Nonnormality of remainders of some topological groups

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Abstract. It is known that every remainder of a topological group is Lindelöf or pseudocompact. Motivated by this result, we study in this paper when a topological group G has a normal remainder. In a previous paper we showed that under mild conditions on G, the Continuum Hypothesis implies that if the Čech-Stone remainder G^* of G is normal, then it is Lindelöf. Here we continue this line of investigation, mainly for the case of precompact groups. We show that no pseudocompact group, whose weight is uncountable but less than $\mathfrak{c}$, has a normal remainder under $\mathsf{MA}+\neg\mathsf{CH}$. We also show that if a precompact group with a countable network has a normal remainder, then this group is metrizable. We finally show that if $C_p(X)$ has a normal remainder, then X is countable (Corollary 4.10) This result provides us with many natural examples of topological groups all remainders of which are nonnormal.

Keywords: remainder; compactification; topological group; normal space

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1. Introduction

All topological spaces under discussion are Tychonoff.

By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. Among the best known remainders are the Čech-Stone remainders $X^* = \beta X \setminus X$ for arbitrary spaces X and the 1-point remainders $\alpha Y \setminus Y$ for locally compact spaces Y.

Remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. An example demonstrating this is Arhangel'skii's Theorem from [3]: every remainder of a topological group is Lindelöf or pseudocompact. All remainders of locally compact groups are compact, hence both Lindelöf and pseudocompact. For non-locally compact groups there is a dichotomy: every remainder is either Lindelöf or pseudocompact.

If X is a separable metrizable space, then it has a separable metrizable compactification. The remainder of this compactification is separable metrizable as well. This implies that the Čech-Stone remainder X^* of X is a Lindelöf p-space, being a perfect preimage of a separable metrizable space. Hence all remainders of X are Lindelöf p-spaces since every remainder is an image of X^* under a perfect

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mapping. Similarly, if a space X has at least one Lindelöf remainder, then all remainders are Lindelöf. (This is folklore.)

In this paper, we are interested in the question when the normality of a remainder of a topological group G forces that remainder to be Lindelöf, or forces other remainders of G to be normal.

As we saw above, this is always the case for separable metrizable groups. But not always so, as can be demonstrated by an example which was brought to our attention by Buzyakova (for a different reason). Supply $G = \{0,1\}^{\omega_1}$ with the topology generated by all boxes that are determined by countably many coordinates. Then G is a topological group, is linearly ordered and hence has a linearly ordered compactification. That G is indeed linearly ordered can be easily checked directly. Alternatively, apply Theorem 6 in Nyikos and Reichel [11]. Hence the remainder of G in this compactification is monotonically normal and therefore, hereditarily normal. But that remainder is not Lindelöf, simply observe that G is a P-space and that any P-space with a Lindelöf remainder is discrete.

We showed in [4] that the Čech-Stone remainder of this topological group G is not normal. Hence, *the normality of a specific remainder of a topological group does not imply in general that all remainders are normal.* Thus, normal remainders behave differently compared to Lindelöf remainders.

We also showed in [4] that for G a nowhere locally compact topological group that contains a nonempty compact G_δ-subset, if the character of G is at most $\mathfrak{c}$, and G^* is normal, then G^* is Lindelöf under the Continuum Hypothesis (abbreviated CH).

Many open problems remain, we address some of them here. Our results show that it is very rare that a precompact group has a normal remainder. For example, if a countable precompact group has a normal remainder, then its weight must be countable. We also show that no pseudocompact group whose weight is uncountable but less than $\mathfrak{c}$, has a normal remainder under $\mathsf{MA}+\neg\mathsf{CH}$. As an application of our methods we show that if $C_p(X)$ has a normal remainder, then X is countable.

2. Preliminaries

A topological group G is *precompact* if for every neighborhood U of the identity element $e \in G$ there is a finite subset F of G such that $FU = G$. It is known that a topological group G is precompact if and only if it is a dense subgroup of a compact group $\overline{G}$. This was shown by Weil [15] and Raĭkov [12]. For details, and more references, see Arhangel'skii and Tkachenko [5]. The topological group $\overline{G}$ is called the *Weil completion* of G and is the same thing as the Raĭkov completion of G (see about this the comments on page 251 in [5]). It is unique up to a natural topological isomorphism, and it is not difficult to show that G and $\overline{G}$ have the same weight. Indeed, both G and $\overline{G}$ are precompact and hence, ω-narrow topological groups. Therefore, $w(G) = \chi(G)$ and $w(\overline{G}) = \chi(\overline{G})$, by Corollary 5.2.4 in [5]. It remains to observe that $\chi(G) = \chi(\overline{G})$, since G is dense in $\overline{G}$.

We refer to Juhász [9] and Arhangel'skii and Tkachenko [5] for undefined terminology on cardinal functions.

3. Main results

Theorem 3.1 ($\mathsf{MA}+\neg\mathsf{CH}$). *If G is a precompact topological group such that $\omega_1 \leq \mathrm{w}(G) < \mathfrak{c}$, then either no remainder of G is normal, or G, and every remainder of G, is a Lindelöf p-space.*

PROOF: Let $\overline{G}$ be the Weil completion of G. Then the weight of $\overline{G}$ is uncountable and less than $\mathfrak{c}$. Since $\overline{G}$ is a compact group, it follows from the last inequality that $\overline{G}$ is separable ([5, 5.2.7]). Hence, the Souslin number $c(G)$ of G is countable.

Case 1. The remainder $Y = \overline{G} \setminus G$ is Lindelöf.

Then G is a paracompact p-space, by Arhangel'skii [2, Theorem 4.1]. Since $c(G) \leq \omega$, it follows that G is a Lindelöf p-space. Then every remainder of G is a Lindelöf p-space by Arhangel'skii [2, Theorem 2.1].

By the Dichotomy Theorem, it remains to consider the following

Case 2. The remainder $Y = \overline{G} \setminus G$ is pseudocompact and not Lindelöf.

Clearly, G and Y are dense in $\overline{G}$. Let us fix a countable subset D which is dense in $\overline{G}$. We claim that Y is separable. If G is separable, then so is its remainder, since it contains a translate of G, and this translate is dense in Y. So assume that G is not separable. Since G is precompact, this means that no nonempty open subset of G is separable. Therefore, $D \cap G$ is nowhere dense in G. Since D is dense in $\overline{G}$, and $G \cup Y = \overline{G}$, it follows that $D \cap Y$ is dense in Y. Hence, Y is separable.

By Booth's Lemma (see [13, p. 20]) which follows from $\mathsf{MA}+\neg\mathsf{CH}$, we get a sequence in Y converging to a point in G. Therefore, Y is not countably compact, hence not normal. Clearly, every remainder of G is pseudocompact and not countably compact. Hence, no remainder of G is normal. $\square$

Consider the subgroup $\Sigma = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x_\alpha = 1\}| \leq \omega\}$ of 2^{ω_1}. It is known that Σ is normal (Kombarov and Malyhin [10]). It clearly follows from Theorem 3.1 that no remainder of Σ is normal under $\mathsf{MA}+\neg\mathsf{CH}$.

Theorem 3.2. *If G is a precompact noncompact group such that the cardinality of every discrete in itself subspace of it is less than the weight of G, then no remainder of G is normal.*

PROOF: The weight of G is clearly uncountable and hence the weight of its Weil completion $\overline{G}$ is the same uncountable cardinal number. Let $\kappa = \mathrm{w}(\overline{G})$. Then $\overline{G}$ contains a copy K of the Cantor cube 2^κ. Let $S = \big(W(\omega{+}1) \times W(\omega_1{+}1)\big) \setminus \{(\omega, \omega_1)\}$, and let T be the one-point compactification of the topological sum of κ many copies of S. Here $W(\omega_1{+}1)$ denotes the space of all ordinals not greater than ω_1 endowed with the order-topology. Hence the space S is the well-known Tychonoff plank. Then T is zero-dimensional and has weight κ, hence it embeds in K and we may assume that the point at infinity of T coincides with the neutral element e of $\overline{G}$. Clearly, $T \setminus \{e\}$ is closed in $\overline{G} \setminus \{e\}$. Since the cardinality of every discrete in itself subspace of G is strictly less than κ, there is a closed copy of the Tychonoff plank in $Y = \overline{G} \setminus G$. Therefore, Y is not normal and not countably compact. Hence Y is not Lindelöf, and it follows from the Dichotomy Theorem that Y is pseudocompact. Therefore, every remainder of G is pseudocompact and not countably compact, which implies that every remainder of G is nonnormal. $\square$

Corollary 3.3. *Every precompact group with a countable network and a normal remainder is metrizable.*

In particular, we have:

Corollary 3.4. *Every countable precompact group with a normal remainder is metrizable.*

Corollary 3.5. *Every hereditarily Lindelöf precompact group with a normal remainder is metrizable.*

Corollary 3.6. *Every hereditarily separable precompact group with a normal remainder is metrizable.*

Among examples to which the last set of results in this section apply, are the countable dense subgroups of the Cantor cube $2^{\mathfrak{c}}$. They are precompact, have weight $\mathfrak{c}$ and have a countable network. Hence no remainder of such a group is normal.

4. The Dichotomy Theorem improved

The proof of Theorem 3.2 given in the preceding section suggests that the following modification of Dichotomy Theorem can be quite helpful in some arguments.

Theorem 4.1. *Suppose that G is a topological group. Then at least one of the following conditions is satisfied.*

(1) *Every remainder of G is Lindelöf.*
(2) *Every remainder of G is countably compact.*
(3) *Every remainder of G is nonnormal.*

Proof: Assume that none of the conditions (1) and (2) holds. Then, by the Dichotomy Theorem and by the invariance of countable compactness under perfect mappings, every remainder of G is pseudocompact and not countably compact. This implies that every remainder of G is nonnormal. $\square$

Corollary 4.2. *Suppose that G is a topological group such that the Tychonoff plank is homeomorphic to a closed subspace of some remainder Y of G. Then every remainder of G is nonnormal.*

Proof: Conditions (1) and (2) in Theorem 4.1 are not satisfied, since the Tychonoff plank is not Lindelöf and is not countably compact. Therefore, condition (3) holds, that is, every remainder of G is nonnormal. □

A similar simple argument shows that the next statement is true:

Corollary 4.3. *Suppose that G is a topological group such that some remainder Y of G has an uncountable closed in Y discrete subspace. Then every remainder of G is nonnormal.*

Corollary 4.4. *Suppose that G is a topological group and Y is a normal remainder of G with an infinite closed discrete subspace. Then Y is Lindelöf.*

Proof: The space Y is not countably compact, since Y has an infinite closed discrete subspace. Thus, conditions (2) and (3) in Theorem 4.1 are not satisfied. It follows that Y is Lindelöf. □

Clearly, the last statement can be reformulated as follows:

Theorem 4.5. *Suppose that G is a non-locally compact topological group and Y is a remainder of G. Then the following conditions are equivalent.*

(a) *Y is normal and not countably compact.*
(b) *Y is Lindelöf.*

The next theorem also generalizes some results in the preceding section.

Theorem 4.6. *Suppose that G is a topological group which has a dyadic compactification bG and satisfies the condition that the cardinality of G is less than the weight of G. Then every remainder of G is nonnormal.*

Proof: Notice that, clearly, $|G| < \mathrm{w}(G) \leq \mathrm{w}(bG)$. Since G is dense in bG, by Efimov's Theorem from [6], $\chi(a, bG) \geq \tau$, for some $a \in G$ and some infinite cardinal τ such that $|G| < \tau \leq \mathrm{w}(bG)$. By a result of Engelking [7], there exists a subspace $A(\tau)$ of bG which is homeomorphic to the Alexandroff's 1-point compactification of the discrete space of cardinality τ such that the only non-isolated point of $A(\tau)$ is a. Since G does not contain a topological copy of $A(\tau)$, it follows that $|G \cap A(\tau)| < \tau$. Hence, for $C = A(\tau) \setminus G$ we have $|C| = \tau > \omega$. Clearly, C is a closed discrete subspace of the remainder $Y = bG \setminus G$. Since C is uncountable and G is a topological group, it follows from Corollary 4.3 that every remainder of G is nonnormal. □

It is enough in this result to assume that G does not contain a topological copy of $A(\tau)$, for some uncountable cardinal τ such that $\tau \leq \mathrm{w}(bG)$.

Again, an example of a topological group to which this result applies is any countable dense subgroup of $2^{\mathfrak{c}}$.

Problem 4.7. Can a remainder of a (countable) non-discrete extremely disconnected topological group be normal?

We now apply our methods and results in some special cases, for example, to groups of the form $C_p(X)$ for some space X. See the books of Arhangel'skii [1] and Tkachuk [14] for details and references on C_p-theory.

Theorem 4.8. *Let X be a Tychonoff space which is a dense subspace of a product of spaces, τ-many of which are noncompact. Then the extent of every remainder of X is not less than τ, that is, every remainder of X has a closed discrete subspace of cardinality τ.*

Proof: One can put all the compact factors of the product into the first noncompact factor, so we can assume without loss of generality that all factors are noncompact. Hence we may assume that X is a dense subspace of $\prod_{i\in I} X_i$, where $|I| \geq \tau$ and X_i is noncompact for every $i \in I$. Clearly, $bX = \prod_{i\in I} \beta X_i$ is a compactification of X. For every $i \in I$, take a point $p_i \in X_i^* = \beta X_i \setminus X_i$, and let $x = (x_i)_{i\in I}$ be an arbitrary point in X. For every $i \in I$, let $y(i)$ in $\prod_{i\in I} \beta X_i$ be given by $y(i)_j = x_j$ if $j \neq i$, and $y(i)_i = p_i$. Then $\{y(i) : i \in I\}$ is a discrete subspace of the remainder of X in bX, and

$$\{x\} \cup \{y(i) : i \in I\}$$

is the one-point compactification of this discrete subspace. Hence, $bX \setminus X$ contains a closed discrete subset of size at least τ. From this it is easy to see that every remainder of X has a closed discrete subset of size at least τ. □

Corollary 4.9. *Let G be a topological group which is homeomorphic to a dense subspace of a product of topological spaces, uncountably many of which are noncompact. Then every remainder of G is nonnormal.*

Proof: Since G is a topological group, we are done by Theorem 4.8 and Corollary 4.3. □

Of course, it follows from Corollary 4.9 that if a topological group G is homeomorphic to a dense subspace of $\mathbb{R}^X$ for some uncountable set X, then every remainder of G is nonnormal. It is known that the space $C_p(X)$ of continuous real-valued functions on a space X is a dense subspace of $\mathbb{R}^X$. Therefore, we have established the following fact:

Corollary 4.10. *If X is uncountable, then every remainder of $C_p(X)$, as well as any remainder of any dense subspace Y of $C_p(X)$, is nonnormal.*

There are also some interesting corollaries to Theorem 4.8 outside the class of all topological groups.

Theorem 4.11 (V=L). *Let X be a nowhere locally compact space which is a dense subspace of a product of a family consisting of $\mathfrak{c}$ many noncompact spaces each of weight at most $\mathfrak{c}$. Suppose also that the Souslin number of X is countable. Then every remainder Y of X in any compactification bX is nonnormal.*

For the proof of Theorem 4.11, we need the following result.

Lemma 4.12. *Suppose that every remainder of a space X in any compactification bX with $\mathrm{w}(bX) = \mathrm{w}(X)$ is nonnormal. Then every remainder of X is nonnormal.*

Proof: Fix a remainder Y_1 of X in a compactification b_1X, and put $\tau = \mathrm{w}(X)$. Obviously, there exists a continuous mapping f of b_1X onto some compactification bX of X such that $\mathrm{w}(bX) = \tau$ and $f^{-1}(X) = X$ (just take the diagonal product of an appropriate family of continuous real-valued functions on b_1X). Then the remainder Y_1 of X in b_1X is mapped onto the remainder Y of X in bX by a perfect mapping. Therefore, if Y_1 is normal, then Y is also normal, a contradiction, which completes the proof of the Lemma. $\square$

Proof of Theorem 4.11: Because of Lemma 4.12, we can assume that

$$\mathrm{w}(bX) = \mathrm{w}(X).$$

Then, clearly, $\mathrm{w}(bX) \le \mathfrak{c}$. The extent of every remainder of X is uncountable by Theorem 4.8. Thus, the remainder Y of X has an uncountable closed discrete subspace. Clearly, the character of Y does not exceed $\mathfrak{c}$. Assume now that Y is normal. Fleissner [8] has shown that, under $\mathsf{V}{=}\mathsf{L}$, every normal space with character $\le \mathfrak{c}$ is collectionwise Hausdorff. Using this result, we conclude that there exists an uncountable disjoint family of nonempty open sets in Y. However, the Souslin number of Y is countable, since obviously Y is dense in bX and the Souslin number of bX is countable, since the Souslin number of X is countable. This contradiction completes the proof. $\square$

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