8) and (2.9) together we deduce that $(Ty)(t) \geq 0$ for each $t \in [\mu-2, \mu+b]_{\mathbb{N}_{\mu-2}}$. And this completes the proof that $T(\mathcal{K}) \subseteq \mathcal{K}$. $\square$

Finally, in the proof of the existence result we make use of Fréchet derivatives in the context of asymptotically linear operator theory; this is a general approach that we have used previously in [32], [36]. Therefore, we conclude this section by presenting the general results we utilize in the existence proof in Section 3.

Definition 2.7 ([58, Definition 7.32.b]). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$. Suppose that $\mathcal{X}$ has an order cone $\mathcal{K}$, and that $T : \mathcal{K} \to \mathcal{K}$ is an operator. Then the operator $T'(+\infty) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the collection of all linear transformations between $\mathcal{X}$ and $\mathcal{Y}$, is called the **positive Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$** if and only if

$$\frac{\|Tx - T'(+\infty)x\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to +\infty \quad \text{for} \quad x \in \mathcal{K}.$$

Lemma 2.8 ([58, Corollary 7.34])**.** *Suppose that*

(1) $T : \mathcal{K} \subseteq \mathcal{X} \to \mathcal{K}$ *is a compact operator on the Banach space* $\mathcal{X}$ *with order cone* $\mathcal{K}$*; and*

(2) $T'(+\infty) : \mathcal{K} \to \mathcal{K}$ *exists as a positive Fréchet derivative of* T *at* $+\infty$ *along the cone* $\mathcal{K}$*, and if* μ *is an eigenvalue for* $T'(+\infty)$*, then* $|\mu| < 1$*.*

Then the operator T *has a fixed point in the cone* $\mathcal{K}$*.*

3. Main results and discussion

3.1 Existence result. In this section we begin by stating and proving the existence result for the abstract summation equation (1.1). We then spend some time detailing the application of this result to specific BVPs. As indicated in Section 1 we shall accomplish this by selecting the maps γ_1, γ_2, and G in a particular way, thereby relating solutions of specific incarnations of (1.1) to particular BVPs.

Theorem 3.1. *Assume that conditions* (A1)–(A5) *hold. In addition, suppose both that*

$$\lim_{y \to +\infty} \frac{f(t,y)}{y} = 0,$$

uniformly for $t \in [\mu - 2, \mu + b]_{\mathbb{N}_{\mu-2}}$*, and that*

$$C_0 > \widetilde{A}_0 C_1 \left(\sum_{i=1}^{n} a_i \gamma_1 \left(\xi_i\right) \right) + \widetilde{B}_0 D_1 \left(\sum_{i=1}^{n} a_i \gamma_2 \left(\xi_i\right) \right),$$

where the numbers $\widetilde{A}_0$ *and* $\widetilde{B}_0$ *are such that*

$$\lim_{z \to +\infty} \frac{H_1(z)}{z} = \widetilde{A}_0 \text{ and } \lim_{z \to +\infty} \frac{H_2(z)}{z} = \widetilde{B}_0.$$

Finally, for fixed $\lambda \in (0, \lambda_0)$*, where* λ_0 *is as in condition* (A5)*, assume that there exists* $t_0 \in [\mu - 2, \mu + b]_{\mathbb{N}_{\mu-2}}$ *such that*

$$\gamma_1\left(t_0\right) H_1(0) + \gamma_2\left(t_0\right) H_2(0) + \lambda \sum_{s=0}^{b} G\left(t_0, s\right) f(s + \mu - 1, 0) > 0.$$

Then problem (1.1) *has at least one positive solution.*

Proof: Letting, as in the statement of the theorem, the numbers $\widetilde{A}_0$ and $\widetilde{B}_0$ represent the positive, finite limits of $\frac{H_i(z)}{z}$ as $z \to +\infty$, for each $i = 1, 2$, respectively, we begin the proof by identifying the operator $L : \mathcal{B} \to \mathcal{B}$ defined by

$$(3.1)\qquad (Ly)(t) := \gamma_1(t)\widetilde{A}_0 \sum_{i=1}^{n} a_i y(\xi_i) + \gamma_2(t)\widetilde{B}_0 \sum_{i=1}^{m} b_i y(\zeta_i)$$

as the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$. It is obvious that the operator L defined by (3.1) is a linear operator in y. Moreover, it is also easy to see that $L(\mathcal{K}) \subseteq \mathcal{K}$.

So, to show that L is, in fact, the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$, we begin by writing

$$(3.2)\qquad \begin{aligned} \frac{|(Ty)(t) - (Ly)(t)|}{\|y\|} &\le \frac{\left|\widetilde{A}_0 \sum_{i=1}^{n} a_i y(\xi_i) - H_1\left(\sum_{i=1}^{n} a_i y(\xi_i)\right)\right|}{\|y\|}\|\gamma_1\| \\ &\quad + \frac{\left|\widetilde{B}_0 \sum_{i=1}^{m} b_i y(\zeta_i) - H_2\left(\sum_{i=1}^{m} b_i y(\zeta_i)\right)\right|}{\|y\|}\|\gamma_1\| \\ &\quad + \frac{\lambda}{\|y\|}\sum_{s=0}^{b} |G(t,s)| f(s+\mu-1, y(s+\mu-1)) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Let $\varepsilon > 0$ be a constant to be fixed later. Then using the fact that $\lim_{z\to+\infty} \frac{H_1(z)}{z} = \widetilde{A}_0$ and $\lim_{z\to+\infty} \frac{H_2(z)}{z} = \widetilde{B}_0$ yields both that

$$(3.3)\qquad I_1 \le \frac{\|\gamma_1\|}{\|y\|}\varepsilon \sum_{i=1}^{n} a_i y(\xi_i) \le \varepsilon\|\gamma_1\| \sum_{i=1}^{n} |a_i| = \varepsilon\|\gamma_1\| C_1$$

and, similarly, that

$$(3.4)\qquad I_2 \le \frac{\|\gamma_2\|}{\|y\|}\varepsilon \sum_{i=1}^{m} b_i y(\zeta_i) \le \varepsilon\|\gamma_2\| \sum_{i=1}^{m} |b_i| = \varepsilon\|\gamma_2\| D_1,$$

whenever $\|y\| \ge \rho_1$ for some $\rho_1 := \rho_1(\varepsilon) > 0$ sufficiently large. At the same time since $\lim_{y\to+\infty} \frac{f(t,y)}{y} = 0$ uniformly in t, it follows that $f(t,y) \le \varepsilon y$ whenever $y \ge \rho_1$ and $t \in [\mu-1, \mu+b-1]_{\mathbb{N}_{\mu-1}}$, perhaps by selecting ρ_1 even larger if necessary. We then write

$$(3.5)\qquad I_3 \le \frac{\lambda}{\|y\|}\left[\sum_{\{s\,:\,y(s+\mu-1)\ge\rho_1\}} |G(t,s)| f(s+\mu-1, y(s+\mu-1))\right.$$

$$
\begin{aligned}
&\left.+\sum_{\{s\,:\,y(s+\mu-1)<\rho_1\}}|G(t,s)|f(s+\mu-1,y(s+\mu-1))\right]\\
&\le\frac{\lambda}{\|y\|}\left[\varepsilon\sum_{s=0}^{b}|G(t,s)|\|y\|+\sum_{s=0}^{b}|G(t,s)|\widetilde{f}_{[0,\rho_1]}^{M}\right]\\
&\le\lambda\left[\varepsilon+\frac{\widetilde{f}_{[0,\rho_1]}^{M}}{\|y\|}\right]\sum_{s=0}^{b}\mathcal{G}(s)\\
&\le 2\varepsilon\lambda\sum_{s=0}^{b}\mathcal{G}(s),
\end{aligned}
$$

where we use the fact (see, for example, [36, Lemma 3.2]) that

$$
\lim_{\rho\to+\infty}\frac{\widetilde{f}_{[0,\rho]}^{M}}{\rho}=0
$$

so that for ρ_1 sufficiently large we have for each y with $\|y\|\ge\rho_1$ that

$$
\frac{\widetilde{f}_{[0,\rho_1]}^{M}}{\|y\|}\le\frac{\widetilde{f}_{[0,\rho_1]}^{M}}{\rho_1}<\varepsilon.
$$

All in all, then, putting estimates (3.3)–(3.5) into estimate (3.2) we deduce that

$$
\frac{|(Ty)(t)-(Ly)(t)|}{\|y\|}\le\varepsilon\underbrace{\left[\|\gamma_1\|C_1+\|\gamma_2\|D_1+2\lambda\sum_{s=0}^{b}\mathcal{G}(s)\right]}_{=:\kappa_0}=\varepsilon\kappa_0. \tag{3.6}
$$

Now, observe that $\kappa_0:=\kappa_0(\gamma_1,\gamma_2,G,\lambda,\{a_i\}_{i=1}^{n},\{b_i\}_{i=1}^{m})$. In particular, the number κ_0 is a constant that depends only on initial data; it does not depend on ρ_1 or ε itself. Therefore, we can make the right-hand side of (3.6) as small as we like by simply selecting ε sufficiently small. Since this estimate holds for each $t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$, it thus follows that L is the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$, as claimed.

In order to deduce by means of Lemma 2.8 that T has a fixed point, it remains to argue that L has no eigenvalue greater than or equal unity. To this end, suppose for contradiction that there exists $\mu\ge 1$ such that (μ,y) is an eigenpair for the operator L. Then it holds that $\mu y(t)=(Ly)(t)$, for each $t\in[\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}}$, with $\|y\|\ne 0$. Consequently, we may write

$$
\mu\sum_{i=1}^{n}a_iy\left(\xi_i\right)=\widetilde{A}_0\left(\sum_{i=1}^{n}a_i\gamma_1\left(\xi_i\right)\right)\left(\sum_{i=1}^{n}a_iy\left(\xi_i\right)\right) \tag{3.7}
$$

$$+\widetilde{B}_0\left(\sum_{i=1}^{n} a_i\gamma_2\left(\xi_i\right)\right)\left(\sum_{i=1}^{m} b_i y\left(\zeta_i\right)\right).$$

Now using both the fact that $1 \leq \mu$ and, seeing as $y \in \mathcal{K}$, the coercivity condition, we thus obtain from (3.7) the estimate

$$C_0\|y\| \leq \left[\widetilde{A}_0\left(\sum_{i=1}^{n} a_i\gamma_1\left(\xi_i\right)\right)\left(\sum_{i=1}^{n} |a_i|\right)+\widetilde{B}_0\left(\sum_{i=1}^{n} a_i\gamma_2\left(\xi_i\right)\right)\left(\sum_{i=1}^{m} |b_i|\right)\right]\|y\|,$$

whence

$$C_0 \leq \widetilde{A}_0 C_1\left(\sum_{i=1}^{n} a_i\gamma_1\left(\xi_i\right)\right)+\widetilde{B}_0 D_1\left(\sum_{i=1}^{n} a_i\gamma_2\left(\xi_i\right)\right),$$

which is a contradiction. Therefore, we conclude that L cannot have an eigenvalue greater than or equal to unity, as claimed.

All in all, then, we deduce from Lemma 2.8 that there exists $y_0 \in \mathcal{K}$ such that $Ty_0 = y_0$. Finally, to demonstrate that y_0 is not identically zero and hence is not a trivial solution, we note that if it held that $y_0 \equiv 0$, then we would obtain

$$0 = y_0(t) = \left(Ty_0\right)(t) = \gamma_1(t)H_1(0)+\gamma_2(t)H_2(0)+\lambda\sum_{s=0}^{b} G(t,s)f(s+\mu-1,0), \tag{3.8}$$

for each $t \in [\mu-2, \mu+b]_{\mathbb{N}_{\mu-2}}$. But then putting $t = t_0$, where t_0 is as in the statement of this theorem, we obtain from (3.8) that

$$0 = \gamma_1\left(t_0\right)H_1(0)+\gamma_2\left(t_0\right)H_2(0)+\lambda\sum_{s=0}^{b} G\left(t_0,s\right)f(s+\mu-1,0) > 0,$$

which is a contradiction. Consequently, the fixed point is nontrivial, and this completes the proof of the theorem. □

Remark 3.2. By altering the auxiliary condition

$$\gamma_1\left(t_0\right)H_1(0)+\gamma_2\left(t_0\right)H_2(0)+\lambda\sum_{s=0}^{b} G\left(t_0,s\right)f(s+\mu-1,0) > 0$$

appearing in the statement of Theorem 3.1, one can easily write down several variants of this existence theorem. We omit the statements of these, however.

3.2 Applications. We conclude this paper by presenting some applications of the existence theorem. In particular, the examples demonstrate that by choosing γ_1, γ_2, and G we can identify solutions of (1.1) with solutions of specific boundary value problems. We give a example worked in detail in the case where $\mu = 2$.

For notational convenience in the sequel, just as in [28] here we put

$$\Omega_0 := (b+\mu)^{\underline{\mu-1}} - \alpha(K+\mu)^{\underline{\mu-1}} > 0.$$

In our applications we will make use of the function $G : [\mu-2,\mu+b]_{\mathbb{N}_{\mu-2}} \times [0,b]_{\mathbb{N}_0} \to \mathbb{R}$ defined by

$$G(t,s) := \begin{cases} g_1(t,s), & 0 \le s \le \min\{t-\mu, K\} \\ g_2(t,s), & 0 \le t-\mu < s \le K \le b \\ g_3(t,s), & 0 < K < s \le t-\mu \le b \\ g_4(t,s), & \max\{t-\mu, K\} < s \le b \end{cases}, \tag{3.9}$$

where

$$\begin{aligned} g_1(t,s) &:= \frac{1}{\Gamma(\mu)}\Bigg[-(t-s-1)^{\underline{\mu-1}} + \frac{t^{\underline{\mu-1}}}{\Omega_0}\Big[(b+\mu-s-1)^{\underline{\mu-1}} \\ &\qquad -\alpha(K+\mu-s-1)^{\underline{\mu-1}}\Big]\Bigg] \\ g_2(t,s) &:= \frac{1}{\Gamma(\mu)}\left[\frac{t^{\underline{\mu-1}}}{\Omega_0}\Big[(b+\mu-s-1)^{\underline{\mu-1}} - \alpha(K+\mu-s-1)^{\underline{\mu-1}}\Big]\right] \\ g_3(t,s) &:= \frac{1}{\Gamma(\mu)}\left[-(t-s-1)^{\underline{\mu-1}} + \frac{t^{\underline{\mu-1}}}{\Omega_0}(b+\mu-s-1)^{\underline{\mu-1}}\right] \\ g_4(t,s) &:= \frac{1}{\Gamma(\mu)}\left[\frac{t^{\underline{\mu-1}}}{\Omega_0}(b+\mu-s-1)^{\underline{\mu-1}}\right]. \end{aligned}$$

The function G was studied extensively in [28, Theorems 4.2, 4.3, 4.4] in the case where $\alpha \ge 0$. There it was shown — see [28, Lemma 3.1] — that G was the Green's function associated to the boundary value problem

$$\begin{aligned} -\Delta^{\mu}_{\mu-2} y(t) &= f(t+\mu-1, y(t+\mu-1)),\ t \in [0,b]_{\mathbb{N}_0} \\ y(\mu-2) &= 0 \\ y(\mu+b) &= \alpha y(\mu+K), \end{aligned} \tag{3.10}$$

where α was nonnegative, subject to some upper bound restrictions detailed in [28], and $K \in [1, b-1]_{\mathbb{N}_0}$. Moreover, the order of the fraction difference, μ, was required to satisfy $1 < \mu \le 2$.

By contrast, here we shall take $\alpha < 0$. This causes the Green's function to be *negative* on part of its domain. Consequently, the analysis performed in [28] is no longer valid and must be modified. To facilitate and streamline the presentation of the example, we shall first present some relevant lemmata, which characterize the properties of G that are needed to apply Theorem 3.1 in this special case.

Lemma 3.3. *Let G be defined as in* (3.9) *and let $\mu \in (1,2]$ be fixed but otherwise arbitrary. Then it holds that*

(1) *the partial map $t \mapsto g_1(t,s)$, for each fixed admissible s, is decreasing on its domain;*
(2) *the partial map $t \mapsto g_2(t,s)$, for each fixed admissible s, is increasing on its domain;*

(3) *the partial map $t \mapsto g_3(t,s)$, for each fixed admissible s, is decreasing on its domain;*
(4) *the partial map $t \mapsto g_4(t,s)$, for each fixed admissible s, is increasing on its domain;*
(5) *the map $(t,s) \mapsto g_2(t,s)$ is nonnegative on its domain; and*
(6) *the map $(t,s) \mapsto g_4(t,s)$ is nonnegative on its domain.*

PROOF: Essentially the proof is similar to that provided in [28, Theorem 4.3]. However, we cannot repeat wholesale the argument given there since $\alpha < 0$ is assumed here, whereas in [28] it was assumed that $\alpha \geq 0$. This means that while the argument has the same basic structure, in some cases certain calculations that were utilized in [28] are not valid in this new setting.

Therefore, we begin by arguing that claims (2) and (4) hold, as these are the easier cases, just as in the proof of [28, Theorem 4.3]. To this end we notice that

$$\Gamma(\mu)\Delta_t g_4(t,s) = (\mu-1)\Omega_0^{-1} t^{\underline{\mu-2}}(b+\mu-s-1)^{\underline{\mu-1}} > 0,$$

which establishes claim (4). On the other hand, we also calculate

$$\Gamma(\mu)\Delta_t g_2(t,s) = (\mu-1)\Omega_0^{-1} t^{\underline{\mu-2}}\left[(b+\mu-s-1)^{\underline{\mu-1}} - \alpha(K+\mu-s-1)^{\underline{\mu-1}}\right],$$

which since $-\alpha > 0$ evidently establishes claim (2). Thus, it remains to argue that claims (1) and (3) hold, which are the more technical cases.

So, we next argue that $\Gamma(\mu)\Delta_t g_1(t,s) \leq 0$ for each pair (t,s) in the domain of g_1. To this end we calculate

$$\begin{aligned} &\Gamma(\mu)\Delta_t g_1(t,s) = \\ &(\mu-1)\left[-(t-s-1)^{\underline{\mu-2}} + \frac{t^{\underline{\mu-2}}}{\Omega_0}\left[(b+\mu-s-1)^{\underline{\mu-1}} - \alpha(K+\mu-s-1)^{\underline{\mu-1}}\right]\right]. \end{aligned} \tag{3.11}$$

From (3.11) we deduce that $\Delta_t g_1(t,s) \leq 0$ if and only if

$$\begin{aligned} &t^{\underline{\mu-2}}\left[(b+\mu-s-1)^{\underline{\mu-1}} - \alpha(K+\mu-s-1)^{\underline{\mu-1}}\right] \\ &\qquad < (t-s-1)^{\underline{\mu-2}}\left[(b+\mu)^{\underline{\mu-1}} - \alpha(K+\mu)^{\underline{\mu-1}}\right]. \end{aligned} \tag{3.12}$$

To establish (3.12) we demonstrate that each of the following inequalities hold:

$$\begin{aligned} t^{\underline{\mu-2}}(b+\mu-s-1)^{\underline{\mu-1}} &< (t-s-1)^{\underline{\mu-2}}(b+\mu)^{\underline{\mu-1}} \\ -\alpha t^{\underline{\mu-2}}(K+\mu-s-1)^{\underline{\mu-1}} &< -\alpha(t-s-1)^{\underline{\mu-2}}(K+\mu)^{\underline{\mu-1}}, \end{aligned} \tag{3.13}$$

whereupon by addition we shall obtain inequality (3.12).

To establish $(3.13)_1$ we note that it is equivalent to

$$\frac{t^{\underline{\mu-2}}}{(t-s-1)^{\underline{\mu-2}}} < \frac{(b+\mu)^{\underline{\mu-1}}}{(b+\mu-s-1)^{\underline{\mu-1}}},$$

which is true only if

$$\frac{\prod_{j=-s}^{0}(t+j)}{\prod_{j=-s}^{0}(t-\mu+2+j)}\cdot\frac{\prod_{j=-s}^{0}(b+1+j)}{\prod_{j=-s}^{0}(b+\mu+j)}<1.$$

But the latter inequality is true, since

$$\min\left\{\frac{\prod_{j=-s}^{0}(t+j)}{\prod_{j=-s}^{0}(t-\mu+2+j)},\frac{\prod_{j=-s}^{0}(b+1+j)}{\prod_{j=-s}^{0}(b+\mu+j)}\right\}<1.$$

Thus, $(3.13)_1$ holds. Similarly, using the fact that $-\alpha>0$ we see that $(3.13)_2$ is equivalent to the inequality

$$\frac{t^{\underline{\mu-2}}}{(t-s-1)^{\underline{\mu-2}}}<\frac{(K+\mu)^{\underline{\mu-1}}}{(K+\mu-s-1)^{\underline{\mu-1}}},$$

which, likewise, is true only if

$$\frac{\prod_{j=-s}^{0}(t+j)}{\prod_{j=-s}^{0}(t-\mu+2+j)}\cdot\frac{\prod_{j=-s}^{0}(K+1+j)}{\prod_{j=-s}^{0}(K+\mu+j)}<1.$$

But, once again, since

$$\min\left\{\frac{\prod_{j=-s}^{0}(t+j)}{\prod_{j=-s}^{0}(t-\mu+2+j)},\frac{\prod_{j=-s}^{0}(K+1+j)}{\prod_{j=-s}^{0}(K+\mu+j)}\right\}<1,$$

it follows that $(3.13)_2$ holds. All in all, we see that (3.12) is true, and so, we conclude that $\Delta_t g_1(t,s)\le 0$, as claimed, and claim (1) follows.

Finally, we argue that claim (3) holds. To this end we calculate

$$\Gamma(\mu)\Delta_t g_3(t,s)=(\mu-1)\left[-(t-s-1)^{\underline{\mu-2}}+\frac{t^{\underline{\mu-2}}}{\Omega_0}(b+\mu-s-1)^{\underline{\mu-1}}\right].$$

But notice that since $-\alpha > 0$ it follows from (3.12) that

$$\begin{aligned}
&-(t-s-1)^{\underline{\mu-2}} + \frac{t^{\underline{\mu-2}}}{\Omega_0}(b+\mu-s-1)^{\underline{\mu-1}} \\
&\quad \le -(t-s-1)^{\underline{\mu-2}} + \frac{t^{\underline{\mu-2}}}{\Omega_0}\left[(b+\mu-s-1)^{\underline{\mu-1}} - \alpha(K+\mu-s-1)^{\underline{\mu-1}}\right] \le 0.
\end{aligned}$$

Thus, because $\mu - 1 > 0$ it follows that since $\Gamma(\mu)\Delta_t g_1(t,s) \le 0$, we have that $\Gamma(\mu)\Delta_t g_3(t,s) \le 0$. Consequently, claim (3) is true.

Finally, the proof of claims (5)–(6) is nearly immediate. In the case of (5) we merely use the fact that $-\alpha > 0$, whereas in the case of (6) it is a triviality. And this completes the proof. ☐

Remark 3.4. One may observe that the conclusion of Lemma 3.3 is identical to that of [28, Theorem 4.3]. However, it should be noted that in [28] an additional constraint was imposed on the value of α — see [28, (4.15)]. Here, by contrast, for any $\alpha < 0$ the result holds. Thus, we need not impose a lower bound on α in order to obtain Lemma 3.3.

If $\mu = 2$, then we can provide some additional analysis of G beyond that which Lemma 3.3 establishes. In particular, we obtain the following result in this integer-order setting. As mentioned in Section 1, so far as we are aware, problem (3.10), in the case where $\alpha < 0$, has not been studied even in case $\mu = 2$. For reference in the sequel, let us note that if we put $\mu = 2$, then (3.9) becomes

$$G(t,s) := \begin{cases} g_1(t,s), & 0 \le s \le \min\{t-2, K\} \\ g_2(t,s), & 0 \le t-2 < s \le K \le b \\ g_3(t,s), & 0 < K < s \le t-2 \le b \\ g_4(t,s), & \max\{t-2, K\} < s \le b \end{cases},$$

where

$$\begin{aligned}
g_1(t,s) &:= -(t-s-1) + \frac{t}{b+2-\alpha(K+2)}\left[(b+1-s) - \alpha(K+1-s)\right] \\
g_2(t,s) &:= \frac{t}{b+2-\alpha(K+2)}\left[(b+1-s) - \alpha(K+1-s)\right] \\
g_3(t,s) &:= -(t-s-1) + \frac{t}{b+2-\alpha(K+2)}(b+1-s) \\
g_4(t,s) &:= \frac{t}{b+2-\alpha(K+2)}(b+1-s),
\end{aligned}$$

using the fact that $t^{\underline{1}} = t$.

Lemma 3.5. *Suppose that $\mu = 2$. Then*

(1) *$g_1(t,s)$ is negative whenever $t > \frac{b+2-\alpha(K+2)}{1-\alpha}$; and*

(2) $g_3(t,s)$ *is negative whenever* $s < \frac{(t-1)\left(b+2-\alpha(K+2)\right)-t(b+1)}{b+2-\alpha(K+2)-t}$.

Proof: In order to prove claim (1) we begin by noting that $g_1(t,s) < 0$ if and only if

$$\begin{aligned}&-(t-s-1)+\frac{t}{b+2-\alpha(K+2)}\left[(b+1-s)-\alpha(K+1-s)\right]\\&\qquad=\frac{-(t-s-1)\big[b+2-\alpha(K+2)\big]+t\big[(b+1-s)-\alpha(K+1-s)\big]}{\underbrace{(b+2)-\alpha(K+2)}_{>0}}<0.\end{aligned}$$

So, we see that $g_1(t,s)$ is negative if and only if

$$-(t-s-1)\big[b+2-\alpha(K+2)\big]+t\big[(b+1-s)-\alpha(K+1-s)\big]<0.$$

From the above inequality a routine exercise yields

$$t>\frac{b+2-\alpha(K+2)}{1-\alpha},$$

and this proves claim (1).

Similarly, to prove claim (2) we notice that $g_3(t,s) < 0$ if and only if

$$-(t-s-1)\big[b+2-\alpha(K+2)\big]+t(b+1-s)<0$$

if and only if

$$s\big(-t+b+2-\alpha(K+2)\big)<(t-1)\big[b+2-\alpha(K+2)\big]-t(b+1).$$

Solving for s in the above inequality we deduce that

$$s<\frac{(t-1)\big(b+2-\alpha(K+2)\big)-t(b+1)}{b+2-\alpha(K+2)-t},$$

which proves claim (2). And this completes the proof. □

Remark 3.6. In the case where $\mu = 2$, Lemmata 3.3 and 3.5 completely characterize the subsets of $[0,b+2]_{\mathbb{N}_0}\times[0,b]_{\mathbb{N}_0}$ on which G is positive, negative, or zero.

By putting the preceding lemmata together, we can generate the following example, which will also illustrate the application of the existence theory developed in subsection 3.1 earlier.

Example 3.7. Let us consider summation equation (1.1) in case we put $\gamma_1(t) := 1-\frac{t}{12}$, $\gamma_2(t) := \frac{t}{12}$, and G as in (3.9) with $\mu = 2$, $\alpha := -1$, $b := 10$, and $K := 3$.

Note also that $\Gamma_0 = 1$. We shall also assume for definiteness that (with, evidently, $n = m = 2$)

$$\sum_{i=1}^{2} a_i y(\xi_i) := 2y(7) - \frac{1}{10}y(2) \text{ and } \sum_{i=1}^{2} b_i y(\zeta_i) := \frac{2}{3}y(6) - \frac{1}{9}y(1).$$

In addition, define the maps H_1 and H_2 as follows.

$$\begin{aligned} H_1(z) &:= \frac{1}{30}z, \\ H_2(z) &:= \frac{1}{20}z + 1. \end{aligned}$$

Then, recalling that $b = 10$, summation equation (1.1) becomes

$$\begin{aligned} y(t) = &\left(1 - \frac{t}{12}\right) H_1\left(2y(7) - \frac{1}{10}y(2)\right) + \frac{t}{12}H_2\left(\frac{2}{3}y(6) - \frac{1}{9}y(1)\right) \\ &+ \lambda\sum_{s=0}^{10} G(t,s)f(s+1, y(s+1)) \\ = &\left(1 - \frac{t}{12}\right)\left(\frac{1}{15}y(7) - \frac{1}{300}y(2)\right) + \frac{t}{12}\left(\frac{1}{30}y(6) - \frac{1}{180}y(1) + 1\right) \\ &+ \lambda\sum_{s=0}^{10} G(t,s)f(s+1, y(s+1)), \end{aligned}$$

for each $t \in [0, 12]_{\mathbb{N}_0}$. Recalling the choices above for K and b, it is easy to show that each solution of the preceding integral equation is likewise a solution of the nonlocal boundary value problem

$$\begin{aligned} -\Delta^2 y(t) &= f(t+1, y(t+1)),\ t \in [0, 10]_{\mathbb{N}_0} \\ y(0) &= H_1\left(2y(7) - \frac{1}{10}y(2)\right) \\ y(12) + y(5) &= \frac{7}{12}H_1\left(2y(7) - \frac{1}{10}y(2)\right) + \frac{17}{12}H_2\left(\frac{2}{3}y(6) - \frac{1}{9}y(1)\right). \end{aligned} \tag{3.14}$$

With the choices for H_1 and H_2 as above, problem (3.14) becomes

$$\begin{aligned} -\Delta^2 y(t) &= f(t+1, y(t+1)),\ t \in [0, 10]_{\mathbb{N}_0} \\ y(0) &= \frac{1}{15}y(7) - \frac{1}{300}y(2) \\ y(12) + y(5) &= -\frac{17}{2160}y(1) - \frac{7}{3600}y(2) + \frac{17}{360}y(6) + \frac{7}{180}y(7) + \frac{17}{12}. \end{aligned}$$

Thus, in particular, the boundary conditions are linear at $t = 0$ and affine at $t = 12$.

By means of Lemmata 3.3 and 3.5 we know that each of g_2 and g_4 is a nonnegative function everywhere on its respective domain, whereas g_1 is negative whenever

$$t > \frac{17}{2},$$

and g_3 is negative whenever

$$s < \frac{17(t-1)-11t}{17-t} = \frac{6t-17}{17-t}.$$

In fact, this latter result implies that, as collected in the following table, for each of the following t-values it follows that $g_3(t,s) < 0$ for the associated collection of s-values.

$t =$	12	11	10	9	8	7	6	5	4
$s \in$	$\mathbb{N}_4^{10}$	$\mathbb{N}_4^{8}$	$\mathbb{N}_4^{6}$	$\mathbb{N}_4^{4}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

In light of this table as well as the definition of the map $(t,s) \mapsto g_1(t,s)$, we thus calculate

$$\begin{aligned}
&\max_{t\in[0,12]_{\mathbb{N}_0}} \sum_{s=0}^{10} G^-(t,s)\\
&\quad = -\min\Bigg\{\sum_{s=0}^{3} g_1(12,s) + \sum_{s=4}^{10} g_3(12,s), \sum_{s=0}^{3} g_1(11,s) + \sum_{s=4}^{8} g_3(11,s),\\
&\qquad\qquad \sum_{s=0}^{3} g_1(10,s) + \sum_{s=4}^{6} g_3(10,s), \sum_{s=0}^{3} g_1(9,s) + \sum_{s=4}^{4} g_3(9,s)\Bigg\}\\
&\quad = \frac{210}{17},
\end{aligned}$$

which will be used in the calculation of the number λ_0^*. Furthermore, for *any* nonnegative function f we calculate

$$\gamma_1(1)H_1(0) + \gamma_2(1)H_2(0) + \lambda \sum_{s=0}^{b} G(1,s)f(s+1,0) > 0$$

so that the auxiliary condition in Theorem 3.1 holds. In addition, here we may set

$$\widetilde{A}_0 = \frac{1}{30} \text{ and } \widetilde{B}_0 = \frac{1}{20}.$$

Then we see that

$$\Gamma_0 \min\{A_0C_0, B_0D_0\} > 0.$$

We also compute both that $C_1 = \frac{21}{10}$, $D_1 = \frac{7}{9}$,

$$C_0 = \min_{s\in[0,10]_{\mathbb{N}_0}} \frac{1}{\mathcal{G}(s)}\left[2G(7,s) - \frac{1}{10}G(2,s)\right] = \frac{13}{150},$$

and

$$D_0 = \min_{s\in[0,10]_{\mathbb{N}_0}} \frac{1}{\mathcal{G}(s)}\left[\frac{2}{3}G(6,s)-\frac{1}{9}G(1,s)\right] = \frac{1}{9}\,.$$

Observe, therefore, that

$$\sum_{i=1}^{2} a_i\gamma_j\left(\xi_i\right) = \begin{cases} \frac{19}{24}, & j=1 \\ \frac{23}{20}, & j=2 \end{cases} \geq \frac{13}{150} = C_0\|\gamma_j\| \text{ and}$$

$$\sum_{i=1}^{2} b_i\gamma_j\left(\zeta_i\right) = \begin{cases} \frac{25}{108}, & j=1 \\ \frac{35}{108}, & j=2 \end{cases} \geq \frac{1}{9} = D_0\|\gamma_j\|.$$

Thus, condition (A4) holds. Finally, we note that

$$\frac{13}{150} = C_0 > \widetilde{A}_0C_1\left(\sum_{i=1}^{n} a_i\gamma_1\left(\xi_i\right)\right) + \widetilde{B}_0D_1\left(\sum_{i=1}^{n} a_i\gamma_2\left(\xi_i\right)\right) = \frac{151}{1800}\,,$$

from which it follows that the first auxiliary condition in the statement of Theorem 3.1 is satisfied.

All in all, then, with the preceding computations in hand we estimate, to three decimal places of accuracy, that

$$\lambda_0 := \min\left\{\frac{221}{2205000}\min\left\{1,\frac{1}{\widetilde{f}^M_{[0,\rho_1]}}\right\},\frac{17}{210}\cdot\frac{\widetilde{H}^m_{1,\left[0,\frac{21}{10}\rho_1^*\right]}+\widetilde{H}^m_{2,\left[0,\frac{7}{9}\rho_1^*\right]}}{\widetilde{f}^M_{\left[0,\rho_1^*\right]}}\right\},$$

where we have taken $A_0 := \frac{1}{35}$ and $B_0 := \frac{1}{25}$. Thus, for example, if we put $f(t,y) := t\sqrt{y}$, then we compute that $\rho_1 = 1$ so that $\rho_1^* = \frac{150}{13}$. Thus, in this specific case we estimate

$$\begin{aligned}\lambda_0 &:= \min\left\{\frac{221}{2205000}\min\left\{1,\frac{1}{\widetilde{f}^M_{[0,1]}}\right\},\frac{17}{210}\cdot\frac{\widetilde{H}^m_{1,\left[0,\frac{21}{10}\cdot\frac{150}{13}\right]}+\widetilde{H}^m_{2,\left[0,\frac{7}{9}\cdot\frac{150}{13}\right]}}{\widetilde{f}^M_{\left[0,\frac{150}{13}\right]}}\right\}\\ &= \min\left\{\frac{221}{2205000},\frac{17}{210}\sqrt{\frac{13}{150}}\right\}\\ &= \frac{221}{2205000}\\ &\approx 0.0001,\end{aligned}$$

to three decimal places of accuracy. Consequently, when f is so selected, by means of Theorem 3.1 we deduce that for each

$$\lambda \in \left(0,\frac{221}{2205000}\right)$$

boundary value problem (3.14) has at least one positive solution.

Remark 3.8. On the one hand, in general the number λ_0 may be rather small, as is the case in Example 3.7. In part, this is due to the fact that C_0 and D_0 may be quite small in some cases, and, if they are not, then C_1 and D_1 may be correspondingly large, in which case to ensure that the auxiliary inequality in Theorem 3.1 holds it may be necessary to require $\widetilde{A}_0$ and $\widetilde{B}_0$ to be rather small.

On the other hand, it is certainly possible to find examples in which λ_0 is larger than in Example 3.7. For instance, if one chooses the nonlocal elements

$$\sum_{i=1}^{2} a_i y\left(\xi_i\right) = 10y(7) - \frac{1}{100}y(2)$$

and

$$\sum_{i=1}^{2} b_i y\left(\zeta_i\right) = 20y(6) - \frac{1}{90}y(1),$$

then one calculates $C_0 \approx 1.9913$ and $D_0 \approx 6.6556$. If, in addition, we may select $A_0 = \frac{1}{30}$, as in Example 3.7, and $B_0 = \frac{1}{200}$, then we find that $\lambda_0 \approx 0.0013$.

Remark 3.9. We wish to emphasize, as Example 3.7 demonstrates and as was mentioned in Section 1, that the existence results do *not* require that either H_1 or H_2 be nonlinear. They can be linear or affine, as the example demonstrates.

Remark 3.10. In the preceding example we focused on the case where G is defined by (3.9), $\gamma_1(t) := 1 - \frac{t}{12}$, and $\gamma_2(t) := \frac{t}{12}$. Of course, by choosing these maps in different ways we would recover existence of solution to a variety of other boundary value problems with associated sign-changing Green's functions in both the integer- and fractional-order setting. However, as mentioned in Section 1 since the problem defined by (3.10) has not even been analyzed when $\mu = 2$ and $\alpha < 0$, we felt it best to focus on this one particular application. Needless to say, however, the existence theory is widely applicable due to the very general form of the summation equation in (1.1).

Remark 3.11. Considering problem (1.2), we wish to note that in this work we have considered only the case where b is finite. If one allows $b \to +\infty$, then (1.2) becomes a half-line problem, and the methodology utilized in this work is no longer entirely valid. This could form the basis for further analysis of problem (1.2), but we do not consider this type of investigation in this paper.

Moreover, although we chose only one particular value of α in Example 3.7, one could study the effect of changing the value of α on the Green's function G in (3.9). For example, notice that

$$\lim_{\alpha \to -\infty} \frac{b + 2 - \alpha(K+2)}{1 - \alpha} = K + 2, \tag{3.15}$$

whereas

$$\left.\frac{b+2-\alpha(K+2)}{1-\alpha}\right|_{\alpha=0} = b+2. \tag{3.16}$$

Thus, from part (1) of Lemma 3.5 (i.e., in case $\mu = 2$) we see that (3.15)–(3.16) suggest that as $\alpha \to -\infty$ the set of points (t, s) for which $g_1(t, s) < 0$ "enlarges", seeing as the collection of t-values for which $g_1(t, s) < 0$ holds increases in size. A similar analysis may be performed for the map $(t, s) \mapsto g_3(t, s)$ by means of part (2) of Lemma 3.5. All in all, then, some additional analysis could be performed analyzing the behavior of G, in a sort-of asymptotic sense, as $\alpha \to -\infty$. But we omit this detailed analysis in the present work.

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