# Remarks on $LBI$-subalgebras of $C(X)$

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*Abstract.* Let $A(X)$ denote a subalgebra of $C(X)$ which is closed under local bounded inversion, briefly, an $LBI$-subalgebra. These subalgebras were first introduced and studied in Redlin L., Watson S., *Structure spaces for rings of continuous functions with applications to realcompactifications*, Fund. Math. **152** (1997), 151–163. By characterizing maximal ideals of $A(X)$, we generalize the notion of $z_A^\beta$-ideals, which was first introduced in Acharyya S.K., De D., *An interesting class of ideals in subalgebras of $C(X)$ containing $C^*(X)$*, Comment. Math. Univ. Carolin. **48** (2007), 273–280 for intermediate subalgebras, to the $LBI$-subalgebras. Using these, it is simply shown that the structure space of every $LBI$-subalgebra is homeomorphic with a quotient of $\beta X$. This gives a different approach to the results of Redlin L., Watson S., *Structure spaces for rings of continuous functions with applications to realcompactifications*, Fund. Math. **152** (1997), 151–163 and also shows that the Banaschewski-compactification of a zero-dimensional space $X$ is a quotient of $\beta X$. Finally, we consider the class of complete rings of functions which was first defined in Byun H.L., Redlin L., Watson S., *Local invertibility in subrings of $C^*(X)$*, Bull. Austral. Math. Soc. **46**(1992), 449–458. Showing that every such subring is an $LBI$-subalgebra, we prove that the compactification of $X$ associated to each complete ring of functions, which is identified in Byun H.L., Redlin L., Watson S., *Local invertibility in subrings of $C^*(X)$*, Bull. Austral. Math. Soc. **46**(1992), 449–458 via the mapping $\mathcal{Z}_A$, is in fact, the structure space of that subring. Henceforth, some statements in Byun H.L., Redlin L., Watson S., *Local invertibility in subrings of $C^*(X)$*, Bull. Austral. Math. Soc. **46**(1992), 449–458 could be proved in a different way.

*Keywords:* local bounded inversion; structure space; $z_A^\beta$-ideal; complete ring of functions

*Classification:* 54C30, 46E25

## 1. Introduction

Throughout this paper all topological spaces are assumed to be completely regular and Hausdorff. For a given topological space $X$, $C(X)$ denotes the algebra of all real-valued continuous functions on $X$, $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded continuous functions. For each $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ denotes the zero-set of $f$ and $Coz(f)$ denotes the complement of $Z(f)$ with respect to $X$. For each element $f$ of an intermediate

DOI 10.14712/1213-7243.2015.158

subalgebra $A(X)$ (i.e., $C^*(X) \subseteq A(X) \subseteq C(X)$), $\mathcal{Z}_A(f)$ denotes $\{E \in Z(X) : \exists g \in A(X) : fg|_{X\setminus E} = 1\}$ (refer to [6] for more details about the mapping $\mathcal{Z}_A$). By a realcompactification of $X$ we mean a realcompact space containing $X$ as a dense subspace. For a topological space $X$, $\beta X$ is the Stone-Čech compactification of $X$ and $\upsilon X$ is the Hewitt-realcompactification of $X$. Every $f \in C(X)$ may be considered as a continuous function from $X$ into the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ and thus it has a Stone extension $f^* : \beta X \to \mathbb{R}^*$. Clearly, if $f$ is bounded, then $f^*$ is the same as $f^\beta$. The set of all points in $\beta X$ where $f^*$ takes real values is denoted by $\upsilon_f X$, i.e., $\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}$. For a subring $R$ of $C(X)$ we set $\upsilon_R X = \{p \in \beta X : f^*(p) < \infty, \forall f \in R\} = \bigcap_{f\in R} \upsilon_f X$. It follows that $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_R X$ for each subring $R$ of $C(X)$, see [1] for more details. A maximal ideal $M$ of a subalgebra $A(X)$ is called real maximal, if $A(X)/M \cong \mathbb{R}$. If the field $A(X)/M$ properly contains a copy of $\mathbb{R}$, then $M$ is called a hyper-real maximal ideal. A subalgebra $A(X)$ of $C(X)$ is called closed under bounded inversion, briefly, a $BI$-subalgebra, if $f$ is invertible in A$(X)$ whenever $f \in A(X)$ with $f \geq 1$. Also, $A(X)$ is called a $\beta$-subalgebra, if the structure space of $A(X)$ is homeomorphic with $\beta X$ ([13, Definition 2.5]). It is shown in [13, Theorem 2.8] that every $\beta$-subalgebra is a $BI$-subalgebra. However, the converse is not true, in general. For example, let $p, q \in \beta X \setminus \upsilon X$ and $I = M^p \cap M^q$, then [17, Remark 1.7 and Remark 4.1] and [13, Theorem 2.9] show that $I + \mathbb{R}$ is a $BI$-subalgebra which is not a $\beta$-subalgebra. It is easy to see that every intermediate subalgebra $A(X)$ is a $\beta$-subalgebra. However, a $\beta$-subalgebra need not be an intermediate subalgebra. For example, whenever $p \in \beta X \setminus \upsilon X$, then $M^p + \mathbb{R}$ is a $\beta$-subalgebra which is not an intermediate subalgebra (refer to [13] and [17]). Note that [13, Theorem 2.9] shows that the $\beta$-subalgebras which are also closed under uniform topology are precisely the intermediate subalgebras. A subalgebra $A(X)$ of $C(X)$ is called closed under local bounded inversion, briefly, $LBI$-subalgebras, if whenever $f \in A(X)$ is bounded away from zero on some cozero-set $E$, then $f$ is $E$-regular in $A(X)$; i.e., if $f \geq c > 0$ on $E$, then there exists $g \in A(X)$ such that $fg|_E = 1$. These subalgebras were introduced and studied in [15]. It is easy to see that every $LBI$-subalgebra is a $BI$-subalgebra. However, the converse of this statement does not hold, in general (see Example 2.2 in the next section). In [13, Theorem 2.8] it is stated that the collection of all maximal ideals of a $\beta$-subalgebra $A(X)$ is $\{M^p_A : p \in \beta X\}$, in which $M^p_A = \{f \in A(X) : (fg)^*(p) = 0, \forall g \in A(X)\}$. Moreover, it follows from [13, Proposition 2.7] that every maximal ideal of a $BI$-subalgebra $A(X)$ is of the form $M^p_A$, for some $p \in \beta X$. Following [13] we set $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$ for each $f$ in a subalgebra $A(X)$; thus, $M^p_A = \{f \in A(X) : p \in S_A(f)\}$. It is easy to see that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2 + g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$, for each $f, g \in A(X)$ and each $n \in \mathbb{N}$. Furthermore, $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$. It is evident that $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ and $S_{C^*}(f) = Z(f^\beta)$. For terms and notations not defined here we follow the standard text [9].

The aim of this paper is to investigate a different approach to the results of [5] and [15]. This is done via characterizing maximal ideals of the subalgebras which are considered in the mentioned papers. Moreover, we generalize the notion of $z_A^\beta$-ideals, which was first defined in [2] for intermediate subalgebras, to the $LBI$-subalgebras. Furthermore, we show that $z_A^\beta$-ideals coincide with $z$-ideals in $LBI$-subalgebras. Note that an ideal $I$ in a commutative ring $R$ is called a $z$-ideal, if $M_f \subseteq I$ whenever $f \in I$, where $M_f$ is the intersection of all maximal ideals of $R$ containing $f$. This paper consists of three sections. Section 1 is the introduction as we have already noticed. In Section 2, we consider the class of $LBI$-subalgebras of $C(X)$. By characterizing maximal ideals of these subalgebras, we generalize the notion of $z_A^\beta$-ideals to the $LBI$-subalgebras. Using these, we give another proof of the fact that the structure space of each $LBI$-subalgebra is homeomorphic with a quotient of $\beta X$, which is proved in [15] via the mapping $\mathcal{Z}_A$. In Section 3, we consider the class of complete rings of functions which is introduced in [5]. It simply follows that every complete ring of functions is an $LBI$-subalgebra and thus the compactification associated with each complete ring of functions, which is identified in [5] via the mapping $\mathcal{Z}_A$, is just the structure space of that subring. Thus, some results of [5] could be achieved in a different way.

## 2. $LBI$-subalgebras of $C(X)$

As noted in the introduction, every $LBI$-subalgebra is a $BI$-subalgebra. Thus, [13, Proposition 2.7] implies that each maximal ideal of an $LBI$-subalgebra $A(X)$ has the form $M_A^p$ for some $p \in \beta X$. The following statement shows that in an $LBI$-subalgebra $A(X)$, the ideal $M_A^p$ is always maximal for each $p \in \beta X$. Note that in this paper $LBI$-subalgebras are assumed to separate points and closed sets of $X$.

**Lemma 2.1.** *For each $p \in \beta X$, the ideal $M_A^p$ is maximal in the $LBI$-subalgebra $A(X)$.*

Proof: Assume that $M_A^p$ is not a maximal ideal. As $A(X)$ is an $LBI$-subalgebra, there exists $q \in \beta X$ such that $M_A^q$ is a maximal ideal in $A(X)$ and $M_A^p \subset M_A^q$. Let $f \in M_A^q \setminus M_A^p$, thus, there exists $g \in A(X)$ such that $(fg)^*(p) \neq 0$; i.e, $p \notin Z((fg)^*)$. Therefore, there exists $h \in C(X)$ such that $p \in cl_{\beta X} Z(h)$ and $cl_{\beta X} Z(h) \cap Z((fg)^*) = \emptyset$. It follows that $h \in M^p$ and $f(x)g(x) > c > 0$ for each $x \in Z(h)$ where $c \in \mathbb{R}^{\geq 0}$. Set $F = \{x \in X : f(x)g(x) > c\}$, clearly, $F$ is a cozero-set containing $Z(h)$ on which $fg$ is bounded away from zero. Thus, there exists $k \in A(X)$ such that $fgk|_F = 1$, since $A(X)$ is an $LBI$-subalgebra. Hence, $fgk|_{Z(h)} = 1$ which implies that $1 - fgk|_{Z(h)} = 0$. As $p \in \mathrm{cl}_{\beta X} Z(h)$, we have $(1 - fgk)^*(p) = 0$ and hence for each $t \in A(X)$ we have $((1 - fgk)t)^*(p) = 0$, since $Z(h) \subseteq Z((1 - fgk)t)$ and thus if $p \notin Z(((1 - fgk)t)^*)$. Then there exists $l \in C(X)$ such that $p \in cl_{\beta X} Z(l)$ and $Z(l) \cap Z((1 - fgk)t) = \emptyset$. This implies that $Z(l) \cap Z(h) = \emptyset$, however, $l, h \in M^p$ which is a contradiction. Therefore, $1 - fgk \in M_A^p \subseteq M_A^q$ and thus $1 \in M_A^q$ which is a contradiction. $\square$

Note that Lemma 2.1 does not hold for $BI$-subalgebras, in general, as the following example shows. This example investigates a $BI$-subalgebra which is not an $LBI$-subalgebra.

*Example* 2.2. Let $X$ be a topological space and $p, q \in \beta X \setminus \upsilon X$ with $p \neq q$. Also, let $I = M^p \cap M^q$ and $A_I = I^u + \mathbb{R}$. It follows from [17, Lemma 2.2] that $A_I$ is a $BI$-subalgebra. Moreover, using [16, Lemma 5.1], we have $M^p \subseteq M^p_{A_I}$ and thus $I \subseteq M^p_{A_I}$. Therefore, if $M^p_{A_I}$ is maximal in $A_I$, then [17, Theorem 2.7] implies that $M^p_{A_I} = I^u$ which means that $(M^p)^u = (M^q)^u$. This contradicts $p \neq q$. Therefore, $M^p_{A_I}$ is not maximal in $A_I$ and hence we can infer from Lemma 2.1 that $A_I$ is not an $LBI$-subalgebra.

The concept of $z^\beta_A$-ideal was first introduced in [2] for intermediate subalgebras. It follows from Lemma 2.1 that this concept could be applied for $LBI$-subalgebras, see Definition 2.5 in the following. The next statement generalizes [2, Lemma 2.2] to $LBI$-subalgebras.

**Notation.** For a subalgebra $A(X)$ of $C(X)$, $S(A)$ denotes $\{S_A(f) : f \in A(X)\}$; for an ideal $I$ of $A(X)$, $S_A[I]$ denotes $\{S_A(f) : f \in I\}$ and for a subcollection $\mathcal{F}$ of $S(A)$, $S^{-1}_A[\mathcal{F}]$ denotes $\{f \in A(X) : S_A(f) \in \mathcal{F}\}$.

**Lemma 2.3.** *Let $A(X)$ be an $LBI$-subalgebra of $C(X)$, then $S_A(f) = \emptyset$ if and only if $f$ is an invertible element in $A(X)$.*

Proof: It is clear that if $f$ is invertible in $A(X)$, then $S_A(f) = \emptyset$. Let $f \in A(X)$ and $S_A(f) = \emptyset$, therefore, $f \notin M^p_A$ for each $p \in \beta X$. As $A(X)$ is an $LBI$-subalgebra, [13, Proposition 2.7] implies that $f$ misses each maximal ideal of $A(X)$. Hence, $f$ is invertible in $A(X)$. □

**Definition 2.4.** A non-empty subcollection $\mathcal{F}$ of $S(A)$ is called a $z^\beta_A$-filter on $\beta X$, whenever

1) $\emptyset \notin \mathcal{F}$;
2) if $S_1$, $S_2$ are in $\mathcal{F}$, then $S_1 \cap S_2 \in \mathcal{F}$;
3) if $S_1 \in \mathcal{F}$, $S_2 \in S(A)$ and $S_1 \subseteq S_2$, then $S_2 \in \mathcal{F}$.

Also, $z^\beta_A$-ultrafilters and prime $z^\beta_A$-filters are defined similarly to $z$-ultrafilters and prime $z$-filters, respectively.

**Definition 2.5.** An ideal $I$ in an $LBI$-subalgebra $A(X)$ is called a $z^\beta_A$-ideal if $S^{-1}_A S_A[I] = I$ in which $S^{-1}_A S_A[I] = \{f \in A(X) : S_A(f) \in S_A[I]\}$.

The definition of $z^\beta_A$-ideal, evidently, implies that every maximal ideal of $A(X)$ is a $z^\beta_A$-ideal. The next statement, which is a generalization of [2, Theorem 2.3 and Theorem 2.6] to $LBI$-subalgebras, indicates the close connection between $z^\beta_A$-ideals and $z^\beta_A$-filters.

**Proposition 2.6.** *Let $A(X)$ be an $LBI$-subalgebra of $C(X)$, then*

1) *if $I$ is a proper ideal of $A(X)$, then $S_A[I]$ is a $z^\beta_A$-filter on $\beta X$;*
2) *if $\mathcal{F}$ is a $z^\beta_A$-filter on $\beta X$, then $S^{-1}_A[\mathcal{F}]$ is a $z^\beta_A$-ideal in $A(X)$;*

3) *if $M$ is a maximal ideal in $A(X)$, then $S_A[M]$ is a $z_A^\beta$-ultrafilter on $\beta X$;*
4) *if $\mathcal{U}$ is a $z_A^\beta$-ultrafilter on $\beta X$, then $S_A^{-1}[\mathcal{U}]$ is a maximal ideal in $A(X)$.*

Proof: Using Lemmas 2.1 and 2.3, and also [2, Theorem 2.3 and Theorem 2.6], the proof is straightforward. □

The next statement gives an algebraic characterization of $z_A^\beta$-ideals which reveals that the class of $z_A^\beta$-ideals of an $LBI$-subalgebra $A(X)$ coincides with the class of $z$-ideals of $A(X)$. This statement is a generalization of [2, Theorem 3.8] to $LBI$-subalgebras.

**Proposition 2.7.** *Let $A(X)$ be an $LBI$-subalgebra and $f, g \in A(X)$, then $S_A(g) \subseteq S_A(f)$ if and only if $M_f(A) \subseteq M_g(A)$.*

Proof: As $A(X)$ is an $LBI$-subalgebra, $f \in M_g(A)$ if and only if $S_A(g) \subseteq S_A(f)$. Now, let $M_f(A) \subseteq M_g(A)$ and $p \in S_A(g)$, then $g \in M_A^p$ and $f \in M_f(A) \subseteq M_g(A) \subseteq M_A^p$. Thus, $p \in S_A(f)$, which implies that $S_A(g) \subseteq S_A(f)$. Conversely, assume the contrary that $S_A(g) \subseteq S_A(f)$ but $M_f(A) \nsubseteq M_g(A)$. Therefore, there exists $h \in M_f(A)$ such that $h \notin M_g(A)$. Hence, there exists some $M \in Max(A)$ such that $h \notin M$ and $g \in M$. As $A(X)$ is an $LBI$-subalgebra, $M = M_A^p$, for some $p \in \beta X$. Hence, $g \in M_A^p$ and $h \notin M_A^p$, which means that $p \in S_A(g)$ and $p \notin S_A(f)$. This contradiction shows that $M_f(A) \subseteq M_g(A)$. □

It follows from the above proposition that an ideal $I$ in an $LBI$-subalgebra $A(X)$ is a $z_A^\beta$-ideal if and only if it is a $z$-ideal. Therefore, from well-known properties of $z$-ideals, it follows that every maximal ideal in $A(X)$ is a $z_A^\beta$-ideal, every $z_A^\beta$-ideal is an intersection of prime ideals, every minimal prime ideal over a $z_A^\beta$-ideal is also a $z_A^\beta$-ideal and hence every minimal prime ideal of $A(X)$ is a $z_A^\beta$-ideal. These facts are generalizations of [2, Theorem 3.2, Theorem 3.3, Theorem 5.5 and Theorem 3.8] to $LBI$-subalgebras. Using the notion of $z_A^\beta$-ideals, we show that the structure space of each $LBI$-subalgebra is Hausdorff. Let $A(X)$ be an $LBI$-subalgebra, then it is clear that $S(A)$ constitutes a base for the closed subsets of a topology on $\beta X$ which we call $S(A)$-topology and denote by $\tau_{S(A)}$. $X$ is a dense subspace of $(\beta X, \tau_{S(A)})$, since $A(X)$ separates points and closed sets in $X$. If $\tau$ denotes the usual topology on $\beta X$, then, clearly, $\tau_{S(A)} \subseteq \tau$. Therefore, $(\beta X, \tau_{S(A)})$ is compact. This fact leads to the next statement which is a reformulation of [15, Theorem 3.5].

**Theorem 2.8.** *The structure space of an $LBI$-subalgebra $A(X)$ is homeomorphic with $\frac{(\beta X, \tau_{S(A)})}{\sim_A}$.*

Proof: Define $\sim_A$ on $\beta X$ as follows $p \sim_A q$ if $M_A^p = M_A^q$, where $p, q \in \beta X$. Clearly, $\sim_A$ defines an equivalence relation on $\beta X$. Therefore, $\frac{\beta X}{\sim_A}$ is a quotient of $\beta X$. Now, define $\varphi : \frac{(\beta X, \tau_{S(A)})}{\sim_A} \to Max(A)$ by $\varphi(p) = M_A^p$. We show that this mapping is a homeomorphism. Let $\mathcal{M}_f = \{M \in Max(A) : f \in M\}$ be a basic closed set in $Max(A)$ and $p \notin \varphi^{-1}(\mathcal{M}_f)$, thus, $M_A^p \notin \mathcal{M}_f$ and hence $p \notin S_A(f)$. Therefore, there exists $g \in A(X)$ such that $p \notin S_A(g)$ and $S_A(f) \subseteq S_A(g)$. Thus, $p \notin S_A(g)$ and $\varphi^{-1}(\mathcal{M}_f) \subseteq S_A(g)$ which means that $\varphi$ is continuous. It is clear that $\varphi$ is also one-one and onto. By showing that $\varphi$ is a closed mapping the proof is completed. Let $S_A(f)$ be a basic closed set and $M_A^p \notin \varphi(S_A(f))$. Thus, $p \notin S_A(f)$ and hence there exists $g \in A(X)$ such that $S_A(f) \subseteq S_A(g)$ and $p \notin S_A(g)$. Therefore, $\varphi(S_A(f)) \subseteq \mathcal{M}_g$ and $M_A^p \notin \mathcal{M}_g$. Hence, $\varphi$ is a closed mapping and we are done. □

The next statement follows from Theorem 2.8 which is a reformulation of [15, Theorem 3.6].

**Theorem 2.9.** *The structure space of each $LBI$-subalgebra of $C(X)$ is a quotient of $\beta X$; precisely, $Max(A)$ is homeomorphic with $\frac{(\beta X, \tau_{S(A)})}{\sim_A}$.*

Proof: Let $A(X)$ be an $LBI$-subalgebra of $C(X)$. At first, we show that $\frac{(\beta X, \tau)}{\sim_A}$ is a compact Hausdorff space. Evidently, this space is compact. Now, assume that $p$ and $q$ are two distinct points $\beta X$ where $\beta X$ is equipped with the $S(A)$-topology and the equivalence relation $\sim_A$ is defined on it. It follows that $M_A^p$ and $M_A^q$ are two distinct maximal ideals in $A(X)$. We claim that there exists $f \in M_A^p$ and $g \in M_A^q$ such that $S_A(f) \cap S_A(g) = \emptyset$. Otherwise, $S_A[M_A^p] \cup S_A[M_A^q]$ constitutes a base for a $z_A^\beta$-filter on $\beta X$, let $\mathcal{F}$ be this $z_A^\beta$-filter. Then clearly $S_A^{-1}[\mathcal{F}]$ is an ideal in $A(X)$ containing both $M_A^p$ and $M_A^q$ which is a contradiction. Therefore, $\frac{(\beta X, \tau_{S(A)})}{\sim_A}$ is Hausdorff. Now, the identity mapping $i : (\beta X, \tau) \to (\beta X, \tau_{S(A)})$ is continuous and hence so is the identity mapping $\mathcal{I} : \frac{(\beta X, \tau)}{\sim_A} \to \frac{(\beta X, \tau_{S(A)})}{\sim_A}$. Thus, $\mathcal{I}$ is a homeomorphism as it is a continuous bijective mapping to a compact Hausdorff space. Therefore, by Theorem 2.8, we are done. □

An immediate consequence of the above statements is the characterization of maximal ideals of invertible lattice-ordered subalgebras of $C(X)$. We call a subalgebra $R$ of $C(X)$ an invertible subalgebra, if $f^{-1} \in R$ whenever $f \in R$ with $Z(f) = \emptyset$. Some well-known examples of invertible lattice-ordered subalgebras are $I+\mathbb{R}$, where $I$ is an absolutely convex ideal in $C(X)$ (refer to [17, Remark 1.8 and Remark 4.1]) and $C_c(X)$, the subalgebra of $C(X)$ consisting of all functions with countable image (refer to [8]). It is easy to see that every invertible lattice-ordered subalgebra $R$ of $C(X)$ is an $LBI$-subalgebra. Indeed, it is clear that this kind of subalgebras are $BI$-subalgebras and if $f \in R$ and $f \geq c > 0$ on a cozero-set $E$, then $g = c \vee f$ is in $R$ and clearly is bounded away from zero on $X$ and hence, has an inverse $h$ in $R$, and it follows that $fh|_E = 1$. Therefore, the collection of all the maximal ideals of an invertible lattice-ordered subalgebra $R$ is $\{M_R^p : p \in \beta X\}$, also, it is easy to see that $M_R^p = M^p \cap R$ for each $p \in \beta X$. Hence, whenever $I$ is an absolutely convex ideal in $C(X)$, then the collection of all the maximal ideals of $I + \mathbb{R}$ is $\{M^p \cap (I + \mathbb{R}) : p \in \beta X\}$ and clearly $M_{I+\mathbb{R}}^p = M_{I+\mathbb{R}}^q$ if and only if $p, q \in \theta(I)$, where $\theta(I) = \bigcap_{f \in I} cl_{\beta X} Z(f)$. Therefore, $I + \mathbb{R}$ is a $\beta$-subalgebra if and only if $I$ is contained in a unique maximal ideal. Applying these facts for the subalgebra $C_K(X)+\mathbb{R}$, where $C_K(X)$ denotes the ideal of $C(X)$ consisting of all functions with compact support (refer to [9, 4D]), implies that $C_K(X)+\mathbb{R}$ has the unique free maximal ideal $C_K(X)$. Thus, whenever $X$ is locally compact, then $Max(C_K(X)+\mathbb{R})$ is homeomorphic with the one-point compactification of $X$. Furthermore, the unique free maximal ideal of $C_\psi(X)+\mathbb{R}$ is $C_\psi(X)$, in which $C_\psi(X)$ denotes the ideal of $C(X)$ consisting of all functions with pseudocompact support (refer to [11]). Hence, whenever $X$ is locally compact, then $Max(C_\psi(X)+\mathbb{R})$ is homeomorphic with the one-point pseudocompactification of $X$. This means that the one-point compactification and the one-point pseudocompactification of locally compact spaces are homeomorphic with quotients of $\beta X$.

Similarly, the characterization of maximal ideals of the subalgebra $C_c(X)$ follows from Lemma 2.1. As earlier noted, $C_c(X)$ is an invertible lattice-ordered subalgebra of $C(X)$ and thus, the collection of all the maximal ideals of $C_c(X)$ is $\{M^p \cap C_c(X) : p \in \beta X\}$. It is well-known that whenever $X$ is a zero-dimensional space, then $Max(C_c(X))$ is homeomorphic with the Banaschewski compactification of $X$ which is denoted by $\beta_o X$ (refer to [3]). Therefore, $C_c(X)$ is a $\beta$-subalgebra if and only if $X$ is strongly zero-dimensional. Moreover, if $X$ is a zero-dimensional space, then $\beta_0 X$ is homeomorphic with a quotient of $\beta X$; in fact, if we define $\sim_c$ on $\beta X$ as $p \sim_c q$ if and only if $M^p_{C_c} = M^q_{C_c}$, then $\beta_0 X$ is homeomorphic with $\frac{\beta X}{\sim_c}$.

Note that a subring $R$ of $C(X)$ is called a $C$-ring, if $R$ is isomorphic with $C(Y)$ for some completely regular Hausdorff space $Y$ (see [15]). $R$ is called an intermediate $C$-algebra, if it is an intermediate subalgebra which is also a $C$-ring. Intermediate $C$-algebras of $C(X)$ are in a $1-1$ correspondence with realcompactifications of $X$ according to the following proposition which is a restatement of [15, Theorem 4.7].

**Proposition 2.10** ([15, Theorem 4.7])**.** *There exists a $1-1$ correspondence between realcompactifications of $X$ and intermediate $C$-algebras of $C(X)$.*

Proof: We first note that every realcompactification of $X$ is homeomorphic with a realcompactification of $X$ which is a subset of $\beta X$, In fact, let $Y$ be a realcompactification of $X$ and set $A_Y(X) = \{f \in C(X) : f \text{ has an extension to } Y\}$. As stated in the proof of part (b) of [14, Theorem 4.6.], $A_Y(X) \cong C(Y)$ and $Y \simeq \upsilon_{A_Y} X$. Also, clearly, $\upsilon_{A_Y} X \subseteq \beta X$. Therefore, it suffices to consider the realcompactifications which are subsets of $\beta X$. Now, it is evident that if $A(X)$ is an intermediate $C$-algebra of $C(X)$, then $\upsilon_A X$ is a realcompactification of $X$. Also, whenever $K$ is a realcompactification of $X$, then $C(K)$ is isomorphic with the intermediate subalgebra $A_K(X) = \{f|_X : f \in C(K)\}$ of $C(X)$. It follows that $A_K(X)$ is an intermediate $C$-algebra of $C(X)$ and $\upsilon_{A_K} X \simeq K$. $\square$

For each $T \subseteq \beta X$, let $B_T$ denote $\{f \in C(X) : f^*(p) < \infty, \forall p \in T\}$, we use $B_p$ instead of $B_{\{p\}}$. It is stated in [7, Theorem 1.2] that an intermediate subalgebra $A(X)$ is a $C$-algebra if and only if there exists a subset $T$ of $\beta X$ such that $A(X) = B_T$. It is clear that $B_T = \bigcap_{p\in T} B_p$ for each subset $T$ of $\beta X$. The next statement shows that, for each $p \in \beta X$, $B_p$ is the intermediate subalgebra generated by the maximal ideal $M^p$ of $C(X)$.

**Proposition 2.11.** *For each $p \in \beta X$, we have $B_p = M^p + C^*(X)$.*

Proof: It is clear that $M^p + C^*(X) \subseteq B_p$. It follows from [13, Theorem 2.9] that each intermediate subalgebra is uniformly closed, thus, $(M^p)^u + C^*(X) = M^p + C^*(X)$. Moreover, [16, Lemma 5.1] implies that $(M^p)^u = \{f \in C(X) : p \in Z(f^*)\}$. Therefore, if $f \in B_p$, then $f^*(p) = r$ for some $r \in \mathbb{R}$, hence, $(f-r)^*(p) = 0$ and thus, $f - r \in (M^p)^u$ which clearly implies that $f \in M^p + C^*(X)$. This completes the proof. $\square$

It follows from the above proposition and [7, Theorem 1.2] that each intermediate $C$-algebra is an intersection of intermediate subalgebras generated by a family of maximal ideals. In fact, whenever $A(X)$ is an intermediate $C$-algebra, then $A(X) = \bigcap_{p\in T}(M^p_A + C^*(X))$, for some $T \subseteq \beta X$.

## 3. Complete ring of functions

Following [5] a subring $A(X)$ of $C^*(X)$ is called a complete ring of functions if $A(X)$ is a uniformly closed subset of $C^*(X)$, contains the constants and separates points and closed sets in $X$. Throughout this section $A(X)$ denotes a subalgebra of $C^*(X)$ which is a complete ring of functions. It follows from Lemma 2.1 and part (c) of [5, Lemma 1.2] that every complete ring of functions is an $LBI$-subalgebra of $C(X)$. As an example of such rings, let $I$ be a free $z$-ideal in $C(X)$, then it is easy to see that $(I^u + \mathbb{R}) \cap C^*(X)$ is a complete ring of functions.

**Lemma 3.1.** *Every maximal ideal in a complete ring of functions $A(X)$ has the form $M^p_A = M^{*p} \cap A(X)$, for some $p \in \beta X$. Moreover, all such ideals are distinct if and only if $A(X) = C^*(X)$.*

Proof: As noted above, every complete ring of functions is an $LBI$-subalgebra. Therefore, the collection of all the maximal ideals of $A(X)$ is $\{M^P_A : p \in \beta X\}$. Moreover, as every complete ring of functions $A(X)$ is a subring of $C^*(X)$, $S_A(f) \subseteq Z(f^\beta)$, for all $f \in A(X)$. Thus, every maximal ideal in $A(X)$ has the form $M^{*p} \cap A(X) = M^p_A$, for some $p \in \beta X$. Also, as such subrings are uniformly closed, [13, Theorem 2.9.] implies that the only complete ring of functions which is a $\beta$-subalgebra is $C^*(X)$. $\square$

As every complete ring of functions is an $LBI$-subalgebra, the structure space of each complete ring of functions is a compactification of $X$ and hence is a quotient of $\beta X$. This means that the compactification which is characterized in [5] via the mapping $\mathcal{Z}_A$ for a complete ring of functions $A(X)$ is, in fact, the structure space of $A(X)$.

**Proposition 3.2.** *Every complete ring of functions is a $C$-ring of $C^*(X)$.*

Proof: Every complete ring of functions is, clearly, a uniformly closed $\Phi$-algebra. Thus, by [10, 3.2], we have $A(X) \cong C(Max(A))$ and hence we are done. □

In [5], the equivalence relation $\sim_A$ is defined on $\beta X$ as $p \sim_A q$ if $\mathcal{Z}_A^{-1}[\mathcal{U}_p] = \mathcal{Z}_A^{-1}[\mathcal{U}_q]$, in which, $\mathcal{U}_p$ is the unique $z$-ultrafilter on $X$ containing $p$ and $\mathcal{Z}_A^{-1} = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{U}_p\}$. It is easy to see that $p \sim_A q$ if and only if $M_A^p = M_A^q$. Therefore, $\beta_A X (= \frac{\beta X}{\sim_A}) \cong Max(A)$. Using this fact, [5, Theorem 2.3] can be proved in a different way.

**Theorem 3.3** ([5, Theorem 2.3]). *Let $f \in C^*(X)$, then $f$ has an extension $f^A$ to $\beta_A X$ if and only if $f \in A(X)$.*

Proof: As $A(X) \cong C(Max(A)) \cong C(\beta_A X)$), the statement is clear. □

**Acknowledgment.** This paper is a part of my Ph.D thesis in Shadid Chamran University of Ahvaz written under the supervision of professor A.R. Aliabad. I would like to express my deep gratitude to him for his kind supervision. I also thank the referee for his/her careful reading and his/her comments which improved the paper.

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(*Received* August 3, 2015, *revised* December 8, 2015)