The σ-property in $C(X)$

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Abstract. The σ-property of a Riesz space (real vector lattice) B is: For each sequence $\{b_n\}$ of positive elements of B, there is a sequence $\{\lambda_n\}$ of positive reals, and $b \in B$, with $\lambda_n b_n \leq b$ for each n. This condition is involved in studies in Riesz spaces of abstract Egoroff-type theorems, and of the countable lifting property. Here, we examine when "σ" obtains for a Riesz space of continuous real-valued functions $C(X)$. A basic result is: For discrete X, $C(X)$ has σ iff the cardinal $|X| < \mathfrak{b}$, Rothberger's bounding number. Consequences and generalizations use the Lindelöf number $L(X)$: For a P-space X, if $L(X) \leq \mathfrak{b}$, then $C(X)$ has σ. For paracompact X, if $C(X)$ has σ, then $L(X) \leq \mathfrak{b}$, and conversely if X is also locally compact. For metrizable X, if $C(X)$ has σ, then X *is* locally compact.

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1. Preliminaries

The σ-property of a Riesz space (as defined in the Abstract) is a feature of the spaces of measurable functions modulo null functions associated with a σ-finite measure ([LZ71, 71.5]), is a component of abstract generalizations of the classical Egoroff theorem ([LZ71, Chapter 10]), and has useful interpretation in terms of relatively uniform convergence ([D74], [LZ71]). Further, if B has the σ-property, then any Riesz space surjection $A \xrightarrow{\varphi} B$ has the "countable lifting property", that is, if $\{b_n\}_{\mathbb{N}} \subseteq B^+$ is disjoint, there is *disjoint* $\{a_n\}_{\mathbb{N}} \subseteq A$ with $\varphi(a_n) = b_n$ $\forall n$ [HR16].

We are considering now the issue of the σ-property in a Riesz space $C(X) = \{f \in \mathbb{R}^X \mid f \text{ continuous}\}$. Here X is a Tychonoff space; $+$ and $\leq$ in $C(X)$ are defined pointwise.

We record some notation, etc., which will be used constantly. The cardinal of a set X is $|X|$. All spaces will be Tychonoff and assumed infinite. For $\{X_i\}_I$ a set of spaces, $\sum_I X_i$ (or just $\sum X_i$) is the topological sum (disjoint union). $\mathbb{N}$ is the positive, or non-negative integers (as convenient), and frequently denotes the countable discrete space. $\{b_n\}_{\mathbb{N}}$ (or just $\{b_n\}$) is a countable set of elements.

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"nbhd" abbreviates "neighborhood". The cozero-set of $f \in \mathbb{R}^X$ is $\operatorname{coz} f = \{x \mid f(x) \neq 0\}$.

The "eventual order" in the set of functions $\mathbb{N}^\mathbb{N}$ is $f \overset{*}{\leq} g$, meaning $\exists k$ such that $f(n) \leq g(n)$ for $n \geq k$. A subset $\mathscr{F}$ of $\mathbb{N}^\mathbb{N}$ is bounded if $\exists g \in \mathbb{N}^\mathbb{N}$ with $f \overset{*}{\leq} g\ \forall f \in \mathscr{F}$. Rothberger's bounding number is $\mathfrak{b} \equiv \min\{|\mathscr{F}| \mid \mathscr{F}$ is unbounded in $\mathbb{N}^\mathbb{N}\}$. What is known is that $\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$, and that little more can be said in ZFC (e.g., they can be all equal (under CH), or they can be all different, or ...). See [D84], [J02].

The Lindelöf number of a space X is $L(X) \equiv \min\{m \mid$ each open cover of X has a subcover of cardinal $< m\}$. X is Lindelöf iff $L(X) \leq \aleph_1$. See [E89].

Theorem 2.1 below (and in the Abstract) says, for discrete X, $C(X)$ has σ iff $|X| < \mathfrak{b}$, and clearly, $L(X) = |X|^+$ here. The following notes that for general X, "$C(X)$ has σ" has little to do with $|X|$; subsequent results show "much to do with $L(X)$".

Proposition 1.1. (a) *If X is compact, then $C(X)$ has σ. Thus, "$C(X)$ has σ" puts no upper bound on the cardinal $|X|$ (or, on the cellular number).*

(b) *Let $\{X_n\}_\mathbb{N}$ be a family of non-pseudocompact spaces, and let $X \equiv (\sum X_n) \cup \{\rho\}$, where a nbhd of ρ contains $\sum_{n \geq k} X_n$ for some k. Then $C(X)$ fails σ. Taking $X_n = \mathbb{N}\ \forall n$ shows $|X| = \aleph_0 \not\Rightarrow C(X)$ has σ.*

Proof: (a) Any $f \in C(X)$ is bounded. Given $\{b_n\} \subseteq C(X)^+$, $b_n \leq m_n$, $0 < m_n \in \mathbb{N}$, and $\frac{1}{m_n} b_n \leq 1 = b \in C(X)$.

(b) For each n, choose unbounded $\overline{b}_n \in C(X_n)$, and define $b_n \in C(X)$ as $b_n \equiv [\overline{b}_n$ on $X_n; 0$ off $X_n]$. Then $\{b_n\}$ witnesses $C(X)$ failing σ. Note that for any choices $z_n \in X_n$, $z_n \longrightarrow \rho$. Suppose $\{\lambda_n\} \subseteq (0, +\infty)$. Choose $z_n \in X_n$ with $b_n(x_n) \geq n/\lambda_n$. Inequalities $\lambda_n b_n \leq b \in C(X)$ would force $b(\rho) = +\infty$. ☐

Remark 1.2. The example 1.1(b) is from [HR16], as an example of a $C(X)$ failing the "disjoint σ-property" (defined just as σ, but assuming that $\{b_n\}$ disjoint, i.e. $b_m \wedge b_n = 0$ for $m \neq n$). Not much is known about disjoint σ, either for $C(X)$'s or other Riesz spaces. The property seems interesting, partly because [HR16] a $C(X)$ has it iff $\forall$ disjoint $\{b_n\} \subseteq C(X)^+$, there is $\{\lambda_n\} \subseteq (0, +\infty)$ for which $\bigwedge_n \lambda_n b_n$ exists in $C(X)$ ("weakly laterally σ-complete").

2. Discrete spaces

For X a set, we consider the issue "$\mathbb{R}^X$ has σ?" It is convenient to work with $\mathbb{N}^X$, and say $\mathbb{N}^X$ has σ if $\forall \{b_n\} \subseteq \mathbb{N}^X\ \exists \{\lambda_n\} \subseteq (0, +\infty)$ and $b \in \mathbb{N}^X$ with $\lambda_n b_n(x) \leq b(x)\ \forall n, x$. Evidently, $\mathbb{R}^X$ has σ iff $\mathbb{N}^X$ has σ.

Theorem 2.1. *For X a set, the following are equivalent.*

(a) *$\mathbb{R}^X$ (or $\mathbb{N}^X$) has σ.*

(b) *For each $\beta : \mathbb{N} \times X \longrightarrow \mathbb{N}$, there are $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ and $b : X \longrightarrow \mathbb{N}$ which $\beta(n, x) \leq \gamma(n) b(x)\ \forall n, x$.*

(c) *$|X| < \mathfrak{b}$.*

Proof: In the following, the letters β, γ, b always stand for functions $\beta : \mathbb{N} \times X \longrightarrow \mathbb{N}$, $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$, $b : X \longrightarrow \mathbb{N}$. Call such (β, x, b) "admissible" if $\beta(n,x) \leq \gamma(n)b(x)$ $\forall n, x$.

(a) $\iff$ (b). In a condition "$\lambda_n b_n \leq b$" the λ_n might as well be in $\mathbb{N}^{-1} = \{1, 1/2, \dots, 1/n, \dots\}$, i.e., the function $\lambda : \mathbb{N} \longrightarrow (0, +\infty)$ might as well range in $\mathbb{N}^{-1}$. Then, replace such λ by $1/\lambda \equiv \gamma : \mathbb{N} \longrightarrow \mathbb{N}$. So $\lambda_n b_n(x) \leq b(x)$ means $b_n(x) \leq \gamma(n)b(x)$. Conversely, a $\{b_n\} \subseteq \mathbb{N}^X$ is the same as a $\beta : \mathbb{N} \times X \longrightarrow \mathbb{N}$.

(b) $\Longrightarrow$ (c). Note: (#) If (β, γ, b) is admissible, then $\forall x_0 \in X$, $\{\beta(n, x_0)/\gamma(n)\}_{\mathbb{N}}$ is upper bounded in $\mathbb{N}$ (by $b(x_0)$).

We show $|X| \geq \mathfrak{b}$ implies that (b) fails. It suffices to consider $|X| = \mathfrak{b}$; identify X with unbounded $\mathscr{F} \subseteq \mathbb{N}^{\mathbb{N}}$. Now define for $f \in \mathscr{F}$, $\beta(n, f) \equiv nf(n)$. Take any γ. Then, using (#) there is no b with (β, γ, b) admissible, because there is $f_0 \in \mathscr{F}$ for which $f_0 \overset{*}{\leq} \gamma$ fails, i.e., $\{n \mid \gamma(n) \leq f_0(n)\}$ is infinite, so $\{\beta(n, f_0)/\gamma(n)\}_{\mathbb{N}}$ is not upper bounded.

(c) $\Longrightarrow$ (b). Note: (##) If β has $\{\beta(\bullet, x)\}_X$ bounded in $(\mathbb{N}^{\mathbb{N}}, \overset{*}{\leq})$, then there are γ, b with (β, γ, b) admissible. For, if $\{\beta(\bullet, x)\}_X \overset{*}{\leq} \gamma$ (we can suppose $1 \leq \gamma$), then, $\forall x$, $\exists$ finite $F_x \subseteq \mathbb{N}$ for which $\beta(n, x) \leq \gamma(n)$ $\forall n \notin F_x$, and we define $b(x) \equiv \sup\{\beta(k, x) \mid k \in F_x\} \wedge 1$.

Now, any $\mathscr{F} \subseteq \mathbb{N}^{\mathbb{N}}$ with $|\mathscr{F}| < \mathfrak{b}$ is bounded (for $\overset{*}{\leq}$). So, if $|X| < \mathfrak{b}$ and β is given, then $\{\beta(\bullet, x)\}_X$ is bounded and (##) applies. □

Proposition 2.2. (a) *Suppose that $A \overset{\varphi}{\twoheadrightarrow} B$ is a surjection of Riesz spaces. If A has σ, then B has σ.*

(b) *Suppose that Y is a C-embedded subspace of X* ([GJ60, 1.16]). *If $C(X)$ has σ, then $C(Y)$ has σ.*

Proof: (a) If $\forall n$ $\varphi(a_n) = b_n \geq 0$, we can have $a_n \geq 0$, so, if $\lambda_n a_n \leq a$ $\forall n$, then $\lambda_n b_n \leq \varphi(a)$ $\forall n$.

(b) The inclusion $Y \subseteq X$ yields a Riesz space surjection $C(X) \overset{\rho}{\twoheadrightarrow} C(Y)$ by restriction, $\rho(f) = f|Y$. Apply (a). □

Corollary 2.3. *If $C(X)$ has σ, then X has no discrete C-embedded subspace of size $\mathfrak{b}$.*

Proof: 2.2(b) and 2.1. □

Question 2.4. For X realcompact, $C(X)$ has $\sigma \overset{?}{\Longrightarrow} L(X) \leq \mathfrak{b}$? 2.3 tends this way. Partial answers appear below, especially in §4. Regarding realcompactness here, note that "$C(X)$ has σ" is a property of the Riesz space $C(X)$, and $C(X) \approx C(\upsilon X)$ (υX the Hewitt realcompactification [GJ60]). In particular, for X the countable ordinals $C(X) \approx C(\upsilon X) = C(\beta X)$ has σ while $L(X) = \aleph_2$ and $L(\beta X) = \aleph_0$. Here βX is the Čech-Stone compactification.

Remarks 2.5. (a) 2.1 includes the following, which are extractable from [LZ71, Chapter 10]: $\mathbb{R}^{\mathbb{N}}$ has σ; $\mathbb{R}^{\mathbb{R}}$ does not.

(b) 2.1 ((b) $\Longleftrightarrow$ (c)) is to be compared with the result of [J80] (paraphrased slightly): $|X| \leq \aleph_0$ iff $\forall \beta : X \times X \longrightarrow \mathbb{N}\ \exists \gamma, b : X \longrightarrow \mathbb{N}$ with $\beta(x,y) \leq \gamma(x)b(y)\ \forall x,y$.

We now make some comparative remarks about the properties that a Riesz space might have, called Egoroff and strongly Egoroff. Beyond the present remarks, this paper shall not concern them, so we just refer to [LZ71] and [H68] for the definitions and discussion.

(c) 2.1 is to be compared with the result of [BJ86]: $\mathbb{R}^X$ has the Egoroff property iff $|X| < \mathfrak{b}$. Now, neither of σ and Egoroff implies the other.

  (i) $C([0,1])$ fails Egoroff ([LZ71, 68.6]), but has σ (for any compact X, $C(X)$ has σ (1.1)).

  (ii) $\mathrm{PD}(\mathbb{N})$ = the polynomial dominated functions on $\mathbb{N}$ has Egoroff, but fails σ.

  Egoroff: $C(\mathbb{N})$ has Egoroff ([LZ71, 75.1]) and Egoroff is inherited by Riesz ideals ([H68, 2.1]).

  σ fails: Let $b_n(x) = x^n$ $(x \in \mathbb{N})$. Suppose $\forall n\ \lambda_n b_n \leq b \in \mathrm{PD}(\mathbb{N})$. We can suppose b is eventually polynomial, i.e., $b(x) = \alpha x^d$, for x large. Then $\lambda_n x^n \overset{*}{\leq} kx^d\ \forall n$. In particular, $\lambda_{d+1}x^{d+1} \leq kx^d$, which means $x \overset{*}{\leq} k/\lambda_{d+1}$, a contradiction.

  (But, I do not know if, for a $C(X)$, Egoroff implies σ.)

(d) [HM15] shows that, for any compact quasi-F space K, $D(K)$ has σ iff it is strongly Egoroff. Here,

$$D(K) = \{f \in C(K, [-\infty, +\infty]) \mid f^{-1}(-\infty, +\infty) \text{ dense in } K\}.$$

This is a Riesz space exactly because K is quasi-F (which means each dense cozero-set is C^*-embedded).

Now, K is basically disconnected iff $D(K)$ is σ-complete, and in this case, if $D(K)$ is merely Egoroff, then it has σ (using [LZ71], §'s 30 and 74, and the above result from [HM15]).

Any $\mathbb{R}^X$ is of this form, with $K = \beta X$ (X discrete here). From this point of view, 2.1 and [BJ86] say the same thing, though the route to the "sameness" is quite complicated.

3. Sums, and P-spaces

Proposition 3.1. *Suppose $X = \sum_I X_i$, all $X_i \neq \varnothing$. If $C(X)$ has σ, then each $C(X_i)$ has σ, and $|I| < \mathfrak{b}$.*

Proof: Suppose $C(X)$ has σ. Each X_i is C-embedded in X, so $C(X_i)$ has σ by 2.2(b). Now, $\forall i$ choose $y_i \in X_i$. Then $Y = \{y_i\}_I$ is discrete and C-embedded in X. Thus $|I| = |Y| < \mathfrak{b}$, by 2.3. $\square$

Question 3.2. Does the converse of 3.1 hold? (The following tends in that direction.)

Lemma 3.3. *Suppose X has the property: $\forall \{b_n\} \subseteq C(X)^+$, X can be decomposed as $X = \sum_I X_i$, with $|I| < \mathfrak{b}$ and $\forall n, i$, $b_n | X_i$ bounded (the decomposition depending upon $\{b_n\}$). Then, $C(X)$ has σ.*

Proof: $b_n | X_i \leq m_{ni} \in \mathbb{R}$. Define $\{g_n\} \subseteq \mathbb{R}^I$ as $g_n(i) = m_{ni}$. By 2.1, $\mathbb{R}^I$ has σ, so there are $\{\lambda_n\}$ and g with $\lambda_n g_n \leq g$ $\forall n$. Define $b \in C(X)$ as $b|X_i = g(i)$ $\forall i$. Then, $\lambda_n b_n \leq b$ $\forall n$. □

Corollary 3.4. (a) *Suppose $X = \sum_I X_i$, with all X_i compact, and $|I| < b$. Then $C(X)$ has σ. (Thus, for all X_i compact, $\neq \varnothing$, $C(\sum_I X_i)$ has σ iff $|I| < \mathfrak{b}$. We improve this in* 4.1 *and* 4.2 *below.)*

(b) *Suppose X is a P-space (G_δ's are open* [GJ60]*). If $L(X) \leq \mathfrak{b}$, then $C(X)$ has σ.*

Proof: (a) Apply 3.3 (the decomposition *not* depending on $\{b_n\}$).

(b) Suppose $\{b_n\} \subseteq C(X)^+$. For each $x \in X$, set $U_x \equiv \bigcap_n b_n^{-1}(b_n(x))$. Then, $\forall n, x$, $b_n | U_x$ is constant, thus bounded. If X is a P-space, each U_x is a zero-set, thus clopen. Evidently, $U_x \cap U_y \neq \varnothing$ implies $U_x = U_y$ (not $x = y$), so $\{U_x\}_X$ is a clopen partition, i.e., $X = \sum_X U_x$, and if $L(X) \leq \mathfrak{b}$, then $\{U_x\}_X$ has size $< \mathfrak{b}$. Now apply 3.3. □

Question 3.5. Does the converse of 3.4(b) hold?

Examples 3.6. Illustrating 3.4(b):

(a) (A familiar space) $\lambda D = D \cup \{\lambda\}$, with $\lambda \notin D$ and D discrete, with nbhds U of λ having $|D - U| \leq \aleph_0$. This λD is a Lindelöf P-space, and $C(\lambda D)$ has σ by 3.4(b).

(b) (Generalization of (a)) Suppose $\aleph_1 \leq \gamma \leq \mathfrak{b}$, and γ has uncountable cofinality. ($\aleph_1$ and $\mathfrak{b}$ are such γ [D84].)

Let $X = D \cup \{\rho\}$, with $\rho \notin D$ and D discrete, with nbhds U of ρ having $|D - U| < \gamma$. Then, X is a P-space with $L(X) \leq \gamma \leq \mathfrak{b}$, and $C(X)$ has σ by 3.4(a). Evidently, $L(X) \leq \gamma$. If $|D| < \gamma$, then X is discrete, thus P. If $\gamma \leq |D|$, then, for $\{U_n\}$ nbhds of ρ, $\bigcap U_n$ is too because $D - \bigcap U_n = \bigcup (D - U_n)$ and γ has uncountable cofinality.

Note that $\aleph_1 = \gamma$ (*a fortiori*, assuming $\aleph_1 = \mathfrak{b}$) yields $X = \lambda D$.

Remark 3.7. There would seem to be more to the connection of "P-like" properties of X with "$C(X)$ has σ" than is in the above. [HM15] has many examples of $C(X)$ with σ, with X almost P (no dense cozero-sets other than X). These βX are also Lindelöf F-spaces, and some are connected. The connection of this with the present paper is completely unclear.

4. Paracompactness, metrizability, and local compactness

We first state the results, then prove them.

Theorem 4.1. *Suppose X is paracompact. If $C(X)$ has σ, then $L(X) \leq \mathfrak{b}$.*

The converse to 4.1 fails by 1.1(b), or 4.4 below. However,

Theorem 4.2. *Suppose X is paracompact and locally compact. If $L(X) \leq \mathfrak{b}$, then $C(X)$ has σ.*

Note that for X paracompact, $C(X)$ has $\sigma \not\Rightarrow X$ locally compact, by examples in 3.6.

On the other hand, a metrizable space is paracompact (A.H. Stone, see [E89, 4.4.1, 5.1.3]), and we have

Theorem 4.3. *Suppose X is metrizable. If $C(X)$ has σ, then X is locally compact. Thus, using* 4.1, *$C(X)$ has σ iff $L(X) \leq \mathfrak{b}$ and X is locally compact.*

Corollary 4.4. *For $X =$ the rationals, or the irrationals, $C(X)$ fails σ.*

We turn to the proofs.

Proof of 4.1: We show the contrapositive. Suppose $\mathfrak{b} < L(X)$, so there is an open cover with no subcover of size $< \mathfrak{b}$. With X paracompact, we can pass to a locally finite open refinement, say $\mathcal{U}$, and clearly $\bar{\mathfrak{b}} \leq |\mathcal{U}|$.

For each $U \in \mathcal{U}$, choose $p_U \in \mathscr{U}$, and set $B = \{p_U \mid U \in \mathcal{U}\}$. We have a surjection $\mathcal{U} \overset{p}{\twoheadrightarrow} B$ $(U \mapsto p_U)$, and $\forall x \in B$, $|p^{-1}(x)| < \aleph_0$ because $\mathcal{U}$ is locally finite. Thus, $|\mathcal{U}| = \sum\{|p^{-1}(x)| \mid x \in B\} \leq |B| \cdot \aleph_0 \leq |\mathcal{U}| \cdot \aleph_0 = |\mathcal{U}|$. Thus $\mathfrak{b} \leq |B|$.

Now, B is closed and discrete (we claim), thus C-embedded (because X is normal), so $C(X)$ fails σ (by 1.3).

To prove the claim: Any $x \in X$ has a nbhd G for which $U \in \mathcal{U}$ has $U \cap G \neq \varnothing$ for only say $U_1, \ldots, U_n \in \mathcal{U}$. If $x \notin B$, then $H = G - \{p_{U_i} \mid i = 1, \ldots, n\}$ is a nbhd of x with $H \cap B = \varnothing$; so B is closed. If $x \in B$, then $H = G - \{p_{U_i} \mid x \neq p_{U_i}\}$ is a nbhd of x with $H \cap B = \{x\}$; so B is discrete. □

For the proof of 4.2, we employ the fact that X is paracompact and locally compact iff $X = \sum_I X_i$, with each X_i σ-compact and locally compact (Morita, see [E89, 5.1.27]). That is, 4.2 is really the following (again, a "sum theorem").

Theorem 4.2'. *Suppose $X = \sum_I X_i$, with each X_i σ-compact and locally compact, and $|I| < \mathfrak{b}$. Then, $C(X)$ has σ.*

Note that this includes the fact that for X σ-compact and locally compact, $C(X)$ has σ ($|I| = 1$). This can be shown directly (more easily) from the fact that $\mathbb{R}^{\mathbb{N}}$ has σ (2.1).

The proof of 4.2', will use the following.

Lemma 4.5. *Suppose Y is σ-compact locally compact. Then, there is a countable family of compact subsets $\{K_j\}_{\mathbb{N}}$ and $\{u_j\}_{\mathbb{N}} \subseteq C(Y, [0,1])$ with $u_j | K_j = 1\ \forall j$, with $Y = \bigcup_{\mathbb{N}} K_j$ and $\{\operatorname{coz} u_j\}_{\mathbb{N}}$ locally finite (and a cover of Y).*

Proof: $Y = \operatorname{coz} g$ for some $g \in C(\beta Y)^+$. Then, $f = 1/g \in C(Y)$; define $K_j = f^{-1}[j, j+2] \subseteq f^{-1}(j-1, j+3) \equiv U_j$. Here, K_j is compact and U_j is open, so by normality of Y, there is $u_j \in C(Y, [0,1])$ with $u_j | K_j = 1$ and $u_j | (Y - u_u) = 0$. Note that $U_m \cap U_n \neq \varnothing$ implies $|u - m| \leq 3$, so we have the local finiteness.

Proof of 4.2': By 3.8, write each $X_i = \bigcup_j K_j^i$, with associated $\{u_j^i\}_j \subseteq C(X_i,[0,1])$. So $X = \bigcup\{K_j^i \mid i \in I, j \in \mathbb{N}\}$.

Now suppose $\{b_n\} \subseteq C(X)^+$. Then, $\forall n,i,j$, $b_n|K_j^i \leq$ some $c(n,i,j) \in \mathbb{N}$. Define $\{c_n\} \subseteq \mathbb{N}^{I\times\mathbb{N}}$ as $c_n(i,j) = c(n,i,j)$.

Since $|I \times \mathbb{N}| < \mathfrak{b}$, there are $\{\lambda_n\}$ and $c \in \mathbb{N}^{I\times\mathbb{N}}$ with $\lambda_n c_n \leq c$ $\forall n$ (by 1.1). That is, $\forall n$ $\lambda_n c_n(i,j) \leq c(i,j)$ $\forall i,j$.

Define $b \in C(X)$ by defining $b|X_i = d_i$ $\forall i$, where $d_i = \sum_j c(i,j)u_j^i$ (with apologies for "d"). This is well-defined by the local finiteness feature of $\{\operatorname{coz} u_j^i\}_j$.

We claim $\lambda_n b_n \leq b$ $\forall n$. Take $n \in \mathbb{N}$, and $x \in X$. Then for unique i, $x \in X_i = \bigcup_j K_j^i$, and $b(x) = d_i(x)$. There is j_0 with $x \in K_{j_0}^i$. Then, $b_n(X) \leq c(n,i,j_0) = c_n(i,j_0)$, so $\lambda_n b_n(x) \leq \lambda_n c_n(i,j_0) \leq c(i,j_0) = c(i,j_0)u_{j_0}^i(x) \leq \sum_j c(i,j)u_j^i = d_i(x) = b(x)$. □

Proof of 4.3: We show the contrapositive.

By [GJ60, 1.21], Y is pseudocompact iff Y contains no C-embedded copy of $\mathbb{N}$. Thus if X is metrizable, and $p \in X$ has no compact nbhd then for any nbhd G of p, $\overline{G}$ contains a countable closed discrete set $\{y_j\}_{\mathbb{N}} = Y(G) \not\ni p$ with $b(G) \in C(X)^+$ for which $b(G)(y_j) = j$ $\forall j$. (For the metrizable $\overline{G}$, pseudocompact = compact.)

Now suppose X is not locally compact, $p \in X$ having no compact nbhd. Take any metric for X, "diameter" meant with respect to it. Choose a nbhd G_1 with $\operatorname{diam} G_1 \leq 1$, and then (per above) $Y(G_1) = \{y_j^1\}_j$ and $b_1 = b(G_1)$ with $b_1(y_j^1) = j$ $\forall j$.

Inductively, choose nbhds G_n of p with $\operatorname{diam} G_n \leq 1/n$ with $p \notin Y(G_n) = \{y_j^n\}_j$, $G_n \cap Y(G_i) = \varnothing$ for $i < n$, and $b_n = b(G_n)$ with $b_n(y_j^n) = j$ $\forall j$.

Note: For any choices $z_n \in Y(G_n)$, $z_n \longrightarrow p$.

Toward contradiction, suppose $\exists\{\lambda_n\}, b$ with $\lambda_n b_n \leq b$ $\forall n$. Then, $\forall n$, choose $z_n \in Y(G_n)$ with $n/\lambda_n \leq b_n(z_n)$. Thus $n \leq b(z_n)$ and $z_n \longrightarrow p$, so $b(p) = +\infty$, which is not possible. □

4.4 follows immediately from 4.3.

One notes a similarity in the arguments showing 4.3 and 1.1(b). We do not pursue this now.

5. A characterization

We show that "$C(X)$ has σ" has equivalents in terms of the position of X in its Čech-Stone compactification βX, and a covering condition.

Let $\operatorname{coz}(\beta X, X) = \{S \mid S \text{ is cozero in } \beta X \text{ and } S \supseteq X\}$, and $\operatorname{coz}_\delta(\beta X, X)$ the family of countable intersections from $\operatorname{coz}(\beta X, X)$.

Theorem 5.1. *The following conditions on X are equivalent.*

(a) *$C(X)$ has σ.*
(b) *$\operatorname{coz}(\beta X, X)$ is co-initial in $\operatorname{coz}_\delta(\beta X, X)$ (for $\subseteq$).*
(c) *For each $\{b_n\} \subseteq C(X)^+$, there is a countable cozero cover $\{U_j\}$ of X with $b_n|U_j$ bounded $\forall n, j$.*

This is essentially very easy, but (b) $\Longrightarrow$ (a) uses that $C(Y)$ will have σ if Y is σ-compact and locally compact (which uses "$\mathbb{R}^{\mathbb{N}}$ has σ"); this is 4.2' with $|I| = 1$.

Proof: Note that $S \in \operatorname{coz}(\beta X, X)$ iff there is $f \in C(\beta X, [0, +\infty])$ with $S = f^{-1}([0, +\infty))$. (Given $S = \operatorname{coz} u$, let $f \equiv 1/u$. Given f let $u = 1/f$, so $\operatorname{coz} u = f^{-1}([0, +\infty])$.) Such S is σ-compact locally compact.

(b) $\Longrightarrow$ (a). Given $\{b_n\}$ in (a), let $f_n \in C(\beta X, [0, +\infty])$ have $f_n|X = b_n$. Take $S \subseteq \bigcap_n f_n^{-1}[0, +\infty)$ by (b). Let $\overline{b}_n \equiv f_n|S \in C(S)$. Now $C(S)$ has σ (as noted above), so there are $\{\lambda_n\}$ and $\overline{b} \in C(S)$ with $\lambda_n \overline{b}_n \leq \overline{b}$ $\forall n$. Thus $\lambda_n b_n \leq \overline{b}|X \equiv b \in C(S)$.

(a) $\Longrightarrow$ (c). Given $\{b_n\}$ in (c), we have $\lambda_n b_n \leq b$ by (a). Let $U_j \equiv b^{-1}(j - 1, j + 1)$. If $x \in U_j$, we have $\lambda_n b_n(x) \leq b(x) \leq j + 1$, so $b_n(x) \leq (j + 1)/\lambda_n$.

(c) $\Longrightarrow$ (b). Given $\{S_n\}$ in (b), take f_n as above with $f_n = f_n^{-1}[0, +\infty)$. Let $b_n = f_n|X$. Take $\{U_j\}$ by (c). Take $V_j \in \operatorname{coz} \beta X$ with $V_j \cap X = U_j$. (X is C^*-embedded, thus z-embedded, in βX.) Let $S \equiv \bigcup_j U_j$; $S \in \operatorname{coz}(\beta X, X)$. Then $S \subseteq \bigcap_n S_n$ because $\forall n, j$ $b_n|U_j$ is bounded, so $f_n|V_j$ is bounded (because U_j is dense in V_j). Thus $U_j \subseteq S_n$. □

5.1 provides an easy partial answer to our Question 3.2.

Corollary 5.2. *Suppose $X = \sum_I X_i$ and all $C(X_i)$ have σ. If $|I| \leq \aleph_0$, then $C(X)$ has σ.*

Proof: (sketch) We can suppose $I = \mathbb{N}$. Note that, $\forall n$ βX_n is (equivalent to) the closure of X_n in βX. If $T \in \operatorname{coz}_\delta(\beta X, X)$, then $\forall n$ $T_n \equiv T \cap \beta X_n \in \operatorname{coz}_\delta(\beta X_n, X_n)$ and there is $S_n \in \operatorname{coz}(\beta X_n, X_n)$ with $S_n \subseteq T_n$ (by 5.1). Set $S \equiv \bigcup S_n$. We have $X \subseteq S \subseteq T$, and $S \in \operatorname{coz}(\beta X, X)$ since it is σ-compact and locally compact. □

Corollary 5.3. *Suppose X is Lindelöf and Čech-complete. If $C(X)$ has σ, then X is locally compact.*

Proof: The hypothesis is equivalent to $X = \bigcap S_n$ for some $\{S_n\} \subseteq \operatorname{coz}(\beta X, X)$; see [E89]. From 5.1, choose $S \subseteq \bigcap_n S_n$. We must have $S = X$. □

Remarks 5.4. (a) 5.3 and 3.7 appear to be "incomparable". (But, a locally compact space is Čech-complete.) See [E89].

- (b) "Čech-complete" cannot be dropped in 5.3, because for an infinite Lindelöf P-space X, $C(X)$ has σ but X is not locally compact.
- (c) It follows from 5.1 and 2.1 that discrete X satisfies 5.1(b) iff $|X| < \mathfrak{b}$. This seems interesting.
- (d) It follows from 5.1 and 3.4(b) that a Lindelöf P-space satisfies 5.1(b). This is given a direct proof in [BGHTZ09], 3.2 to a seemingly unrelated purpose.
- (e) While there is an obvious similarity of 5.1(c) with 3.3, I do not see a real connection.
- (f) For X the irrationals, 4.4 showed that $C(X)$ fails σ. 5.3 provides another proof.
- (g) [HM15] contains a result similar to 5.1 for $D(K)$, K quasi-F.

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