A weighted inequality for the Hardy operator involving suprema

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Abstract. Let u be a weight on $(0,\infty)$. Assume that u is continuous on $(0,\infty)$. Let the operator S_u be given at measurable non-negative function φ on $(0,\infty)$ by

$$S_u\varphi(t) = \sup_{0<\tau\leq t} u(\tau)\varphi(\tau).$$

We characterize weights v, w on $(0,\infty)$ for which there exists a positive constant C such that the inequality

$$\left(\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t)\,dt\right)^{\frac{1}{p}}$$

holds for every $0 < p, q < \infty$. Such inequalities have been used in the study of optimal Sobolev embeddings and boundedness of certain operators on classical Lorenz spaces.

Keywords: Hardy operators involving suprema; weighted inequalities

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1. Introduction

In [1], it was characterized when the *Hardy–Littlewood maximal operator* M is bounded on the so-called classical Lorentz spaces. We recall that the operator M is defined at every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(Mf)(x) = \sup_{Q\ni x} |Q|^{-1}\int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|E|$ denotes the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. To prove this result, two ingredients have been used. First of them was the well-known two-sided estimate for the non-increasing rearrangement of Mf in terms of the maximal non-increasing rearrangement. This result is due to Riesz, Wiener, Stein and Herz (cf. [2, Chapter 3, Theorem 3.8]). Second key ingredient was the characterization of the boundedness of the *Hardy averaging operator*

$$Af(t) := \frac{1}{t}\int_0^t f(s)\,ds$$

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on the cone of non-increasing functions in a weighted Lebesgue space. An analogous problem was later in [4] considered for the *fractional maximal operator*. This operator, denoted by M_γ, where $\gamma \in (0,n)$, is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$M_\gamma f(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It turned out that in order to handle the fractional maximal operator one needs to characterize a weighted inequality involving a substantially different operator than the Hardy's average integral operator. Namely, the operator R_γ was employed, which is defined at a measurable and positive on $(0,\infty)$ function g by

$$R_\gamma g(t) = \sup_{t \le s < \infty} s^{\frac{\gamma}{n}-1} g(s), \quad t \in (0,\infty).$$

The operator R_γ is a typical example of what we may call *a Hardy-type operator involving suprema*. In [10], a more general (weighted) version of such operator was studied. We recall that by a *weight* we shall throughout understand a positive measurable function on $(0,\infty)$. For a weight u, the operator R_u was defined in [10] at each non-negative measurable function g by

$$R_u g(t) = \sup_{t \le s < \infty} u(s) g(s), \quad t \in (0,\infty).$$

An analogous, in a certain sense, dual operator, denoted by S_u and defined by

$$S_u g(t) = \sup_{0 < s \le t} u(s) g(s), \quad t \in (0,\infty),$$

has been recently proved useful in various applications. These cover, for example, the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds either in the Euclidean space (see e.g. [11], [12]) or in the product probability spaces of which the Gaussian space is a key example ([5], [6]). They further constitute a useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding ([13]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [8], [9], [7] or [14].

Although both the operators R_u and S_u are of interest, a comprehensive study was so far devoted only to the operator R_u. In this paper we characterize a weighted inequality for the operator S_u, restricted to the cone of non-increasing functions. The method of the proof is in some sense similar to that used in [10] but the characterizing conditions are different in nature and the technical steps of the proof had to be modified in a corresponding way.

Let $0 < p, q < \infty$ and let u be a continuous weight. Our principal goal is to give a characterization of weights v and w such that inequality

$$\left(\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t)\,dt\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all non-negative and non-increasing functions φ on $(0,\infty)$. It will be useful to observe that, for every non-negative function φ, the function $S_u\varphi$ is non-decreasing on $(0,\infty)$.

We treat the cases $p \le q$ and $p > q$ separately since the techniques we use in their proofs are quite different.

As usual, here and below, by $A \lesssim B$ we mean that $A \le CB$, where C is a positive constant independent of appropriate quantities involved in the expressions A and B.

2. Main results

Theorem 1. *Let $0 < p \le q < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)\,dt < \infty$ and $0 < \int_x^\infty w(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then inequality* (1.1) *is satisfied for all non-negative and non-increasing functions φ on $(0,\infty)$ if and only if*

$$\sup_{a\in(0,\infty)} \frac{\left(\int_0^a (\bar{u}(t))^q w(t)\,dt\right)^{\frac{1}{q}} + \bar{u}(a)\left(\int_a^\infty w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^a v(t)\,dt\right)^{\frac{1}{p}}} < +\infty, \tag{2.1}$$

where $\bar{u}(t) = \sup_{0<\tau\le t} u(\tau)$.

Proof: *Sufficiency.* We distinguish several cases. First, let $\int_0^\infty w(t)\,dt = \infty$ and $\int_0^\infty v(t)\,dt = \infty$. We define sequences $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y'_s\}_{s\in\mathbb{Z}}$ by

$$\int_{x_k}^\infty w(t)\,dt = 2^{-k} \quad \text{and} \quad \int_0^{y'_s} v(t)\,dt = 2^s. \tag{2.2}$$

Then we have

$$(0,\infty) = \bigcup_{k\in\mathbb{Z}} [x_k, x_{k+1}) = \bigcup_{s\in\mathbb{Z}} [y'_s, y'_{s+1}). \tag{2.3}$$

Consequently, using (2.3), the definition of the operator S_u, its monotonicity and (2.2), we get

$$\begin{aligned}\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} [S_u\varphi(t)]^q w(t)\,dt \\ &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} [\sup_{0<\tau\le t} u(\tau)\varphi(\tau)]^q w(t)\,dt\end{aligned}$$

$$\leq \sum_{k\in\mathbb{Z}} \sup_{0<\tau\leq x_{k+1}} [u(\tau)\varphi(\tau)]^q \int_{x_k}^{x_{k+1}} w(t)\,dt$$
$$\leq \sum_{k\in\mathbb{Z}} 2^{-k-1} \sup_{-\infty<i\leq k} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q.$$

Using a simple upper estimate of a supremum by a corresponding sum, (2.2) and (2.3) again, and interchanging the sums, we obtain

$$\begin{aligned}\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt &\leq \sum_{k\in\mathbb{Z}} 2^{-k-1} \sum_{i=-\infty}^{k} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &= \sum_{i\in\mathbb{Z}} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \sum_{k=i}^{\infty} 2^{-k-1} \\ &= \sum_{i\in\mathbb{Z}} 2^{-i} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &= \sum_{i\in\mathbb{Z}} \int_{x_i}^{\infty} w(t)\,dt \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &\lesssim \sum_{i\in\mathbb{Z}} \int_{x_{i+1}}^{x_{i+2}} w(t)\,dt \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q.\end{aligned}$$

Now, given $i\in\mathbb{Z}$, let us find points $z_i \in [x_i, x_{i+1}]$ such that

$$\sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \leq 2[u(z_i)\varphi(z_i)]^q. \tag{2.4}$$

Thus, $[x_{i+1}, x_{i+2}] \subseteq [z_i, z_{i+2}]$ and

$$\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt \lesssim \sum_{i\in\mathbb{Z}} \left(\int_{z_i}^{z_{i+2}} w(t)\,dt\right) [u(z_i)\varphi(z_i)]^q.$$

For a technical reason we divide the sum in two parts, write

$$\sum_{k\in\mathbb{Z}} \left(\int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\right) [u(z_{2k})\varphi(z_{2k})]^q =: S_{even},$$
$$\sum_{k\in\mathbb{Z}} \left(\int_{z_{2k+1}}^{z_{2k+3}} w(t)\,dt\right) [u(z_{2k+1})\varphi(z_{2k+1})]^q =: S_{odd}.$$

We shall estimate S_{even}. First, we reduce the sequence $\{y'_s\}$. Fix $k\in\mathbb{Z}$. If the interval $[z_{2k}, z_{2k+2})$ contains more than one element of the sequence $\{y'_s\}$, we delete from this sequence all such elements except the one which lies nearest to z_{2k}. Thus, every interval $[z_{2k}, z_{2k+2})$, $k\in\mathbb{Z}$, now contains at most one element of the reduced sequence, which we denote by $\{y_n\}_{n\in\mathbb{Z}}$. More formally, we denote $Y_k = \{s\in\mathbb{Z}; y'_s \in [z_{2k}, z_{2k+2})\}, k\in\mathbb{Z}$, further $J = \{k\in\mathbb{Z}; Y_k \neq 0\}$, $\theta_k =$

$\min\{y'_s; s \in Y_k\}$, $k \in J$, and finally $Y = \{\theta_k\}_{k\in J}y$. Then Y is a subsequence of $\{y'_s\}$, which we enumerate as $\{y_n\}_{n\in\mathbb{Z}}$. Clearly, $y_n < y_{n+1}$ for all $n \in \mathbb{Z}$ and this sequence is a covering sequence having the following properties: Suppose that for some $n, k, s \in \mathbb{Z}$ we have

$$y_n < z_{2k} \le y_{n+1} = y'_s. \tag{2.5}$$

Then one can easily check that

$$y_{n-1} \le y'_{s-2}, \tag{2.6}$$

$$y'_{s-1} < z_{2k}, \tag{2.7}$$

$$y_{n-1} < z_{2k-2}. \tag{2.8}$$

By (2.6) and (2.7), for all $n, k, s \in \mathbb{Z}$ satisfying (2.5),

$$\int_0^{y_{n+1}} v(t)\,dt = 4\int_{y'_{s-2}}^{y'_{s-1}} v(t)\,dt \le 4\int_{y_{n-1}}^{z_{2k}} v(t)\,dt.$$

We need to estimate $\varphi^p(z_{2k})$ and to use this estimate in inequality for S_{even}. So, since the function φ is non-increasing, we have

$$\varphi^p(z_{2k}) = \frac{\int_{y_{n-1}}^{z_{2k}} v(t)\,dt}{\int_{y_{n-1}}^{z_{2k}} v(t)\,dt}\,\varphi^p(z_{2k}) \le \left(\int_{y_{n-1}}^{z_{2k}} v(t)\,dt\right)^{-1} \int_{y_{n-1}}^{z_{2k}} \varphi^p(t)v(t)\,dt. \tag{2.9}$$

Hence

$$\varphi^q(z_{2k}) \lesssim \left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-\frac{q}{p}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt\right)^{\frac{q}{p}}. \tag{2.10}$$

Let us still write

$$u^q(x) \le \left(\sup_{0<\tau\le t} u(\tau)\right)^q = [\bar{u}(t)]^q \quad \text{for all} \quad t \ge x.$$

Denote $A_n = \{k \in \mathbb{Z}; y_n < z_{2k} \le y_{n+1}\}$, $n \in \mathbb{Z}$. Then

$$S_{even} = \sum_{n\in\mathbb{Z}}\sum_{k\in A_n}\int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\;[u(z_{2k})\varphi(z_{2k})]^q.$$

Fix $n \in \mathbb{Z}$ and define numbers $l_1^n = \min\{k; k \in A_n\}$ and $l_2^n = \max\{k; k \in A_n\}$. Thanks to (2.4), the definition of l_1^n and l_2^n and the fact that φ is non-increasing, we get

$$\sum_{k\in A_n}\int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\;[u(z_{2k})\varphi(z_{2k})]^q$$

$$\leq \left(\int_{z_{2l_1^n}}^{y_{n+1}} (\bar{u}(t))^q w(t)\,dt + [\bar{u}(y_{n+1})]^q \int_{y_{n+1}}^{z_{2l_2^n+2}} w(t)\,dt \right) [\varphi(z_{2l_1^n})]^q.$$

Thus by (2.5) and (2.10),

$$\begin{aligned}
&\sum_{k\in A_n} \int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\ [u(z_{2k})\varphi(z_{2k})]^q \\
&\qquad \leq \left(\int_0^{y_{n+1}} (\bar{u}(t))^q w(t)\,dt + (\bar{u}(y_{n+1}))^q \int_{y_{n+1}}^{\infty} w(t)\,dt \right) [\varphi(z_{2l_1^n})]^q \\
&\qquad \lesssim \sum_{n\in\mathbb{Z}} \left(\int_0^{y_{n+1}} (\bar{u}(t))^q w(t)\,dt + (\bar{u}(y_{n+1}))^q \int_{y_{n+1}}^{\infty} w(t)\,dt \right) \\
&\qquad\quad \times \left(\int_0^{y_{n+1}} v(t)\,dt \right)^{-\frac{q}{p}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt \right)^{\frac{q}{p}} \\
&\qquad \lesssim \sum_{n\in\mathbb{Z}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt \right)^{\frac{q}{p}},
\end{aligned}$$

where in the last inequality we use the condition (2.1). Since $p \leq q$, we can use the convexity of the function $x^{\frac{q}{p}}$ and we have

$$\begin{aligned}
S_{even} &\lesssim \sum_{n\in\mathbb{Z}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt \right)^{\frac{q}{p}} \\
&\lesssim \left(\sum_{n\in\mathbb{Z}} \int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt \right)^{\frac{q}{p}} \\
&\lesssim \left(\int_0^{\infty} \varphi^p(t)v(t)\,dt \right)^{\frac{q}{p}}.
\end{aligned}$$

In order to estimate S_{odd}, we define a possibly different sequence $\{y_n\}_{n\in\mathbb{Z}}$. Again, we reduce the sequence y_n' in the same way, but this time in intervals $[z_{2k+1}, z_{2k+3})$. Now, it is clear that we can estimate S_{odd} in the same way as S_{even} was estimated. The main reason for the division into sums S_{even} and S_{odd} is to guarantee that the sets A_n are non-empty.

If $\int_0^\infty w(t)\,dt < \infty$, then we can without loss of generality assume that $\int_0^\infty w(t)dt = 1$ and work instead of the sequence $\{x_k\}_{k\in\mathbb{Z}}$ only with the reduced sequence $\{x_k\}_{k=0}^{\infty}$. In the case when moreover $\int_0^\infty v(t)\,dt < \infty$, then we replace the sequence $\{y_n\}_{n\in\mathbb{Z}}$ by a reduced sequence $\{y_n\}_{n=-\infty}^{N}$ with an appropriate $N \in \mathbb{Z}$.

This completes the proof of the sufficiency part.

Necessity. We first observe that

$$S_u\chi_{(0,a]}(t) = \bar{u}(t)\chi_{(0,a]}(t) + \bar{u}(a)\chi_{(a,\infty)}(t).$$

Now, testing the inequality (1.1) with functions $\varphi(t) = \chi_{(0,a]}(t), a \in (0,\infty)$, we get exactly the inequality (2.1). □

Our next aim is to handle the case when $0 < q < p < \infty$. We shall need the following special case of [10, Theorem 4.4].

Theorem 2. *Let U be a continuous weight and let V and W be weights such that $0 < \int_0^x V(t)\,dt < \infty$ and $0 < \int_0^x W(t)\,dt < \infty$ for every $x \in (0,\infty)$. Let $0 < Q < 1$ and let R be defined by*

$$\frac{1}{R} = \frac{1}{Q} - 1.$$

Then the inequality

$$\left(\int_0^\infty \left(\sup_{t\le s<\infty} \frac{U(s)}{s}\int_0^s g(y)\,dy\right)^Q W(t)\,dt\right)^{\frac{1}{Q}} \lesssim \int_0^\infty g(t)V(t)\,dt$$

holds for every non-negative measurable function g if and only if

$$\left(\int_0^\infty \left(\int_t^\infty \left(\frac{\tilde{U}(s)}{s}\right)^Q W(s)\,ds\right)^R \left(\frac{\tilde{U}(t)}{t}\right)^Q \left[\operatorname*{ess\,sup}_{a<t<b} \frac{1}{V(t)}\right]^R W(t)\,dt\right)^{\frac{1}{R}} < \infty$$

and

$$\left(\int_0^\infty \left(\int_0^t W(s)\,ds\right)^R \left[\sup_{t\le\tau<\infty} \frac{\tilde{U}(\tau)}{\tau} \operatorname*{ess\,sup}_{a<t<b} \frac{1}{V(t)}\right]^R W(t)\,dt\right)^{\frac{1}{R}} < \infty,$$

where

$$\tilde{U}(t) = t \sup_{t\le\tau<\infty} \frac{U(\tau)}{\tau}, \quad t \in (0,\infty).$$

Theorem 3. *Let $0 < q < p < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)\,dt < \infty$ and $0 < \int_x^\infty w(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then inequality (1.1) is satisfied for all non-negative and non-increasing functions φ on $(0,\infty)$ if and only if the following two conditions are* *satisfied:*

$$\begin{aligned}(2.11)\qquad &\int_0^\infty \left(\int_0^t \sup_{0<\tau\le s} u(\tau)^{\frac{q}{p}} w(s)\,ds\right)^{\frac{q}{q-p}} \sup_{0<y\le t} u(y)^{\frac{q}{p}}\\ &\qquad\times w(t)\left(\int_0^t v(s)\,ds\right)^{-\frac{q}{p-q}} dt<\infty\end{aligned}$$

and

$$(2.12)\qquad \int_0^\infty \left(\int_t^\infty w(y)\,dy\right)^{\frac{q}{p-q}} \left(\sup_{0<\tau\le t} \frac{\sup_{0<z\le\tau} u(z)}{\int_0^\tau v(y)\,dy}\right)^{\frac{q}{p-q}} w(t)\,dt<\infty.$$

Proof: Changing variables ($y=\frac{1}{t}$) on both sides of the inequality (1.1), we get

$$\left(\int_0^\infty \left(\sup_{0<\tau\le\frac{1}{y}} u(\tau)\varphi(\tau)\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}.$$

On denoting $z=\frac{1}{\tau}$, we arrive at the inequality

$$\left(\int_0^\infty \left(\sup_{0<\frac{1}{z}\le\frac{1}{y}} u(\tfrac{1}{z})\varphi(\tfrac{1}{z})\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}$$

for every non-increasing positive function φ. Noting that $0<\frac{1}{z}\le\frac{1}{y}$ is equivalent to $y\le z<\infty$, we actually have

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u(\tfrac{1}{z})\varphi(\tfrac{1}{z})\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}.$$

By a simple re-scaling, this is equivalent to

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\varphi^p(\tfrac{1}{z})\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}.$$

Since φ is a non-increasing positive function, the function $z\mapsto\varphi^p(\frac{1}{z})$ is positive and non-decreasing on $(0,\infty)$ in the variable z. By a standard approximation argument based on the Monotone Convergence Theorem (see, e.g., [3]), one can equivalently reduce the last inequality to the same one but restricted only to functions of the form

$$\varphi^p(\tfrac{1}{z})=\int_0^z h(s)\,ds.$$

We thus get

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty \int_0^t h(s)ds\, v(\tfrac{1}{t})\frac{dt}{t^2}$$

for every measurable non-negative function h on $(0,\infty)$. By the Fubini theorem, this is nothing else than

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(s)\int_s^\infty v(\tfrac{1}{t})\frac{dt}{t^2}\,ds,$$

that is,

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(s)\int_0^{\frac{1}{s}} v(y)\,dy\,ds.$$

Theorem 2 applied to parameters

$$Q=\tfrac{q}{p},\ U(z)=zu^p(\tfrac{1}{z}),\ W(y)=w(\tfrac{1}{y})y^{-2},\ V(s)=\int_0^{\frac{1}{s}} v(y)\,dy$$

now shows that the latter inequality holds if and only if the conditions (2.11) and (2.12) are satisfied. The proof is complete. □

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