Local convergence of a one parameter fourth-order Jarratt-type method in Banach spaces

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Abstract. We present a local convergence analysis of a one parameter Jarratt-type method. We use this method to approximate a solution of an equation in a Banach space setting. The semilocal convergence of this method was recently carried out in earlier studies under stronger hypotheses. Numerical examples are given where earlier results such as in [Ezquerro J.A., Hernández M.A., *New iterations of R-order four with reduced computational cost*, BIT Numer. Math. **49** (2009), 325–342] cannot be used to solve equations but our results can be applied.

Keywords: Banach space; Newton's method; local convergence; radius of convergence

Classification: 65D10, 65D99

1. Introduction

Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces and $\mathbf{D}$ be a convex open subset of $\mathbf{X}$.

In this study, we are concerned with the problem of approximating a solution $x^\star$ of the nonlinear equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F} : \mathbf{D} \to \mathbf{Y}$ is a Fréchet-differentiable operator. Many problems can be formulated as equations like (1.1) using Mathematical Modelling [1]–[22]. The solutions of such equations can rarely be found in closed form. That is why most solution methods for such equations are usually iterative. The study about convergence of iterative methods is usually centered on two types: semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point to give criteria ensuring the convergence of iterative procedures. While the local analysis is based on the information around a solution to find estimates of the radii of convergence balls. There exist many studies which deal with the local and the semilocal convergence analysis of Newton-like methods such as [1]–[22].

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J.A. Ezquerro and M.A. Hernández in [11] studied the semilocal convergence of the R-order four method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{cases} y_n = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \\ z_n = y_n + \dfrac{1}{3}\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ H_n = \mathcal{F}'(x_n)^{-1}(\mathcal{F}'(z_n) - \mathcal{F}'(x_n)) \\ x_{n+1} = y_n - \dfrac{3\alpha}{4}(I + \dfrac{3}{2}H_n)^{-1}H_n(y_n - x_n), \quad \alpha \neq 0 \end{cases} \tag{1.2}$$

where x_0 is an initial point. They assumed that there exist constants β_1, β_2, β_3, β_4 and $\mathcal{M}$ such that

($\mathbf{C}_1$): $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \beta_1$,
($\mathbf{C}_2$): $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x)\right\| \leq \beta_2$ for each $x \in \mathbf{D}$,
($\mathbf{C}_3$): $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'''(x)\right\| \leq \beta_3$ for each $x \in \mathbf{D}$,
($\mathbf{C}_4$): $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'''(x) - \mathcal{F}'''(y))\right\| \leq \beta_4\|x - y\|$ for each $x, y \in \mathbf{D}$,
($\mathbf{C}_5$): $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x)\right\| \leq \mathcal{M}$.

The assumptions used in [11] were given in non-affine invariant form but we present them here in an affine invariant form.

The conditions for the local convergence analysis are obtained from the preceding ones by replacing x_0 by $x^\star$. However some of the ($\mathbf{C}$) conditions may not be satisfied even for simple scalar functions. As a motivational example, let us define function f on $\mathbf{D} = [-1/2, 5/2]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{1.3}$$

Then, we have

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that conditions ($\mathbf{C}_3$) or ($\mathbf{C}_4$) are not satisfied. Hence, the results depending on ($\mathbf{C}_3$) or ($\mathbf{C}_4$) cannot be applied in this case.

In the present paper, we drop the conditions ($\mathbf{C}_2$)–($\mathbf{C}_4$) to study the local convergence of method (1.2). This way we expand the applicability of these methods. The conditions used involve only the first Fréchet-derivative (see (2.8)–(2.11)).

The rest of the paper is organized as follows. In Section 2, we present the local convergence analysis for method (1.2). The numerical examples are given in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section.

Let L_0, $\mathsf{L} > 0$, $\alpha \neq 0$ and $\mathcal{M} \geq 1$ be given parameters. It is convenient for the local convergence analysis of method (1.2) that follows to define functions on $[0, 1/\mathsf{L}_0)$ by

$$\begin{aligned}
g_1(r) &= \frac{\mathsf{L}\, r}{2(1-\mathsf{L}_0\, r)}\,,\\
g_2(r) &= \frac{1}{1-\mathsf{L}_0 r}\left(\frac{\mathsf{L} r}{2}+\frac{\mathcal{M}}{3}\right),\\
g_3(r) &= \frac{3\mathsf{L}_0(1+g_2(r))}{2(1-\mathsf{L}_0 r)}\,,\\
g_4(r) &= g_3(r) r,\\
h_4(r) &= g_4(r) - 1,\\
g_5(r) &= \frac{1}{2}\left[\frac{\mathsf{L}}{1-\mathsf{L}_0 r}+\frac{|\alpha|\, g_3(r)(1+g_1(r))}{1-g_3(r) r}\right] r\\
\text{and}\quad h_5(r) &= g_5(r) - 1.
\end{aligned}$$

Notice that

$$g_2(r) = g_1(r) + \frac{\mathcal{M}}{3(1-\mathsf{L}_0 r)} = \frac{\mathsf{L} r}{2(1-\mathsf{L}_0 r)} + \frac{\mathcal{M}}{3(1-\mathsf{L}_0 r)}\,.$$

Suppose that

$$1 \leq \mathcal{M} < 3. \tag{2.1}$$

Then, we get

$$r_2 := \frac{3-\mathcal{M}}{3(\frac{\mathsf{L}}{2}+\mathsf{L}_0)} < r_{\mathcal{A}} := \frac{1}{\frac{\mathsf{L}}{2}+\mathsf{L}_0}\,. \tag{2.2}$$

We have that

$$0 \leq g_1(r) < 1 \quad \text{and} \quad 0 < g_2(r) < 1 \quad \text{for each} \quad r \in [0, r_2). \tag{2.3}$$

Using the definition of functions g_4 and h_4, we get that

$$h_4(0) = -1 < 0 \quad \text{and} \quad h_4(r) \to +\infty \quad \text{as} \quad r \to \frac{1^-}{\mathsf{L}_0}\,.$$

It follows from the intermediate value theorem that function h_4 has zeros in the interval $(0, 1/\mathsf{L}_0)$. Denote by r_4 the smallest such root. Then, we have that

$$0 < g_4(r) < 1 \quad \text{for each} \quad r \in (0, r_4). \tag{2.4}$$

Moreover, we have that

$$h_5(0) = -1 < 0 \quad \text{and} \quad h_5(r) \to +\infty \quad \text{as} \quad r \to r_4^-\,. \tag{2.5}$$

Hence, function h_5 has zeros in the interval $(0, r_4)$. Denote by r_5 the smallest such zero. Then, we have that

$$0 < g_5(r) < 1 \quad \text{for each} \quad r \in (0, r_5). \tag{2.6}$$

Define

$$r^\star = \min(r_2, r_5). \tag{2.7}$$

Notice that functions g_i, $i = 1, 2, \ldots, 5$ are increasing.

Denote by $U(q, \rho)$ and $\overline{U}(q, \rho)$ the open and closed balls in $\mathbf{X}$ of center $q \in \mathbf{X}$ and radius $\rho > 0$, respectively.

Next, we present the local convergence analysis of method (1.2).

Theorem 2.1. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ be a Fréchet differentiable operator. Suppose that there exist $x^\star \in \mathbf{D}$ and $\mathsf{L}_0 > 0$ such that for each $x \in \mathbf{D}$*

$$\mathcal{F}(x^\star) = 0, \quad \mathcal{F}'(x^\star)^{-1} \in \mathbf{L}(\mathbf{Y}, \mathbf{X}), \tag{2.8}$$

$$\left\| \mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star)) \right\| \le \mathsf{L}_0 \|x - x^\star\|. \tag{2.9}$$

Moreover, suppose that there exist parameters $\mathsf{L} > 0$ and $\mathcal{M} \in [1, 3)$ such that for all $x \in \mathbf{D}_0 := \mathbf{D} \cap U(x^\star, 1/\mathsf{L}_0)$

$$\left\| \mathcal{F}'(x^\star)^{-1}(\mathcal{F}(x) - \mathcal{F}(x^\star) - \mathcal{F}'(x)(x - x^\star)) \right\| \le \frac{\mathsf{L}}{2} \|x - x^\star\|^2, \tag{2.10}$$

$$\left\| \mathcal{F}'(x^\star)^{-1} \mathcal{F}'(x) \right\| \le \mathcal{M}, \tag{2.11}$$

$$\overline{U}(x^\star, r^\star) \subseteq \mathbf{D} \tag{2.12}$$

hold, where the convergence radius $r^\star$ is defined in equation (2.7). Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^\star, r^\star)$ is well defined, remains in $U(x^\star, r^\star)$ for each $n = 0, 1, 2, \ldots$ and converges to $x^\star$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|y_n - x^\star\| \le g_1(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\| < r^\star, \tag{2.13}$$

$$\|z_n - x^\star\| \le g_2(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\|, \tag{2.14}$$

$$\frac{3}{2}\|H_n\| \le g_3(\|x_n - x^\star\|)\|x_n - x^\star\| < 1, \tag{2.15}$$

$$\|x_{n+1} - x^\star\| \le g_5(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\|, \tag{2.16}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r^\star, 2/\mathsf{L}_0)$, the limit point $x^\star$ is the only solution of equation $\mathcal{F}(x) = 0$ in $\mathbf{D}_1 := U(x^\star, T) \cap \mathbf{D}$.

Proof: Using (2.9), the definition of $r^\star$ and the hypothesis $x_0 \in U(x^\star, r^\star)$, we get that

$$\left\| \mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(x^\star)) \right\| \le \mathsf{L}_0 \|x_0 - x^\star\| < \mathsf{L}_0 r^\star < 1. \tag{2.17}$$

It follows from (2.17) and the Banach lemma on invertible operators [4], [6], [11] that

$$\begin{aligned}&\mathcal{F}'(x_0)^{-1} \in \mathbf{L}(\mathbf{Y},\mathbf{X}) \quad \text{and}\\ &\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \leq \frac{1}{1-\mathsf{L}_0\|x_0-x^\star\|}\,.\end{aligned} \tag{2.18}$$

Hence, y_0 and z_0 are well defined. We have from the first sub-step in method (1.2) for $n=0$ that

$$\begin{aligned}y_0-x^\star &= x_0-x^\star-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\\ &= \mathcal{F}'(x_0)^{-1}[-\mathcal{F}(x_0)+\mathcal{F}'(x_0)(x_0-x^\star)]\\ &= -\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\mathcal{F}'(x^\star)^{-1}[\mathcal{F}(x_0)-\mathcal{F}(x^\star)-\mathcal{F}'(x_0)(x_0-x^\star)]\end{aligned}$$

so,

$$\begin{aligned}\|y_0-x^\star\| &\leq \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\|\left\|\mathcal{F}'(x^\star)^{-1}[\mathcal{F}(x_0)-\mathcal{F}(x^\star)-\mathcal{F}'(x_0)(x_0-x^\star)]\right\|\\ &\leq \frac{\mathsf{L}\|x_0-x^\star\|^2}{2(1-\mathsf{L}_0\|x_0-x^\star\|)} = g_1(\|x_0-x^\star\|)\|x_0-x^\star\| \leq \|x_0-x^\star\| < r^\star,\end{aligned}$$

which shows (2.13) and $y_0 \in U(x^\star, r^\star)$. Consequently, from the second substep in method (1.2) for $n=0$, we obtain that

$$\begin{aligned}\|z_0-x^\star\| &\leq \|y_0-x^\star\| + \frac{1}{3}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\|\\ &\qquad \times\left\|\int_0^1 \mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x^\star+t(x_0-x^\star))(x_0-x^\star)\,dt\right\|\\ &\leq \frac{\mathsf{L}\|x_0-x^\star\|^2}{2(1-\mathsf{L}_0\|x_0-x^\star\|)} + \frac{\mathcal{M}\|x_0-x^\star\|}{3(1-\mathsf{L}_0\|x_0-x^\star\|)}\\ &= g_2(\|x_0-x^\star\|)\|x_0-x^\star\| \leq \|x_0-x^\star\| < r^\star,\end{aligned}$$

which shows (2.14) for $n=0$ and $z_0 \in U(x^\star, r^\star)$.

Next we need an estimate of $\|H_0\|$. We have by (2.4), (2.9), (2.13) and (2.18) that

$$\begin{aligned}\frac{3}{2}\|H_0\| &\leq \frac{3}{2}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\|\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(z_0)-\mathcal{F}'(x_0))\right\|\\ &\leq \frac{3}{2}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\|\Big(\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(z_0)-\mathcal{F}'(x^\star))\right\|\\ &\qquad +\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x_0)-\mathcal{F}'(x^\star))\right\|\Big)\\ &\leq \frac{3\mathsf{L}_0(\|z_0-x^\star\|+\|x_0-x^\star\|)}{2(1-\mathsf{L}_0\|x_0-x^\star\|)}\end{aligned}$$

$$\leq \frac{3\mathsf{L}_0(1+g_2(\|x_0-x^\star\|))\|x_0-x^\star\|}{2(1-\mathsf{L}_0\|x_0-x^\star\|)}$$
$$= g_3(\|x_0-x^\star\|)\|x_0-x^\star\| = g_4(\|x_0-x^\star\|) < 1,$$

which shows (2.15) for $n=0$. It follows that $(1+3/2H_0)^{-1}$ exists and

$$\left\|\left(I+\frac{3}{2}H_0\right)^{-1}\right\| \leq \frac{1}{1-g_3(\|x_0-x^\star\|)\|x_0-x^\star\|}. \tag{2.19}$$

Hence, x_1 is well defined. Then, using the last substep in method (1.2) for $n=0$, (2.13) and (2.19) that

$$\begin{aligned}\|x_1-x^\star\| &\leq \|y_0-y^\star\| + \frac{|\alpha|}{2}\frac{g_3(\|x_0-x^\star\|)\|x_0-x^\star\|\,(\|y_0-x^\star\|+\|x_0-x^\star\|)}{1-g_3(\|x_0-x^\star\|)\|x_0-x^\star\|}\\ &\leq \frac{1}{2}\left[\frac{\mathsf{L}}{1-\mathsf{L}_0\|x_0-x^\star\|} + |\alpha|\,\frac{g_3(\|x_0-x^\star\|)(1+g_1(\|x_0-x^\star\|))}{1-g_3(\|x_0-x^\star\|)\|x_0-x^\star\|}\right]\|x_0-x^\star\|^2\\ &= g_5(\|x_0-x^\star\|)\|x_0-x^\star\|\\ &= c\|x_0-x^\star\| < r^\star,\end{aligned}$$

where $c = g_5(\|x_0-x^\star\|) \in [0,1)$, which shows (2.16) for $n=0$ and $x_1 \in U(x^\star, r^\star)$. By simply replacing y_0, z_0, x_1 by y_k, z_k, x_{k+1} in the preceding estimates we arrive at (2.13)–(2.16). In particular, we have that since function g_5 is increasing,

$$\begin{aligned}\|x_2-x^\star\| &\leq g_5(\|x_1-x^\star\|)\|x_1-x^\star\| \leq g_5(\|x_0-x^\star\|)\|x_1-x^\star\|\\ &= c\|x_1-x^\star\|\end{aligned} \tag{2.20}$$

and by induction for $\|x_k-x^\star\| \leq c\|x_{k-1}-x^\star\|$

$$\begin{aligned}\|x_{k+1}-x^\star\| &\leq g_5(\|x_k-x^\star\|)\|x_k-x^\star\| \leq g_5(\|x_{k-1}-x^\star\|)\|x_k-x^\star\|\\ &\leq g_5(\|x_0-x^\star\|)\|x_k-x^\star\| = c\|x_k-x^\star\|\\ &\leq c^{k+1}\|x_0-x^\star\|,\end{aligned} \tag{2.21}$$

so $\lim_{k\to+\infty} x_k = x^\star$ and $x_{k+1} \in U(x^\star, r)$.

Finally, to show the uniqueness part, let $y^\star \in \mathbf{D}_1$ be such that $\mathcal{F}(y^\star)=0$. Set $Q = \int_0^1 \mathcal{F}'(x^\star+\theta(y^\star-x^\star))\,d\theta$. Then, using (2.9), we get that

$$\left\|\mathcal{F}'(x^\star)^{-1}(Q-\mathcal{F}'(x^\star))\right\| \leq \mathsf{L}_0\int_0^1 \theta\|x^\star-y^\star\|\;d\theta \leq \frac{\mathsf{L}_0 T}{2} < 1.$$

Hence, $Q^{-1} \in \mathbf{L}(\mathbf{Y},\mathbf{X})$. Then, by the identity $0 = \mathcal{F}(x^\star)-\mathcal{F}(y^\star) = Q(x^\star-y^\star)$, we deduce $x^\star = y^\star$. $\square$

It turns out that we can use another approach. Indeed, define functions $\overline{g}_5$ and $\overline{h}_5$ on the interval $[0, r_4)$ by

$$\overline{g}_5(r) = \frac{1}{2(1-\mathsf{L}_0 r)}\left[\mathsf{L} + \frac{3|\alpha|\,\mathcal{M}\mathsf{L}_0(1-\mathsf{L}_0 r+\mathsf{L}r/2+\mathcal{M}/3)}{2(1-\mathsf{L}_0 r)^2 - 3\mathsf{L}_0(1-\mathsf{L}_0 r+\mathsf{L}r/2+\mathcal{M}/3)r}\right] r \tag{2.22}$$

and

$$\overline{h}_5(r) = \overline{g}_5(r) - 1. \tag{2.23}$$

Then, we also have that

$$\overline{h}_5(0) = -1 < 0 \quad \text{and} \quad h_5(r) \to +\infty \quad \text{as} \quad r \to r_4^-.$$

Denote by $\overline{r}_5$ the smallest such zero. Set

$$R^\star = \min\{r_2, \overline{r}_5\}. \tag{2.24}$$

Next, we present a second local convergence result for method (1.2). □

Theorem 2.2. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \to \mathbf{Y}$ be a Fréchet differentiable operator. Suppose that there exist $x^\star \in \mathbf{D}$ and $\mathsf{L}_0 > 0$ such that for each $x \in \mathbf{D}$*

$$\mathcal{F}(x^\star) = 0, \quad \mathcal{F}'(x^\star)^{-1} \in \mathbf{L}(\mathbf{Y}, \mathbf{X}),$$
$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star))\right\| \le \mathsf{L}_0\|x - x^\star\|.$$

Moreover, suppose that there exist parameters $\mathsf{L} > 0$ and $\mathcal{M} \in [0, 3)$ such that for all $x \in \mathbf{D}_0 = \mathbf{D} \cap U(x^\star, 1/\mathsf{L}_0)$

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}(x) - \mathcal{F}(x^\star) - \mathcal{F}'(x)(x - x^\star))\right\| \le \frac{\mathsf{L}}{2}\|x - x^\star\|^2$$
$$\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\right\| \le \mathcal{M}$$

and

$$\overline{U}(x^\star, R^\star) \subseteq \mathbf{D}$$

hold, where the convergence radius $R^\star$ is defined in (2.24). Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^\star, R^\star)$ is well defined, remains in $U(x^\star, R^\star)$ for each $n = 0, 1, 2, \ldots$ and converges to $x^\star$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|y_n - x^\star\| \le g_1(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\| < R^\star,$$
$$\|z_n - x^\star\| \le g_2(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\|,$$
$$\frac{3}{2}\|H_n\| \le g_3(\|x_n - x^\star\|)\|x_n - x^\star\| < 1,$$
$$\|x_{n+1} - x^\star\| \le \overline{g}_5(\|x_n - x^\star\|)\|x_n - x^\star\| \le \|x_n - x^\star\|,$$

where function $\overline{g}_5$ is given in (2.22). Furthermore, for $T \in [R^\star, 2/\mathsf{L}_0)$, the limit point $x^\star$ is the only solution of equation $\mathcal{F}(x) = 0$ in $\mathbf{D}_2 := U(x^\star, T) \cap \mathbf{D}$.

Proof: It follows exactly as in Theorem 2.1 until the computation of the upper bound on $\|x_1 - x^\star\|$ where we use the estimate

$$\begin{aligned}\|y_0 - x_0\| &= \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \\ &\leq \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \left\| \int_0^1 \mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x^\star + t(x_0 - x^\star))(x_0 - x^\star)\, dt\right\| \\ &\leq \frac{\mathcal{M}\|x_0 - x^\star\|}{1 - \mathsf{L}_0\|x_0 - x^\star\|} \leq \frac{\mathcal{M}R^\star}{1 - \mathsf{L}_0 R^\star}\end{aligned}$$

instead of

$$\begin{aligned}\|y_0 - x_0\| &\leq \|y_0 - x^\star\| + \|x_0 - x^\star\| \\ &\leq (1 + g_1(\|x_0 - x^\star\|))\|x_0 - x^\star\|\end{aligned}$$

used in the proof of Theorem 2.1. This change leads to

$$\begin{aligned}\|x_1 - x^\star\| &\leq \|y_0 - x^\star\| + \frac{|\alpha|\,\mathcal{M}}{2} \frac{g_3(\|x_0 - x^\star\|)\|x_0 - x^\star\|^2}{(1 - \mathsf{L}_0\|x_0 - x^\star\|)(1 - g_3(\|x_0 - x^\star\|)\|x_0 - x^\star\|)} \\ &= \overline{g}_5(\|x_0 - x^\star\|)\|x_0 - x^\star\| = c_0\|x_0 - x^\star\| < R^\star,\end{aligned}$$

where $c_0 = \overline{g}_5(\|x_0 - x^\star\|) \in [0, 1)$, instead of the old estimate. The rest of the proof follows as in Theorem 2.1. □

Remarks 2.3. 1. In view of (2.9) and the estimate

$$\begin{aligned}\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\right\| &= \left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star)) + I\right\| \\ &\leq 1 + \left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star))\right\| \\ &\leq 1 + \mathsf{L}_0\|x - x^\star\|\,,\end{aligned}$$

condition (2.11) can be dropped and $\mathcal{M}$ can be replaced by $\mathcal{M}(r) = 1 + \mathsf{L}_0 r$ or simply by $\mathcal{M}(r) = \mathcal{M} = 2$, since $r \in [0, 1/\mathsf{L}_0)$.

2. Condition (2.10) can be replaced by the popular but stronger condition

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(y))\right\| \leq \overline{\mathsf{L}}\|x - y\| \quad \text{for each} \quad x, y \in \mathbf{D}.$$

3. The results obtained here can be used for operators $\mathcal{F}$ satisfying autonomous differential equations [2], [6], [15], [17] of the form

$$\mathcal{F}'(x) = \mathcal{P}(\mathcal{F}(x))$$

where $\mathcal{P}$ is a continuous operator. Then, since $\mathcal{F}'(x^\star) = \mathcal{P}(\mathcal{F}(x^\star)) = \mathcal{P}(0)$, we can apply the results without actually knowing $x^\star$. For example, let $\mathcal{F}(x) = e^x - 1$. Then, we can choose: $\mathcal{P}(x) = x + 1$.

4. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2], [6], [15], [17].

5. The parameter $r_{\mathcal{A}}$ — given in (2.2) — was shown by us to be the convergence radius of Newton's method [2], [6]

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad \text{for each} \quad n = 0, 1, 2, \dots \tag{2.25}$$

under the conditions (2.8)–(2.9). It follows from (2.4) that the convergence radius r of the third order method (1.2) cannot be larger than the convergence radius $r_{\mathcal{A}}$ of the second order Newton's method (2.25). In our earlier studies, we used $\overline{r}_A = \frac{2}{2\mathsf{L}_0`+\overline{\mathsf{L}}}$. As already noted in [2], [4], [6] $r_{\mathcal{A}}$ is at least as large as the convergence ball given by Rheinboldt [21]

$$r_{\mathcal{R}} = \frac{2}{3\,\overline{\mathsf{L}}}\,.$$

In particular, for $\mathsf{L}_0 < \overline{\mathsf{L}}$ we have that $r_{\mathcal{R}} < \overline{r}_{\mathcal{A}}$ and

$$\frac{r_{\mathcal{R}}}{\overline{r}_{\mathcal{A}}} \longrightarrow \frac{1}{3} \quad \text{as} \quad \frac{\mathsf{L}_0}{\overline{\mathsf{L}}} \longrightarrow 0.$$

That is our convergence ball $\overline{r}_{\mathcal{A}}$ is at most three times larger than Rheinboldt's. The same value for $r_{\mathcal{R}}$ was also given by Traub [22]. Notice also that $\overline{r}_{\mathcal{A}} \leq r_{\mathcal{A}}$, since $\mathsf{L} \leq \overline{\mathsf{L}}$.

3. Numerical examples

We present some numerical examples in this section.

Example 3.1. Returning back to the motivational example, we get $\mathsf{L}_0 = \mathsf{L} = \overline{\mathsf{L}} = 96.662907$, $\mathcal{M} = 1.0631$ and $\alpha = 1$. Then, using (2.7) and (2.24), we obtain

$$r^\star = 0.002116\dots, \quad R^\star = 0.002026\dots$$

Example 3.2. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^{m-1}$ for integer $m \geq 2$, $\mathbf{X}$ and $\mathbf{Y}$ are equipped with the max-norm $\|\mathbf{x}\| = \max_{1\leq i\leq m-1}\|x_i\|$. The corresponding matrix norm is

$$\|A\| = \max_{1\leq i\leq m-1} \sum_{j=1}^{m-1} |a_{ij}|$$

for $A = (a_{ij})_{1\le i,j\le m-1}$. On the interval $[0,1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^2 = 0 \\ v(0) = v(1) = 0, \end{cases} \tag{3.1}$$

see [2], [6]. To discretize the above equation, we divide the interval $[0,1]$ into m equal parts with length of each part $h = 1/m$ and coordinate of each point $x_i = i\,h$ with $i = 0,1,2,\dots,m$. A second-order finite difference discretization of equation (3.1) results in the following set of nonlinear equations

$$\mathcal{F}(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^2 - 2v_i + v_{i+1} = 0 \\ \text{for} \quad i = 1,2,\dots,(m-1) \quad \text{and from (3.1)} \quad v_0 = v_m = 0 \end{cases} \tag{3.2}$$

where $\mathbf{v} = [v_1, v_2, \dots, v_{(m-1)}]^{\mathrm{T}}$. For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$\mathcal{F}'(\mathbf{v}) = \begin{bmatrix} \dfrac{2v_1}{m^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \dfrac{2v_2}{m^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \dfrac{2v_3}{m^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \dfrac{2v_{(m-1)}}{m^2} - 2 \end{bmatrix}.$$

Let $m = 101$ and we choose $\alpha = 1.0$. To solve the nonlinear system (3.2) by method (1.2), we implemented the algorithm in MatLab. Performance of method (1.2) is reported in Table 1.

n	$\lVert x_n - x_{n-1} \rVert_2$	$\lVert \mathcal{F}(x) \rVert_2$
0		$10.122106\dots$
1	$1.035814\dots$	$1{,}738\,051 \cdot 10^{-1}$
2	$8{,}099\,007 \cdot 10^{-1}$	$2{,}012\,154 \cdot 10^{-2}$
3	$1{,}507\,597 \cdot 10^{-1}$	$5{,}264\,710 \cdot 10^{-5}$
4	$5{,}063\,846 \cdot 10^{-4}$	$1{,}91 \cdot 10^{-14}$
5	$1{,}25 \cdot 10^{-13}$	$1{,}1 \cdot 10^{-14}$

Table 1. Solving (3.1) by method (1.2) for $x_0 = [1, 1.09, \dots, 10]^{\mathrm{T}}$.

Figure 1 plots our numerical solution.

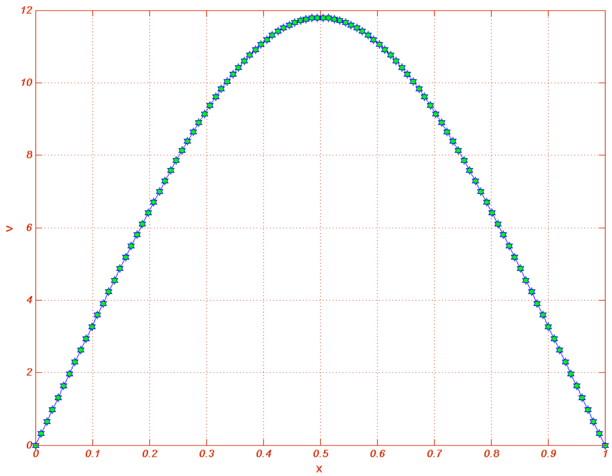

FIGURE 1. Solution of the boundary value problem (3.1).

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