A note on the commutator of two operators on a locally convex space

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Abstract. Denote by C the commutator $AB - BA$ of two bounded operators A and B acting on a locally convex topological vector space. If $AC - CA = 0$, we show that C is a quasinilpotent operator and we prove that if $AC - CA$ is a compact operator, then C is a Riesz operator.

Keywords: locally convex space; commutator; nilpotent operator; compact operator; Riesz operator

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1. Introduction

Let X be a complex Hausdorff locally convex topological vector space. A system of seminorms $P = \{p_\alpha : \alpha \in \Delta\}$ inducing the topology on X will be called a *calibration*. We denote by $\mathcal{P}(X)$ the collection of all calibrations on X. For a given seminorm p_α we denote $U_\alpha = \{x \in X : p_\alpha(x) < 1\}$. A calibration P is directed if for each $p_\alpha, p_\beta \in P$ there is some $p_\gamma \in P$ such that $p_\alpha \leq p_\gamma$ and $p_\beta \leq p_\gamma$. For a given calibration P the system of semiballs $\{\varepsilon U_\alpha : \varepsilon > 0, \alpha \in \Delta\}$ forms a neighborhood base at 0. Let us denote by $\mathcal{L}(X)$ the set of all linear continuous operators on X. An operator $T \in \mathcal{L}(X)$ is *compact* ($T \in \mathcal{K}(X)$) if there is some open neighborhood W at 0 such that $T(W)$ is a relatively compact set, and T is *bounded* ($T \in \mathcal{B}(X)$) if $T(W)$ is a bounded set. If P is some given directed calibration on X we can replace the set W by some semiball U_γ in the above definition. If the set $T(U_\gamma)$ is bounded and $p_\gamma \in P$ is the corresponding seminorm for U_γ, then for each $p_\alpha \in P$ there is some $c_\alpha > 0$ such that $p_\alpha(Tx) \leq c_\alpha p_\gamma(x)$, $x \in X$, $\alpha \in \Delta$. We say that T is bounded with respect to the seminorm p_γ. For a given $P \in \mathcal{P}(X)$ we denote by $B_P(X)$ the collection of all linear operators T on X for which $p_\alpha(Tx) \leq c p_\alpha(x)$, where $x \in X$, $p_\alpha \in P$, and $c > 0$ is independent of $\alpha \in \Delta$. $B_P(X)$ is a unital normed algebra with respect to the norm $\|T\|_P = \sup\{p_\alpha(Tx) : \; p_\alpha(x) \leq 1, \; x \in X, \; p_\alpha \in P\}$. For a given $p_\alpha \in P$ let J_α denote the null space of p_α. The quotient space $X_\alpha = X/J_\alpha$ is a normed space with the norm $\|x_\alpha\|_\alpha = p_\alpha(x)$, where $x_\alpha = x + J_\alpha$, and $\widetilde{X}_\alpha$ denotes the completion of X_α. Let $T \in \mathcal{L}(X)$ be such that $T(J_\alpha) \subseteq J_\alpha$, then the corresponding operator T_α on X_α is well-defined by $T_\alpha(x_\alpha) = Tx + J_\alpha$, its continuous extension

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to $\widetilde{X}_\alpha$ will be denoted by $\widetilde{T}_\alpha$. For a given $T \in \mathcal{L}(X)$ the number $\lambda \in \mathbb{C}$ is in the resolvent set of T iff $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The spectrum $\sigma(T)$ is the complement of the resolvent set. An operator T is *quasinilpotent* if $\sigma(T) = \{0\}$. For an associative algebra $\mathcal{A}$ and any $a, b \in \mathcal{A}$ the commutator $ab - ba$ will be denoted by $[a, b]$ or also by $\delta_a(b)$.

2. The results

Lemma 1. *Let X be a locally convex space and let $\mathcal{F} = \{A_i : i \in N\}$ be a finite family of operators in $\mathcal{B}(X)$, where $N = \{1, 2, \ldots, n\}$. Let $\mathcal{A}$ be the algebra of operators generated by $\mathcal{F}$. Then there exists a calibration $P \in \mathcal{P}(X)$ such that the following hold:*

(i) *$\mathcal{A}$ is contained in $B_P(X)$,*
(ii) *there is some $p_\gamma \in P$ such that all operators from $\mathcal{A}$ are bounded with respect to the seminorm p_γ.*

Proof: (i) Let $P_0 = \{q_\alpha : \alpha \in \Delta\}$ be a directed calibration on X. For any $A_i \in \mathcal{F}$ there exists some $q_\gamma^{(i)} \in P_0$ such that for each $\alpha \in \Delta$ the following holds

$$q_\alpha(A_i x) \leq a_\alpha^{(i)} q_\gamma^{(i)}(x), \quad x \in X$$

for some $a_\alpha^{(i)} > 0$. Write $\lambda_\alpha = \max\{a_\alpha^{(i)} : i \in N\}$, $\alpha \in \Delta$, and let $q_\gamma \in P_0$ be a common successor of $q_\gamma^{(i)}$, $i \in N$. Then, clearly for each $i \in N$ we have $q_\alpha(A_i x) \leq \lambda_\alpha q_\gamma(x)$, $x \in X$, and for any $T \in \mathcal{A}$ there is some $t_\alpha > 0$ such that

$$q_\alpha(Tx) \leq t_\alpha q_\gamma(x), \ x \in X. \tag{1}$$

Let us define a new family of seminorms $P = \{p_\alpha : \alpha \in \Delta\}$, where $p_\alpha(x) = \max\{q_\alpha(x), \lambda_\alpha q_\gamma(x)\}$, $x \in X$, $\alpha \in \Delta$. For each $\alpha \in \Delta$, $q_\alpha \leq p_\alpha$ and $p_\alpha \leq \max\{1, \lambda_\alpha\} \max\{q_\alpha, q_\gamma\}$, thus P is a calibration on X. For any $p_\alpha \in P$ and any $A_i \in \mathcal{F}$ we have $p_\alpha(A_i x) = \max\{q_\alpha(A_i x), \lambda_\alpha q_\gamma(A_i x)\} \leq \max\{\lambda_\alpha q_\gamma(x), \lambda_\alpha \lambda_\gamma q_\gamma(x)\}$ $\leq c_0 \lambda_\alpha q_\gamma(x) \leq c_0 p_\alpha(x)$, $x \in X$, where $c_0 = \max\{1, \lambda_\gamma\}$, hence $A_i \in B_P(X)$. Then we have, for any $T \in \mathcal{A}$, $p_\alpha(Tx) \leq c p_\alpha(x)$, where c is independent of $\alpha \in \Delta$. Thus, $T \in B_P(X)$.

(ii) Choose any $p_\alpha \in P$ and any $T \in \mathcal{A}$. By (1) and by the relationship between P_0 and P we obtain $p_\alpha(Tx) \leq \max\{1, \lambda_\alpha\} \max\{q_\alpha(Tx), q_\gamma(Tx)\} \leq d_\alpha q_\gamma(x) \leq d_\alpha p_\gamma(x)$, $x \in X$, where $d_\alpha = \max\{1, \lambda_\alpha\} \max\{t_\alpha, t_\gamma\}$. □

In the following lemma we specify some properties of the passage to the quotient space on which the induced operators are well-defined.

Lemma 2. *Let X be a locally convex space and let $\mathcal{F}$ be, as above, a finite family of bounded operators. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$ and let $P \in \mathcal{P}(X)$ and $p_\gamma \in P$ be from the previous lemma. Then for each $p_{\gamma'} \in P$ for which $p_\gamma \leq p_{\gamma'}$ the following hold.*

(i) $\widetilde{(S+T)}_{\gamma'} = \widetilde{S}_{\gamma'} + \widetilde{T}_{\gamma'}$, $S, T \in \mathcal{A}$.

(ii) $\widetilde{(ST)}_{\gamma'} = \widetilde{S}_{\gamma'}\widetilde{T}_{\gamma'}$, $S,T \in \mathcal{A}$.
(iii) $\|\widetilde{T}_{\gamma'}\|_{\gamma'} \leq \|T\|_P$, $T \in \mathcal{A}$.
(iv) *If $T \in \mathcal{A}$ is a compact operator for which $T(U_{\gamma'})$ is a relatively compact set, then $\widetilde{T}_{\gamma'}$ is a compact operator, too.*

Proof: By the preceding lemma, and by the assumption $p_\gamma \leq p_{\gamma'}$, each $T \in \mathcal{A}$ is also bounded with respect to the seminorm $p_{\gamma'}$. Especially we have $T(J_{\gamma'}) \subseteq J_{\gamma'}$. Thus, the corresponding operator $T_{\gamma'}$ on $X_{\gamma'}$ and its extension $\widetilde{T}_{\gamma'}$ on $\widetilde{X}_{\gamma'}$ are well-defined and are bounded. By [4, p. 413], we have the equalities (i) and (ii). Let us prove (iii). The algebra $\mathcal{A}$ is contained in $B_P(X)$, hence for each $T \in \mathcal{A}$ it follows

$$p_\alpha(Tx) \leq \|T\|_P p_\alpha(x),\ x \in X,\ p_\alpha \in P.$$

Especially, $p_{\gamma'}(Tx) \leq \|T\|_P p_{\gamma'}(x)$, $x \in X$, then also $\|T_{\gamma'}\|_{\gamma'} \leq \|T\|_P$, and also $\|\widetilde{T}_{\gamma'}\|_{\gamma'} \leq \|T\|_P$. Since each relatively compact set is also totally bounded, the statement (iv) follows by [4, p. 413]. □

Let $\mathcal{A}$ be an associative algebra and $a, b \in \mathcal{A}$ such that $\delta_a^2(b) = [a, \delta_a(b)] = 0$. Then the following is true (see e.g. [1, p. 86])

$$\delta_a^n(b^n) = n!\delta_a(b)^n,\ n \in \mathbb{N}. \tag{2}$$

Proposition 1. *Let $\mathcal{A}$ be an associative algebra and assume that $a, b \in \mathcal{A}$ satisfy the conditions $\delta_a^2(b) = 0$ and $[b, a^n] = 0$ for some $n \in \mathbb{N}$. Then*

$$\delta_a(b)^{2n-1} = 0.$$

Proof: By the assumption $\delta_a^2(b) = 0$ we have $[a, \delta_b(a)] = -[a, \delta_a(b)] = 0$. Then it is easy to show by induction that $[b, a^k] = ka^{k-1}[b, a]$, for each $k \in \mathbb{N}$. Thus for $k = n$, and by the above assumption we obtain $a^n b = a^{n-1}ba$. If we multiply this equality by a on the left, we have $a^{n+1}b = ba^{n+1}$. In the same way we obtain by induction

$$a^{n+k}b = ba^{n+k},\ k = 0, 1, 2, \dots$$

Denoting $c := b^{2n-1}$ we have $\delta_a^{2n-1}(c) = \sum_{j=0}^{2n-1}(-1)^j\binom{2n-1}{j}a^{2n-1-j}ca^j$. For $0 \leq j \leq n-1$ we have $a^{2n-1-j}ca^j = ca^{2n-1} = a^{2n-1}c$, and for $n \leq j \leq 2n-1$ we have $a^{2n-1-j}ca^j = a^{2n-1}c$. Hence it follows $\delta_a^{2n-1}(c) = a^{2n-1}c\sum_{j=0}^{2n-1}(-1)^j\binom{2n-1}{j} = 0$. Then by (2) we obtain $(2n-1)!\delta_a(b)^{2n-1} = \delta_a^{2n-1}(b^{2n-1}) = 0$. □

Corollary 1. *Let $\mathcal{A}$ be an associative algebra. If $a, b \in \mathcal{A}$ are such that $\delta_a^2(b) = 0$ and $a^n = 0$ for some $n \in \mathbb{N}$, then $\delta_a(b)^{2n-1} = 0$.*

The following theorem is the classical Kleinecke-Shirokov theorem if X is a Banach space.

Theorem 1. *Let X be a sequentially complete locally convex space and let $A, B \in \mathcal{B}(X)$ be such that $\delta_A^2(B) = 0$. Then $\delta_A(B)$ is a quasinilpotent operator.*

Proof: It may be supposed that X is not a normed space. By Lemma 1, there is some $P \in \mathcal{P}(X)$ such that $A, B \in B_P(X)$. Since X is sequentially complete, $B_P(X)$ is a Banach algebra (see, e.g. [2]). Write $T = \delta_A(B)$, then by (2) for each $\lambda \neq 0$ there exists $(T - \lambda I)^{-1} \in B_P(X) \subseteq \mathcal{L}(X)$, hence $\sigma(T) \subseteq \{0\}$. Now, T is a bounded operator acting on a non-normable locally convex space, hence, by a consequence of Kolmogorov theorem on normability of topological vector spaces, T is not invertible. Thus, $0 \in \sigma(T)$. $\square$

In the following theorem we shall assume that A is an algebraic operator with the minimal polynomial μ. This means μ is a monic polynomial with minimal degree such that $\mu(A) = 0$. This theorem was formulated and proved in [3] for the algebra of bounded operators on a Banach space, actually, the proof is valid for operators on any complex vector space. We prove the same result by partially alternative arguments based on Proposition 1.

Theorem 2. *Let X be a complex vector space and $A, B \in \mathcal{L}(X)$ be such that $\delta_A^2(B) = 0$. Let A be an algebraic operator with the minimal polynomial $\mu(\lambda) = \prod_{j=1}^n (\lambda - \lambda_j)^{n_j}$, where $\{\lambda_j\}$ are distinct. Then for $m = 2\max\{n_j\} - 1$ holds*

$$\delta_A(B)^m = 0.$$

Proof: For the algebraic operator A with the above minimal polynomial the following decomposition holds $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ on $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$, where, for $j = 1, \ldots, n$, $X_j = ker((A - \lambda_j I)^{n_j})$, $A_j = A|_{X_j}$ and $(A - \lambda_j I)|_{X_j}$ is a nilpotent operator of order n_j (see e.g. [5]). Choose any $j \in \{1, 2, \ldots, n\}$. By the equality $\delta_{A-\lambda_j I}^2(B) X_j = \{0\}$ we can prove in the same way as in [3] that $B(X_j) \subseteq X_j$. Thus, by Corollary 1, $(\delta_{A-\lambda_j I}(B)|_{X_j})^{2n_j - 1} = 0$. Hence $\delta_A(B)^m = 0$, where $m = 2\max\{n_j\} - 1$. $\square$

Corollary 2. *Let X be a complex vector space and let $A, B \in \mathcal{L}(X)$ be such that $\delta_A^2(B) = 0$. Let A be an algebraic operator for which the minimal polynomial has only simple zeroes. Then A commutes with B.*

We can find in [4, p. 405] a definition of a Riesz operator acting on a Hausdorff topological vector space. The following theorem is a generalization to locally convex spaces of a result proven in [6] for the Banach spaces.

Theorem 3. *Let X be a sequentially complete locally convex space and let $A, B \in \mathcal{B}(X)$. If $\delta_A^2(B)$ is a compact operator, then $\delta_A(B)$ is a Riesz operator.*

Proof: Let us denote by $\mathcal{A}$ the algebra of operators generated by A and B. Denoting $C = \delta_A^2(B)$, we shall prove that

$$\delta_A^n(B^n) = n!\delta_A(B)^n + K_n, \ n = 2, 3, \ldots, \tag{3}$$

where K_n can be written as

$$K_n = E_n C + C E_n' + \sum_{i \in M_n} F_i C F_i', \ n = 2, 3, \ldots, \tag{4}$$

where M_2 is an empty set, for $n \geq 3$, M_n is some finite set of natural numbers, and all operators belong to the algebra $\mathcal{A}$. Indeed, for $n = 2$ we have $\delta_A^2(B^2) = B\delta_A^2(B) + 2\delta_A(B)^2 + \delta_A^2(B)B = 2\delta_A(B)^2 + K_2$, where $K_2 = BC + CB$. For a given $n \geq 2$, let (3) be true and let K_n be of the form (4). Then by the Leibniz formula it follows

$$\delta_A^n(B^{n+1}) = n!\delta_A(B)^n B + n\delta_A^{n-1}(B^n)\delta_A(B) + S_n,$$

where $S_n = K_n B + \sum_{k=2}^n \binom{n}{k}\delta_A^{n-k}(B^n)\delta_A^k(B)$. Applying the operator δ_A on both sides of the above equality, and taking into account (3) for the given n, we obtain by a simple calculation

$$\begin{aligned}\delta_A^{n+1}(B^{n+1}) = n!\delta_A(B)^{n+1} + n(n!\delta_A(B)^n + K_n)\delta_A(B) + n!(\delta_A^2(B)\delta_A(B)^{n-1} \\ +\delta_A(B)\delta_A^2(B)\delta_A(B)^{n-2} + \cdots + \delta_A(B)^{n-1}\delta_A^2(B))B + n\delta_A^{n-1}(B^n)\delta_A^2(B) + \delta_A(S_n) \\ = (n+1)!\delta_A(B)^{n+1} + K_{n+1}.\end{aligned}$$

Since (4) is closed for left/right multiplications by elements from $\mathcal{A}$, and δ_A is inner derivation, so K_{n+1} is again of the form (4). Note, that (3) follows directly from the relation (2) considering the quotient algebra $\mathcal{L}(X)/\mathcal{K}(\mathrm{X})$, but we need also the form of operators K_n given in (4). By Lemma 1 there is some $P \in \mathcal{P}(X)$, and $p_\gamma \in P$ such that $\mathcal{A} \subseteq B_P(X)$ and all operators from $\mathcal{A}$ are bounded with respect to the seminorm p_γ. Since $C \in \mathcal{K}(X)$, we can find some semiball $U_{\gamma'} \subseteq U_\gamma$ such that $C(U_{\gamma'})$ is relatively compact. Clearly, $p_\gamma \leq p_{\gamma'}$, hence

$$p_\alpha(Tx) \leq d_\alpha p_\gamma(x) \leq d_\alpha p_{\gamma'}(x),\ \alpha \in \Delta,\ T \in \mathcal{A},$$

for some $d_\alpha > 0$. Especially for $\alpha = \gamma'$ we have $p_{\gamma'}(Tx) \leq d_{\gamma'} p_{\gamma'}(x)$, consequently $T(U_{\gamma'}) \subseteq d_{\gamma'}U_{\gamma'}$, for all $T \in \mathcal{A}$. Now, it is easy to see, by (4), that $K_n(U_{\gamma'})$ is relatively compact set for each $n \geq 2$. The relation (3) implies

$$\delta_A(B)^n - C_n = \frac{1}{n!}\delta_A^n(B^n),\ n = 2, 3, \ldots,$$

where $C_n = -K_n/n!$ are compact operators contained in $\mathcal{A}$. Clearly, $U_{\gamma'}$ is a semiball for which $C_n(U_{\gamma'})$ are relatively compact sets for all n. Fix any $n \geq 2$, then

$$\|\delta_A(B)^n - C_n\|_P = \frac{1}{n!}\|\delta_A^n(B^n)\|_P \leq \frac{1}{n!}\|\delta_A\|^n \|B\|_P^n.$$

Using Lemma 2, we get

$$\|\widetilde{\delta_A(B)}_{\gamma'}^{\,n} - \widetilde{(C_n)}_{\gamma'}\|_{\gamma'} \leq \|\delta_A(B)^n - C_n\|_P \leq \frac{c^n}{n!},$$

where $c = \|\delta_A\|\|B\|_P$, and $\widetilde{(C_n)}_{\gamma'}$ is compact operator. Therefore also holds

$$\inf_{T_{\gamma'} \in \mathcal{K}(\widetilde{X}_{\gamma'})} \|\widetilde{\delta_A(B)}_{\gamma'}^{\,n} - T_{\gamma'}\|_{\gamma'} \leq \frac{c^n}{n!}.$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty}\{\inf_{T_{\gamma'}\in\mathcal{K}(\widetilde{X}_{\gamma'})}\|\widetilde{\delta_A(B)_{\gamma'}}^{\,n} - T_{\gamma'}\|_{\gamma'}\}^{1/n} = 0.$$

Thus, $\widetilde{\delta_A(B)_{\gamma'}}$ is by [8] an asymptotically quasi-compact operator on $\widetilde{X}_{\gamma'}$, which means by [8] that it is a Riesz operator on $\widetilde{X}_{\gamma'}$. Therefore, $\delta_A(B)$ is then by [7, Theorems 6.2, 4.2 and 6.3] a Riesz operator on X. □

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