Functionally countable subalgebras and some properties of the Banaschewski compactification

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Abstract. Let X be a zero-dimensional space and $C_c(X)$ be the set of all continuous real valued functions on X with countable image. In this article we denote by $C_c^K(X)$ (resp., $C_c^\psi(X)$) the set of all functions in $C_c(X)$ with compact (resp., pseudocompact) support. First, we observe that $C_c^K(X) = O_c^{\beta_0 X\setminus X}$ (resp., $C_c^\psi(X) = M_c^{\beta_0 X\setminus \upsilon_0 X}$), where $\beta_0 X$ is the Banaschewski compactification of X and $\upsilon_0 X$ is the $\mathbb{N}$-compactification of X. This implies that for an $\mathbb{N}$-compact space X, the intersection of all free maximal ideals in $C_c(X)$ is equal to $C_c^K(X)$, i.e., $M_c^{\beta_0 X\setminus X} = C_c^K(X)$. By applying methods of functionally countable subalgebras, we then obtain some results in the remainder of the Banaschewski compactification. We show that for a non-pseudocompact zero-dimensional space X, the set $\beta_0 X \setminus \upsilon_0 X$ has cardinality at least $2^{2^{\aleph_0}}$. Moreover, for a locally compact and $\mathbb{N}$-compact space X, the remainder $\beta_0 X \setminus X$ is an almost P-space. These results lead us to find a class of Parovičenko spaces in the Banaschewski compactification of a non pseudocompact zero-dimensional space. We conclude with a theorem which gives a lower bound for the cellularity of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$, whenever X is a zero-dimensional, locally compact space which is not pseudocompact.

Keywords: zero-dimensional space; strongly zero-dimensional space; $\mathbb{N}$-compact space; Banaschewski compactification; pseudocompact space; functionally countable subalgebra; support; cellularity; remainder; almost P-space; Parovičenko space

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1. Preliminaries

We recall that a zero-dimensional topological space is a Hausdorff space with a base consisting of clopen sets. Mrówka showed in [13] that X is zero-dimensional if and only if it can be embedded into the product space $\mathbb{N}^\kappa$, where $\mathbb{N}$ is the set of natural numbers with discrete topology and κ is a cardinal number.

We also recall that a topological space X is $\mathbb{N}$-compact if it can be embedded as a closed subset of the product space $\mathbb{N}^\kappa$, for some cardinal number κ, see [3], [13], [14], [15], [16], [17], [18] for more details on this subject.

For every zero-dimensional space X, there exists an $\mathbb{N}$-compact space $\upsilon_0 X$ such that X is dense in it and every continuous function $f : X \to Y$, with Y being

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an $\mathbb{N}$-compact space, has a unique extension $f^* : \upsilon_0 X \to Y$. We can replace an arbitrary $\mathbb{N}$-compact space Y by the fixed discrete space $\mathbb{Z}$ (the set of integer numbers) and have the following characterization of the $\mathbb{N}$-compactification of a zero-dimensional space, see e.g., [20, 5.4 (d)].

Theorem 1.1. *For an $\mathbb{N}$-compact extension T of a zero-dimensional space X there is a homeomorphism between T and $\upsilon_0 X$ that fixes X pointwise if and only if for each continuous function $f : X \to \mathbb{Z}$, there exists a continuous function $F : T \to \mathbb{Z}$ such that $F|_X = f$.*

We remind the reader that a Tychonoff space X is strongly zero-dimensional if and only if every two disjoint zero-sets in X are separated by a clopen partition. It is well known that every strongly zero-dimensional realcompact space is $\mathbb{N}$-compact. Since every countable subset of $\mathbb{R}$ (the set of real numbers) is Lindelöf and zero-dimensional, it is strongly zero-dimensional. This fact implies that every countable subset of $\mathbb{R}$ is $\mathbb{N}$-compact. So we have the following lemma.

Lemma 1.2. *Let X be a zero-dimensional Hausdorff space. For each continuous function $f : X \to \mathbb{R}$ with countable image, there exists an extension $f^* : \upsilon_0 X \to \mathbb{R}$ such that the image of f^* is equal to the image of f.*

For an arbitrary Tychonoff space X, we denote by $C_c(X)$ the set of all continuous real-valued functions on X with countable image. The set $C_c(X)$ forms a subring of $C(X)$ (i.e., the set of all continuous real valued functions on X) with pointwise addition and multiplication. Ghadermazi, Karamzadeh and Namdari showed in [5] that for a Tychonoff space X there exists a zero-dimensional space Y such that $C_c(X) \cong C_c(Y)$ as rings. ***In view of this fact, in the present article we restrict our attention to zero-dimensional spaces.*** In the same article, the authors gave an example of a space X for which $C_c(X)$ is not isomorphic to any $C(Y)$. They remarked that $C_c(X)$, although not isomorphic to any ring of continuous functions in general, enjoys most of the important properties of $C(X)$. The reader could find all prerequisites and unfamiliar notions for this subring in [5].

For a zero-dimensional space X, by $\beta_0 X$ we mean its Banaschewski compactification. We recall that $\beta_0 X$ is the unique (up to homeomorphism) zero-dimensional compact space which contains X as a dense subset such that every continuous two-valued function $f : X \to \{0, 1\}$ has an extension to $\beta_0 X$, see [20, 4.7, Corollary (f)]. A topological space X is strongly zero-dimensional if and only if $\beta_0 X$ is homeomorphic to βX, i.e., if and only if βX is zero-dimensional (see [2]).

Dowker has given an example of a zero-dimensional space X for which βX is not zero-dimensional and hence $\beta X \neq \beta_0 X$, see [20, Exercise 4V]. The structure of $\beta_0 X$ is related to the clopen ultrafilters defined on X. Indeed $\beta_0 X$ is homeomorphic to the set of all clopen ultrafilters equipped with the Stone topology, see [20, 4.7]. An outline for recovering $\upsilon_0 X$ as a subspace of all clopen ultrafilters on X which have the countable intersection property can be found in [20, Exercise 5E]. Therefore we have $X \subseteq \upsilon_0 X \subseteq \beta_0 X$. Note that for $p \in \beta_0 X \setminus \upsilon_0 X$, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen neighborhoods of p in $\beta_0 X$ such that $\bigcap_{n=1}^{\infty} V_n$ does not meet X.

This article consists of two parts. In the first part, by applying the aforementioned notions, we characterize some important ideals in $C_c(X)$. It is shown that the set of all functions in $C_c(X)$ with compact (resp., pseudocompact) support coincides with the ideal $O_c^{\beta_0 X \setminus X}$ (resp., $M_c^{\beta_0 X \setminus \upsilon_0 X}$). It is shown that for an $\mathbb{N}$-compact space X, the intersection of all free maximal ideals in $C_c(X)$ coincides with the ideal $C_c^K(X)$.

In the second part, we apply the subring $C_c(X)$ and some related notions to find some information about $\beta_0 X$. For example, we give the least cardinality of the remainder $\beta_0 X \setminus \upsilon_0 X$, i.e., the set of all clopen ultrafilters on X which do not have the countable intersection property. It is shown that for a non pseudocompact space X, there are at least $2^{2^{\aleph_0}}$ such clopen ultrafilters on X. We shall show that if X is locally compact and $\mathbb{N}$-compact, then $\beta_0 X \setminus X$ is an almost P-space, i.e., a space for which the interior of every zero-set is nonempty, see [10] and [21]. We show that zero-sets of $\beta_0 X$ which do not meet X are Parovičenko spaces. Finally, we show that whenever X is a locally compact, zero-dimensional space which is not pseudocompact, the cellularity of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$ of $\beta_0 X$ are at least $2^{\aleph_0}$. We close this section with the following results which are useful in the sequel. We recall that if I is a subset of $C_c(X)$, the set $Z[I]$ consists of all zero-sets Z of X for which there exists $f \in I$ such that $Z = Z(f)$. We denote $Z[C_c(X)]$ briefly by $Z_c[X]$.

Lemma 1.3. (a) *For a sequence $\{U_n : n \in \mathbb{N}\}$ of clopen subsets of X, there exists a $Z \in Z_c[X]$ such that $Z = \bigcap_{n=1}^{\infty} U_n$.*

(b) *For $Z \in Z_c[X]$, there exists a sequence $\{W_n : n \in \mathbb{N}\}$ of clopen subsets of X such that $Z = \bigcap_{n=1}^{\infty} W_n$.*

Proof: (a) Without loss of generality, we may assume that $U_1 \supseteq U_2 \supseteq U_3 \cdots$ is a decreasing sequence of clopen sets of X. Now define

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{(X \setminus U_n)},$$

where $\chi_{X \setminus U_n}$ is the characteristic function of the clopen set $X \setminus U_n$. It is easy to see that $f(X) \subseteq \{0\} \bigcup \{\frac{1}{2^n} : n \in \mathbb{N} \bigcup \{0\}\}$. Therefore $f \in C_c(X)$ and $Z(f) = \bigcap_{n=1}^{\infty} U_n$.

(b) Suppose that $Z \in Z_c[X]$. Consider $0 < f \in C_c(X)$ such that $Z = Z(f)$. Choose a decreasing sequence $r_1 > r_2 > \cdots > r_n > \cdots$ of real numbers which tends to zero, and for each $n \in \mathbb{N}$, $r_n \notin f(X)$. For each $n \in \mathbb{N}$, define $W_n = f^{-1}([0, r_n))$. Then each W_n is clopen in X, and $Z(f) = \bigcap_{n=1}^{\infty} W_n$. So we are done. □

Corollary 1.4. *Let X be a zero-dimensional space. For $f \in C_c(X)$ there exists $F \in C_c(\beta_0 X)$ such that $Z(f) = Z(F) \bigcap X$.*

Proof: By part (b) of Lemma 1.3, there exists a sequence $\{W_n : n \in \mathbb{N}\}$ of clopen subsets of X such that $Z(f) = \bigcap_{n=1}^{\infty} W_n$. For each $n \in \mathbb{N}$, $\mathrm{cl}_{\beta_0 X} W_n$ is a clopen subset in $\beta_0 X$. Part (a) of Lemma 1.3 implies that there exists an $F \in C_c(\beta_0 X)$ such that $Z(F) = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta_0 X} W_n$. Clearly $Z(f) = Z(F) \bigcap X$. □

2. Characterization of some special ideals in $C_c(X)$

In the beginning of this section we characterize maximal ideals of $C_c(X)$. This characterization leads us to specify the ideal which consists of all functions in $C_c(X)$ with compact (resp., pseudocompact) support. The reader is reminded that in [1] it is claimed that it can be shown that the space of maximal ideals of $C_c(X)$ with the Stone topology (i.e., the structure space of the ring $C_c(X)$) is isomorphic to $\beta_0 X$. We settle the internal characterization of maximal ideals of $C_c(X)$ which has this result as an immediate consequence. In the sequel we prove a counterpart of Gelfand-Kolmogoroff theorem in rings of continuous functions. First, we need a lemma which is essential for characterizing all maximal ideals of the ring $C_c(X)$. We recall that a subspace of a zero-dimensional space is two-embedded in X if each continuous map f of Y into the two-element discrete space $\{0,1\}$ has a continuous extension $F : X \to \{0,1\}$.

Lemma 2.1. *Let X be zero-dimensional. For any two functions $f, g \in C_c(X)$ we have*

$$\mathrm{cl}_{\beta_0 X}\left(Z(f) \cap Z(g)\right) = \mathrm{cl}_{\beta_0 X} Z(f) \cap \mathrm{cl}_{\beta_0 X} Z(g).$$

Proof: Case 1. Suppose that $Z(f) \cap Z(g) = \emptyset$. There exists an $h \in C_c(X)$ such that $Z(f) = Z(h)$ and $Z(g) = h^{-1}(1)$. Choose some $0 < r < 1$ such that $r \notin h(X)$ and put $U = h^{-1}\left((-\infty, r)\right)$. The subset U is clopen in X, $Z(f) \subseteq U$ and $Z(g) \subseteq X \setminus U$. Note that X is two-embedded in $\beta_0 X$. This implies that $\mathrm{cl}_{\beta_0 X} U \cap \mathrm{cl}_{\beta_0 X}(X \setminus U) = \emptyset$ and then $\mathrm{cl}_{\beta_0 X} Z(f) \cap \mathrm{cl}_{\beta_0 X} Z(g) = \emptyset$.

Case 2. Now suppose that $Z(f) \cap Z(g) \neq \emptyset$. Evidently

$$\mathrm{cl}_{\beta_0 X}\left(Z(f) \cap Z(g)\right) \subseteq \mathrm{cl}_{\beta_0 X} Z(f) \cap \mathrm{cl}_{\beta_0 X} Z(g).$$

Let $p \in \mathrm{cl}_{\beta_0 X} Z(f) \cap \mathrm{cl}_{\beta_0 X} Z(g)$. Assume that p does not belong to $\mathrm{cl}_{\beta_0 X}(Z(f) \cap Z(g))$. There exists a clopen set $U \subseteq \beta_0 X$ such that $p \in U$ and $U \cap (\mathrm{cl}_{\beta_0 X}(Z(f) \cap Z(g))) = \emptyset$. The subset $V = U \cap X$ is clopen in X and also is the zero-set of the characteristic function $\chi_{X \setminus V}$, i.e., $Z(\chi_{X \setminus V}) = V$. Then $V \cap Z(f) \cap Z(g) = \emptyset$. By Case 1, we have

$$\mathrm{cl}_{\beta_0 X}(V \cap Z(f)) \cap \mathrm{cl}_{\beta_0 X}(V \cap Z(g)) = \emptyset,$$

and hence p does not belong to at least one of them; say $p \notin \mathrm{cl}_{\beta_0 X}(V \cap Z(f))$. Choose a neighborhood W of p such that $W \cap V \cap Z(f) = \emptyset$ and hence $W \cap U \cap Z(f) = \emptyset$. But $W \cap U$ is a neighborhood of p and must intersect $Z(f)$. This is a contradiction and the proof is complete. □

Now here is the characterization of maximal ideals of $C_c(X)$. The reader will note the resemblance to the Gelfand-Kolmogoroff theorem that describes maximal ideals of function rings $C(X)$.

Theorem 2.2. *The maximal ideals of $C_c(X)$ are in one-to-one correspondence with the points of $\beta_0 X$ and are given by*

$$M_c^p = \{f \in C_c(X) : p \in \mathrm{cl}_{\beta_0 X} Z(f)\},$$

for $p \in \beta_0 X$.

Proof: First we show that for each $p \in \beta_0 X$, M_c^p is a maximal ideal. By Lemma 2.1, one can verify that M_c^p forms an ideal. Suppose, on the contrary, that there exists a maximal ideal M which properly contains M_c^p. Therefore there exists some $f \in M$ such that $p \notin \mathrm{cl}_{\beta_0 X} Z(f)$. Since $\beta_0 X$ is zero-dimensional, there exists a clopen subset U in $\beta_0 X$ such that $p \in U$ and $U \cap \mathrm{cl}_{\beta_0 X} Z(f) = \emptyset$. The set $V = U \cap X$ is clopen in X, and clearly $p \in \mathrm{cl}_{\beta_0 X} V$. Since V is the zero-set of the characteristic function $\chi_{X \setminus V} \in C_c(X)$, we have $\chi_{X\setminus V} \in M_c^p$. Consider the function $g = \chi_{X\setminus V} + f^2$. Obviously $Z(g) = Z(\chi_{X\setminus V}) \cap Z(f) = V \cap Z(f) = \emptyset$. Therefore M contains a unit of $C_c(X)$ which implies that $M = C_c(X)$.

Now we show that each maximal ideal M in $C_c(X)$ has this form. By Lemma 2.1, the set $\{\mathrm{cl}_{\beta_0 X} Z(f) : f \in M\}$ is a family of closed subsets with the finite intersection property, and since $\beta_0 X$ is compact, there exists some $p \in \bigcap_{f \in M} \mathrm{cl}_{\beta_0 X} Z(f)$. Therefore $M \subseteq M_c^p$ and hence $M = M_c^p$. □

Suppose that $p \in \beta_0 X$. We define the set O_c^p as follows.

$$O_c^p = \{f \in C_c(X) : p \in \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)\}.$$

It is easy to show that, in exact analogy with the $C(X)$ case, O_c^p is an ideal of $C_c(X)$, and the only maximal ideal containing O_c^p is M_c^p. In the following, we show that for $p \in \beta_0 X$, if $Z \in Z[O_c^p]$, there exists a zero-set neighborhood Z' of p in $\beta_0 X$ such that $Z = Z' \cap X$. Note that if X is not strongly zero-dimensional, there exists a bounded real valued function on X which has no extension to $\beta_0 X$. But for any zero-dimensional space, the following proposition allows us to extend zero-sets of the functions in O_c^p.

Proposition 2.3. *For each $p \in \beta_0 X$,*

$$Z[O_c^p] = \{Z' \cap X : Z' \in Z_c[\beta_0 X], p \in \mathrm{int}_{\beta_0 X} Z'\}.$$

Proof: Suppose that $Z \in Z[O_c^p]$, i.e., $p \in \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z$. By Corollary 1.4, there exists some $Z' \in Z_c[\beta_0 X]$ such that $Z' \cap X = Z$. Clearly $p \in \mathrm{int}_{\beta_0 X} Z'$. Hence Z belongs to the right hand side set. Now let $Z' \in Z_c[\beta_0 X]$ and $p \in \mathrm{int}_{\beta_0 X} Z'$. There exists a clopen set $U \subseteq \beta_0 X$ such that $p \in U \subseteq \mathrm{int}_{\beta_0 X} Z'$. Therefore $U \cap X \subseteq Z' \cap X$ and hence $p \in \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X}(Z' \cap X)$. This implies that $Z' \cap X \in Z[O_c^p]$. □

We recall that for a subset $A \subseteq \beta_0 X$, the set O_c^A (resp., M_c^A) is equal to $\bigcap_{p \in A} O_c^p$ (resp., $\bigcap_{p \in A} M_c^p$). The set of all functions in $C_c(X)$ with compact support is denoted by $C_c^K(X)$. In the following result we characterize the set $C_c^K(X)$ as a certain ideal of $C_c(X)$.

Theorem 2.4. *For a zero-dimensional space X, $C_c^K(X) = O_c^{\beta_0 X\setminus X}$.*

Proof: Suppose that $f \in C_c^K(X)$. Then $\mathrm{cl}_X(X \setminus Z(f))$ is compact and hence $\mathrm{cl}_{\beta_0 X}(X \setminus Z(f)) \subseteq X$. By Corollary 1.4, there exists some $Z' \in Z_c[\beta_0 X]$ such that $Z(f) = Z' \cap X$. Therefore $X \cap (\beta_0 X \setminus Z') = X \setminus Z(f)$. Since X is dense and $\beta_0 X \setminus Z'$ is open in $\beta_0 X$, $\mathrm{cl}_{\beta_0 X}(\beta_0 X \setminus Z') \subseteq X$. Thus

$$\beta_0 X \setminus X \subseteq \beta_0 X \setminus \mathrm{cl}_{\beta_0 X}(\beta_0 X \setminus Z') = \mathrm{int}_{\beta_0 X} Z'.$$

Hence by Proposition 2.3, for all $p \in \beta_0 X \setminus X$, $Z' \cap X \in Z_c[O_c^p]$ and therefore $f \in O_c^{\beta_0 X\setminus X}$. For the reverse inclusion, suppose that $f \in O_c^{\beta_0 X\setminus X}$. Then $\beta_0 X \setminus X \subseteq \mathrm{int}_{\beta_o X}\mathrm{cl}_{\beta_0 X} Z(f)$. By Proposition 2.3, there exists some $Z' \in Z_c[\beta_0 X]$ such that $Z' \cap X = Z(f)$ and $\beta_0 X \setminus X \subseteq \mathrm{int}_{\beta_0 X} Z'$. This implies that $\beta_0 X \setminus \mathrm{int}_{\beta_0 X} Z' \subseteq X$. Since

$$\beta_0 X \setminus \mathrm{int}_{\beta_0 X} Z' = \mathrm{cl}_{\beta_0 X}(\beta_0 X \setminus Z') \subseteq X$$

and $X \setminus Z(f) \subseteq \beta_0 X \setminus Z'$, clearly $\mathrm{cl}_X(X \setminus Z(f))$ is compact and hence $f \in C_c^K(X)$. □

The following two results are important for the rest of this section. Recall that a subset $S \subseteq X$ is C_c-embedded in X if for each $f \in C_c(S)$, there exists some $F \in C_c(X)$ such that $F|_X = f$.

Proposition 2.5. *If $S \subseteq X$ is C_c-embedded in X, then it is separated by a clopen partition from every $Z \in Z_c[X]$ disjoint from it.*

Proof: Suppose that $h \in C_c(X)$ and $S \cap Z(h) = \emptyset$. Define $f(s) = \frac{1}{h(s)}$ for all $s \in S$. Let $F \in C_c(X)$ be such that $F|_S = f$. Put $k = hF$. Clearly $k \in C_c(X)$ and $k|_S = 1$ and $k|_{Z(h)} = 0$. It is enough to choose some $0 < r < 1$ such that $r \notin k(X)$. Then $U = k^{-1}((-\infty, r))$ is a clopen subset of X such that $Z(h) \subseteq U$ and $S \subseteq X \setminus U$. □

In the sequel, we investigate conditions under which non-trivial C_c-embedded subsets exist. Note that C-embedded subsets have no benefits for our purpose. Indeed, we know nothing about the cardinality of the image of the extension of a function with countable image. The following lemma gives us a condition under which a C_c- embedded subset exists in a zero-dimensional space.

Lemma 2.6. *Let X be zero-dimensional and suppose $f \in C_c(X)$ carries a subset $S \subseteq X$ homeomorphically to a closed subset $f(S) \subseteq f(X)$. Then S is C_c-embedded in X.*

Proof: Suppose that $f|_S : S \to f(S)$ is a homeomorphism. Therefore $f^{-1} : f(S) \to S$ is continuous. Consider some $g \in C_c(S)$. We observe that the composite $g \circ f^{-1}$ belongs to $C_c(f(S))$. The set $f(X)$ is countable and therefore normal. Hence $g \circ f^{-1}$ has an extension G to $f(X)$. It is obvious that

$G \in C_c(f(X))$. The composite $G \circ f$ belongs to $C_c(X)$. For $s \in S$, we have $G \circ f(s) = G(f(s)) = g \circ f^{-1}(f(s)) = g(s)$. Therefore $G \circ f$ is the extension of g to X which has a countable image. □

The foregoing lemma leads us to the following corollary.

Corollary 2.7. *Let X be a zero-dimensional space and $E \subseteq X$. Suppose the function $h \in C_c(X)$ is unbounded on E. Then E contains a closed copy of $\mathbb{N}$, which is C_c-embedded in X and h approaches infinity on E.*

In what follows, we adapt the original approach of Mandelker in [11] and [12] to functionally countable subalgebras for characterizing the subset consisting of all functions in $C_c(X)$ with pseudocompact support. We denote by $C_c^{\psi}(X)$, the set of all functions in $C_c(X)$ with pseudocompact support. Recall that a subset $S \subseteq X$ is relatively pseudocompact with respect to $C_c(X)$, if for each $f \in C_c(X)$, the function $f|_S$ is bounded. We have the following equivalence for relatively pseudocompact subsets with respect to $C_c(X)$.

Proposition 2.8. *Let X be a zero-dimensional space. A subset $A \subseteq X$ is relatively pseudocompact with respect to $C_c(X)$ if and only if $\mathrm{cl}_{\beta_0 X} A \subseteq \upsilon_0 X$.*

Proof: For the necessity, suppose that $\mathrm{cl}_{\beta_0 X} A \cap (\beta_0 X \setminus \upsilon_0 X) \neq \emptyset$ and choose some $p \in \mathrm{cl}_{\beta_0 X} A \cap (\beta_0 X \setminus \upsilon_0 X)$. There exists a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen neighborhoods of p in $\beta_0 X$ such that $\upsilon_0 X \cap (\bigcap_{n=1}^{\infty} V_n) = \emptyset$. By part (a) of Lemma 1.3, there exists some $F \in C_c(\beta_0 X)$ such that $Z(F) = \bigcap_{n=1}^{\infty} V_n$. Put $f = F|_X$ and define $h = \frac{1}{f}$. Clearly $h \in C_c(X)$ and since p is a limit point of A in $\beta_0 X$, h is unbounded on A, which is a contradiction. For the sufficiency, consider $f \in C_c(X)$. The extension $f^{\upsilon_0} \in C_c(\upsilon_0 X)$ is bounded on the compact subset $\mathrm{cl}_{\beta_0 X} A$. Therefore f is bounded on A. □

For continuing our investigation, we need a proposition which is due to Pierce. The reader can find it in [19, Lemma 1.9.3].

Proposition 2.9. *A zero-dimensional space X is not pseudocompact if and only if there exists a continuous and onto map $f : X \to \mathbb{N}$.*

Theorem 2.10. *If $f \in C_c(X)$ and $X \setminus Z(f)$ is relatively pseudocompact with respect to $C_c(X)$, then $X \setminus Z(f)$ is pseudocompact.*

Proof: Suppose on the contrary that $S = \mathrm{cl}_X(X \setminus Z(f))$ is not pseudocompact. By Proposition 2.9, there exists some $h \in C_c(S)$ with $h \geq 1$ that is unbounded on $X \setminus Z(f)$. By Corollary 2.7, there exists a countable discrete subset $D \subseteq X \setminus Z(f)$ such that D is C_c-embedded in S and h is unbounded on D. Proposition 2.5 implies that there exists a clopen set $O \subseteq S$ such that $D \subseteq O$ and $Z(f) \cap S \subseteq S \setminus O$. Define the function k as follows,

$$k(x) = \begin{cases} \frac{1}{h(s)} \vee \chi_{S\setminus O}(s), & s \in S \\ 1, & s \in \mathrm{cl}_X\,(X \setminus S)\,. \end{cases}$$

Since $\mathrm{cl}_X\,(X \setminus S) = \mathrm{cl}_X \mathrm{int}_X Z(f)$, the function k is well-defined. For, if

$$x \in S \cap \mathrm{cl}_X\,(X \setminus S)\,,$$

then $x \in Z(f) \cap S$ and hence $\chi_{S\setminus O}(x) = 1$. Therefore $\frac{1}{h(x)} \vee \chi_{S\setminus O}(x) = 1$. The pasting lemma implies that k is continuous on X. Clearly $k \in C_c(X)$ and $k > 0$. Hence $\frac{1}{k}$ is unbounded on D. Thus S is not relatively pseudocompact with respect to $C_c(X)$, a contradiction. □

Theorem 2.11. *Let X be a zero-dimensional space and $f \in C_c(X)$. If $\beta_0 X \setminus \upsilon_0 X \subseteq \mathrm{cl}_{\beta_0 X} Z(f)$, then $\beta_0 X \setminus \upsilon_0 X \subseteq \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)$.*

Proof: Suppose that $p \in \beta_0 X \setminus \upsilon_0 X$. There exists a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen neighborhoods of p in $\beta_0 X$ such that $\upsilon_0 X \cap (\bigcap_{n=1}^{\infty} V_n) = \emptyset$. By part (a) of Lemma 1.3, there exists some $h \in C_c(\beta_0 X)$ such that $Z(h) = \bigcap_{n=1}^{\infty} V_n$. Put $T = \beta_0 X \setminus Z(h)$. Consider the function $\frac{1}{h}$ on T. If $Z(h)$ meets $\mathrm{cl}_{\beta_0 X}(X \setminus Z(f))$, then $\frac{1}{h}$ which is continuous on T is unbounded on $X \setminus Z(f)$. Hence by Corollary 2.7, there exists a countable closed set $S \subseteq X \setminus Z(f)$ which is C_c-embedded in T. Thus S is C_c-embedded in X and by Proposition 2.5, there exists a clopen set $O \subseteq X$ such that $S \subseteq O$ and $Z(f) \subseteq X \setminus O$. Therefore $\mathrm{cl}_{\beta_0 X} S \cap \mathrm{cl}_{\beta_0 X} Z(f) = \emptyset$. Since S is closed in T and also is noncompact, there exists some $p \in \mathrm{cl}_{\beta_0 X} S \setminus T$. This implies that $p \in Z(h)$ and $p \notin \mathrm{cl}_{\beta_0 X} Z(f)$ which is a contradiction. Hence $Z(h) \cap \mathrm{cl}_{\beta_0 X}(X \setminus Z(f)) = \emptyset$ and so

$$Z(h) \subseteq \beta_0 X \setminus \mathrm{cl}_{\beta_0 X}(X \setminus Z(f)) \subseteq \mathrm{cl}_{\beta_0 X} Z(f).$$

Therefore $\mathrm{cl}_{\beta_0 X} Z(f)$ is a neighborhood of p. □

Now we are ready to characterize the set $C_c^{\psi}(X)$ as an ideal of $C_c(X)$.

Theorem 2.12. *Let X be a zero-dimensional space. Then*

$$C_c^{\psi}(X) = M_c^{\beta_0 X \setminus \upsilon_0 X} = O_c^{\beta_0 X \setminus \upsilon_0 X}.$$

Proof: By Theorem 2.11, the second equality is clear. Assume that $f \in C_c^{\psi}(X)$. Since $S(f) = X \setminus Z(f)$ is pseudocompact and $(\upsilon_0 X \setminus Z(f^{\upsilon_0})) \cap X = X \setminus Z(f)$, the subset $\upsilon_0 X \setminus Z(f^{\upsilon_0})$ is also pseudocompact. Note that $S(f^{\upsilon_0}) = \upsilon_0 X \setminus Z(f^{\upsilon_0})$ is a cozero-set of the real compact space $\upsilon_0 X$, and hence it is realcompact, see [6, Corollary 8.14]. Therefore $S(f^{\upsilon_0})$ must be compact. We observe that

$$\beta_0 X = \mathrm{cl}_{\beta_0 X} Z(f) \cup \mathrm{cl}_{\beta_0 X} S(f) = \mathrm{cl}_{\beta_0 X} Z(f) \cup S(f^{\upsilon_0}),$$

and hence we have $\beta_0 X \setminus \upsilon_0 X \subseteq \mathrm{cl}_{\beta_0 X} Z(f)$. Thus $f \in M_c^{\beta_0 X \setminus \upsilon_0 X}$. Now if $f \in O_c^{\beta_0 X \setminus \upsilon_0 X}$, there exists a compact set K such that

$$\beta_0 X \setminus \upsilon_0 X \subseteq \beta_0 X \setminus K \subseteq \mathrm{cl}_{\beta_0 X} Z(f),$$

and hence

$$X \setminus Z(f) \subseteq \beta_0 X \setminus \mathrm{cl}_{\beta_0 X} Z(f) \subseteq K \subseteq \upsilon_0 X.$$

Since $X \setminus Z(f)$ is relatively pseudocompact with respect to $C_c(X)$, by Theorem 2.10, the set $X \setminus Z(f)$ is pseudocompact. $\square$

For an $\mathbb{N}$-compact space, Theorems 2.4 and 2.12 imply the following corollary which says that in an $\mathbb{N}$-compact space, $C_c^K(X)$ is equal to the intersection of all free maximal ideals in $C_c(X)$.

Corollary 2.13. *If X is an $\mathbb{N}$-compact space, then $C_c^K(X) = M_c^{\beta_0 X \setminus X}$.*

3. Remainder of the Banaschewski compactification via $C_c(X)$

In this section, first we want to find the least cardinality of the remainder $\beta_0 X \setminus \upsilon_0 X$. In other words, we show that for a zero-dimensional non-pseudocompact space X, we have at least $2^{2^{\aleph_0}}$ clopen ultrafilters on X which do not have the countable intersection property. In the rest of this section we observe that the remainder of the Banaschewski compactification of an $\mathbb{N}$-compact space is an almost P-space, and a connection between the remainder and Parovičenko spaces is given. Finally, for a zero-dimensional, locally compact space X which is not pseudocompact, we give a lower bound for the cellularity of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$ of $\beta_0 X$. The following two lemmas are needed in the sequel. The second one is a consequence of Proposition 2.9.

Lemma 3.1. *Each C_c-embedded subset S of X is two-embedded.*

Proof: Let $f : S \to \{0,1\}$ be a continuous two-valued function, then $f \in C_c(X)$. Therefore there exists some $G \in C_c(X)$ such that $G|_S = f$. The image of G is countable and there exists a real number $0 < r < 1$ such that $r \notin G(X)$. The subset $U = G^{-1}((\infty, r))$ is clopen in X and $f^{-1}(0) \subseteq U$ and $f^{-1}(1) \subseteq X \setminus U$. Define $F : X \to \{0,1\}$ to be 0 on U and 1 on $X \setminus U$. The function F is continuous and two-valued, and its restriction to S is f. $\square$

Lemma 3.2. *Let X be a zero-dimensional space. Then X is pseudocompact if and only if $\beta_0 X = \upsilon_0 X$.*

Proof: For the necessity, suppose that X is pseudocompact. If $\beta_0 X \setminus \upsilon_0 X \neq \emptyset$, for any $p \in \beta_0 X \setminus \upsilon_0 X$, there exists a countable set consisting of clopen neighborhoods of p, say $\{U_n : n \in \mathbb{N}\}$, in $\beta_0 X$ with $\upsilon_0 X \cap (\bigcap_{n=1}^{\infty} U_n) = \emptyset$. By part (a) of Lemma 1.3, there exists a function $f \in C_c(\beta_0 X)$ such that $Z(f) = \bigcap_{n=1}^{\infty} U_n$. If we restrict f to X, then for the function $g = f|_X \in C_c(X)$, we have $Z(g) = \emptyset$. Hence g is a unit of $C_c(X)$. The function $\frac{1}{g}$ belongs to $C_c(X)$ and clearly g is unbounded on X, which is a contradiction.

For the sufficiency, assume that $\beta_0 X = \upsilon_0 X$. If X is not pseudocompact, then by Proposition 2.9, there exists a continuous and onto map $f : X \to \mathbb{N}$. The function f has countable image and therefore has an extension to $\upsilon_0 X$. But $\upsilon_0 X$ is compact and the extension of f is unbounded which is a contradiction. This completes the proof. $\square$

Now we are ready to present one of our main theorems in this section. This result is notable whenever we deal with zero-dimensional spaces which are not strongly zero-dimensional.

Theorem 3.3. *For a zero-dimensional space X, let $f \in C_c(\beta_0 X)$ and $Z(f) \cap X = \emptyset$. Then $Z(f)$ contains a copy of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, and therefore its cardinality is at least $2^{2^{\aleph_0}}$.*

Proof: Denote the restriction of f to X by g. So $Z(g) = \emptyset$ and g belongs to $C_c(X)$. This function has an inverse in $C_c(X)$, say $\frac{1}{g}$. Obviously $\frac{1}{g}$ is unbounded on X. Hence by Corollary 2.7, there exists a C_c-embedded copy of $\mathbb{N}$ in X on which the function $\frac{1}{g}$ tends to infinity. Since $\mathbb{N}$ is C_c-embedded in X, by Lemma 3.1, it is also two-embedded in X and therefore two-embedded in $\beta_0 X$. Thus every continuous two-valued function on $\mathbb{N}$ is two-embedded in $\mathrm{cl}_{\beta_0 X}\mathbb{N}$. By the uniqueness theorem in the existence of the Banaschewski compactification, $\mathrm{cl}_{\beta_0 X}\mathbb{N}$ and $\beta_0\mathbb{N}$ are homeomorphic, and also we have $\mathrm{cl}_{\beta_0 X}\mathbb{N} \setminus \mathbb{N} \subseteq Z(f)$. The discrete space $\mathbb{N}$ is strongly zero-dimensional and hence $\beta_0\mathbb{N} = \mathrm{cl}_{\beta_0 X}\mathbb{N} = \beta\mathbb{N}$. The cardinality of $\beta\mathbb{N}$ is equal to $2^{2^{\aleph_0}}$ and therefore $|Z(f)| \geq 2^{2^{\aleph_0}}$. □

Since each G_δ-point is a zero-set, Lemma 1.3 and Theorem 3.3 imply the following corollary.

Corollary 3.4. *For a zero-dimensional space X, no point $p \in \beta_0 X \setminus X$ is a G_δ-point of $\beta_0 X$.*

In the next result, by applying Theorem 3.3, we give the least cardinality of the remainder $\beta_0 X \setminus \upsilon_0 X$, whenever X is a non-pseudocompact zero-dimensional space.

Proposition 3.5. *Let X be a non-pseudocompact zero-dimensional space. The remainder $\beta_0 X \setminus \upsilon_0 X$ has at least $2^{2^{\aleph_0}}$ points.*

Proof: Since X is zero-dimensional and non-pseudocompact, by Proposition 2.9, there exists a continuous and onto function $f : \upsilon_0 X \to \mathbb{N}$. Therefore the function $g = \frac{1}{f}$ is continuous whose image equals to the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. Note that the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is zero-dimensional and compact. Hence by [20, 4.7, Proposition (d)], g has a continuous extension $G : \beta_0 X \to \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. For each $n \in \mathbb{N}$, choose some x_n such that $g(x_n) = \frac{1}{f(x_n)} = \frac{1}{n}$. Each $g^{-1}(\frac{1}{n})$ is clopen in $\upsilon_0 X$ and hence the cluster points of the set $\{x_n : n \in \mathbb{N}\}$ is contained in $\beta_0 X \setminus \upsilon_0 X$. If we consider a cluster point p of the set $\{x_n : n \in \mathbb{N}\}$, then we observe that $G(p) = 0$. Therefore $G \in C_c(\beta_0 X)$ and $Z(G) \neq \emptyset$. Since $Z(G) \cap \upsilon_0 X = \emptyset$, Theorem 3.3 implies that $|Z(G)| \geq 2^{2^{\aleph_0}}$. Thus the cardinality of $\beta_0 X \setminus \upsilon_0 X$ is at least $2^{2^{\aleph_0}}$. □

We recall that the character of the space X at a point p, denoted $\chi(p, X)$ or $\chi(p)$, is the least cardinal equal to the cardinal number of a (filter) base for the neighborhoods of p. By Proposition 3.5, for a zero-dimensional non pseudocompact space X, we have the following corollary.

Corollary 3.6. *Let X be a zero-dimensional and non pseudocompact space. Then*

(a) *if $p \in \beta_0 X \setminus \upsilon_0 X$, then $\{p\}$ is not a zero-set of $\beta_0 X \setminus \upsilon_0 X$;*
(b) *$\beta_0 X \setminus \upsilon_0 X$ has no isolated point;*
(a) *the character of $\beta_0 X \setminus \upsilon_0 X$ at each of its points is uncountable.*

PROOF: (a) If p is a zero-set of $\beta_0 X \setminus \upsilon_0 X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of clopen neighborhoods of p in $\beta_0 X$ such that $\{p\} = (\beta_0 X \setminus \upsilon_0 X) \cap (\bigcap_{n=1}^{\infty} U_n)$. Also there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen neighborhoods of p in $\beta_0 X$ such that $\upsilon_0 X \cap (\bigcap_{n=1}^{\infty} V_n) = \emptyset$. Hence $\{p\} = (\bigcap_{n=1}^{\infty} U_n) \cap (\bigcap_{n=1}^{\infty} V_n)$. Part (a) of Lemma 1.3 implies that there exists $f \in C_c(\beta_0 X)$ such that $\{p\} = Z(f)$. But this contradicts Proposition 3.5.

(b) Since each isolated point is a zero-set, clearly part (a) implies part (b).

(c) If for some p the character at p is countable, then p is a G_δ-point and hence a zero-set of $\beta_0 X \setminus \upsilon_0 X$, which contradicts part (a). □

We recall that a topological space is scattered if each of its nonempty subsets has an isolated point with the relative topology. Lemma 3.2 together with part (b) of Corollary 3.6 imply the following corollary.

Corollary 3.7. *For a zero-dimensional space X, if $\beta_0 X$ is scattered then X is pseudocompact.*

In the following, we observe some results about connections between the Banaschewski compactification and almost P-spaces. We recall that a topological space X is an almost P-space if each nonempty G_δ-subset of X has a nonempty interior, see [10] and [21].

Proposition 3.8. *$\beta_0 X$ is an almost P-space if and only if X is a pseudocompact almost P-space.*

PROOF: For the necessity, suppose that $\beta_0 X$ is an almost P-space. If X is not pseudocompact, then for each $p \in \beta_0 X \setminus \upsilon_0 X$, there exists a countable collection of clopen subsets of X, say $\{U_n : n \in \mathbb{N}\}$, such that $\bigcap_{n=1}^{\infty} U_n = \emptyset$ and for each $n \in \mathbb{N}$, $p \in \mathrm{cl}_{\beta_0 X} U_n$. Note that for each $n \in \mathbb{N}$, $\mathrm{cl}_{\beta_0 X} U_n$ is a clopen subset of $\beta_0 X$. Hence $G = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta_0 X} U_n$ is a G_δ-subset in $\beta_0 X$. By our hypothesis, $\mathrm{int}_{\beta_0 X} G \neq \emptyset$. Therefore G intersects X. But $G \cap X = \bigcap_{n=1}^{\infty} U_n = \emptyset$, which is a contradiction and therefore X is pseudocompact. Since X is dense in $\beta_0 X$, by Proposition 2.1 in [10], X is an almost P-space.

For the sufficiency, suppose that X is a pseudocompact almost P-space. Let H be a nonempty G_δ-subset of $\beta_0 X$. There exist a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen subsets of $\beta_0 X$ and $p \in H$ such that $p \in \bigcap_{n=1}^{\infty} V_n \subseteq H$. For each $n \in \mathbb{N}$, $V_n \cap X$ is clopen in X and the collection $\Omega = \{V_n \cap X : n \in \mathbb{N}\}$ has the finite intersection property. Inasmuch as $\beta_0 X = \upsilon_0 X$, Ω is contained in a clopen ultrafilter with the countable intersection property. Hence $\bigcap_{n=1}^{\infty} (V_n \cap X)$ is nonempty and therefore a G_δ-subset of X. So there exists a neighborhood O in $\beta_0 X$ such that $O \cap X \subseteq \bigcap_{n=1}^{\infty} (V_n \cap X)$. By taking closure, we observe that $O \subseteq \bigcap_{n=1}^{\infty} V_n$. Hence H has a nonempty interior. □

In the following theorem, we introduce a class of almost P-spaces in connection with the Banaschewski compactification of locally compact and $\mathbb{N}$-compact spaces.

Theorem 3.9. *Let X be a locally compact and $\mathbb{N}$-compact space. Then $\beta_0 X \setminus X$ is an almost P-space.*

Proof: Let G be a nonempty G_δ-set in $\beta_0 X \setminus X$ and $p \in G$. There exists a sequence of open sets in $\beta_0 X$, say $\{V_n : n \in \mathbb{N}\}$, such that $G = (\beta_0 X \setminus X) \cap (\bigcap_{n=1}^{\infty} V_n)$. Also there exists a sequence of clopen sets $\{U_n : n \in \mathbb{N}\}$ of X such that $\bigcap_{n=1}^{\infty} U_n = \emptyset$ and $p \in \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta_0 X} U_n$. Note that the G_δ-set $\bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta_0 X} U_n$ is a subset of $\beta_0 X \setminus X$. Obviously $H = (\bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta_0 X} U_n) \cap (\bigcap_{n=1}^{\infty} V_n)$ is a G_δ-subset of $\beta_0 X$ and $p \in H \subseteq \beta_0 X \setminus X$. There exists a sequence of clopen subsets of $\beta_0 X$, say $\{O_n : n \in \mathbb{N}\}$, such that $p \in \bigcap_{n=1}^{\infty} O_n \subseteq H$. By part (a) of Lemma 1.3, there exists some $F \in C_c(\beta_0 X)$ such that $Z(F) = \bigcap_{n=1}^{\infty} O_n$. Since X is locally compact, for each $i \in \mathbb{N}$, there exists an open set $W_i \subseteq X$ such that $\mathrm{cl}_X W_i$ is compact and for each $x \in W_i$, $F(x) \leq \frac{1}{i}$. It is enough to show that the set $(\beta_0 X \setminus X) \cap \mathrm{cl}_{\beta_0 X} \left(\bigcup_{i \in \mathbb{N}} W_i\right)$ is a subset of $Z(F)$. Consider some $t \in (\beta_0 X \setminus X) \cap \mathrm{cl}_{\beta_0 X} \left(\bigcup_{i \in \mathbb{N}} W_i\right)$. Each neighborhood P of t in $\beta_0 X$ intersects infinitely many W_i's. To see this, assume that for a neighborhood P of t, there exists some $n \in \mathbb{N}$ such that $P \cap W_m = \emptyset$, for all $m > n$. This implies that $t \in \mathrm{cl}_{\beta_0 X} (\bigcup_{i=1}^{n} W_i)$. Each W_i has a compact closure in X, and hence $\mathrm{cl}_{\beta_0 X} (\bigcup_{i=1}^{n} W_i)$ is a compact subset of X which implies that $t \in X$. This contradicts the choice of t. Since each neighborhood P of t in $\beta_0 X$ intersects infinitely many W_i's, the function F vanishes in t. Hence

$$T = (\beta_0 X \setminus X) \cap \mathrm{cl}_{\beta_0 X} \left(\bigcup_{i \in \mathbb{N}} W_i \right) \subseteq Z(F).$$

It is easy to see that T is nonempty, open in $\beta_0 X \setminus X$ and $T \subseteq G$, which shows that G has a nonempty interior. This completes the proof. □

Using Corollary 3.6 together with Theorem 3.9, we can find a class of spaces whose importance is in Boolean algebras; see for example [4]. We recall that a compact zero-dimensional space X is called a Parovičenko space if it has the following properties:

(a) X has no isolated points;
(b) nonempty G_δ-sets have nonempty interiors;
(c) X is an F-space (i.e., for each $f \in C(X)$, the subsets $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated).

The reader should be warned that some authors include the condition that the weight of the space X is $2^{\aleph_0}$. We apply the definition which has no restriction on the weight of the space, see e.g., [4].

The following proposition gives us a large class of Parovičenko spaces in the Banaschewski compactification of a non pseudocompact zero-dimensional space.

Proposition 3.10. *Let Z be a zero-set of $\beta_0 X$ such that $Z \cap X = \emptyset$. Then Z is a Parovičenko space.*

Proof: Evidently Z is compact and zero-dimensional. Since Z is a G_δ-subset of $\beta_0 X$, $W = \beta_0 X \setminus Z$ is σ-compact and therefore strongly zero-dimensional, see [6, 16.17]. Thus $\beta_0 W = \beta W$. The space W is Lindelöf and hence is $\mathbb{N}$-compact. We observe that $Z = \beta_0 W \setminus W = \beta W \setminus W$. So by Theorem 3.9, Z must be an almost P-space. Also since W is locally compact and σ-compact, by Theorem 14.27 of [6], Z is an F-space. Part (b) of Corollary 3.6, implies that Z contains no isolated point. Hence Z must be a Parovičenko space. $\square$

Let X be a zero-dimensional, locally compact space which is not pseudocompact. We close this section by giving a lower bound for the cellularity of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$ of $\beta_0 X$. We recall that the cellularity of a space Y, denoted by $c(Y)$, is the smallest cardinal number κ for which each pairwise disjoint family of nonempty open sets of Y has κ or fewer members. Also the reader is reminded that for a zero-dimensional space X and a cardinal number κ, a partition of X of cardinality κ is a family $\{U_i : i \in \mathbb{I}\}$ of pairwise disjoint clopen subsets of X whose union is X and $|\mathbb{I}| = \kappa$. Note that by Proposition 2.9, every zero-dimensional space which is not pseudocompact, has a partition of cardinality $\aleph_0$. The following theorem which is due to Tarski, is needed for our purpose, see [8].

Theorem 3.11 (Tarski)**.** *Let E be an infinite set. Then there is a collection $\Re$ of subsets of E such that $|\Re| = |E|^{\aleph_0}$, $|R| = \aleph_0$ for each $R \in \Re$ and the intersection of any two distinct members of $\Re$ is finite.*

Theorem 3.12. *Let X be a zero-dimensional and locally compact space which is not pseudocompact. If X has a partition of cardinality κ, then the cellularity of each of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$ of $\beta_0 X$ are at least $\kappa^{\aleph_0}$.*

Proof: We just prove that the cellularity of $\beta_0 X \setminus \upsilon_0 X$ is at least $\kappa^{\aleph_0}$. The second assertion can be derived similarly. Let $\{U_i : i \in \mathbb{I}\}$ be a partition of X with $|\mathbb{I}| = \kappa$. For each $i \in \mathbb{I}$, choose a nonempty clopen and compact subset $W_i \subseteq U_i$. Note that $\{\mathrm{cl}_{\upsilon_0 X} U_i : i \in \mathbb{I}\}$ is a partition of cardinality κ for $\upsilon_0 X$ and for each $i \in \mathbb{I}$, $\mathrm{cl}_{\upsilon_0 X} W_i = W_i$ is clopen in $\upsilon_0 X$. For each subset $J \subseteq \mathbb{I}$, define

$$A(J) = (\beta_0 X \setminus \upsilon_0 X) \cap \mathrm{cl}_{\beta_0 X}\left(\bigcup_{i\in J} W_i\right).$$

Since for each $J \subseteq \mathbb{I}$, $\{W_i : i \in J\}$ is a locally finite family of clopen subsets of $\upsilon_0 X$, then $\bigcup_{i\in J} W_i$ is clopen in $\upsilon_0 X$ (see [2, Theorem 1.1.11]) and hence $\mathrm{cl}_{\beta_0 X}(\bigcup_{i\in J} W_i)$ is clopen in $\beta_0 X$. This implies that for each subset $J \subseteq \mathbb{I}$, $A(J)$ is clopen in $\beta_0 X \setminus \upsilon_0 X$. It is clear that for each two subsets J_1 and J_2 of $\mathbb{I}$, $A(J_1) \cap A(J_2) = A(J_1 \cap J_2)$. For a subset $J \subseteq \mathbb{I}$, we claim that $A(J) = \emptyset$ if and only if J is finite. For, if $J \subseteq \mathbb{I}$ is finite, then $cl_{\beta_0 X}(\bigcup_{i\in J} W_i) = \bigcup_{i\in J} W_i$ and hence $A(J) = \emptyset$. For the converse, if $J \subseteq \mathbb{I}$ is infinite, then for each $j \in J$, choose some $x_j \in W_j$.

Clearly the subset $B = \{x_j : j \in J\}$ is a closed and discrete subset of $\upsilon_0 X$. Since $\beta_0 X$ is compact, the subset B has a cluster point p in $\beta_0 X \setminus \upsilon_0 X$. This implies that $A(J) \neq \emptyset$.

Now apply Theorem 3.12 to find a collection Υ of $\kappa^{\aleph_0}$ infinite subsets of $\mathbb{I}$ such that any two members of Υ have finite intersection. Define $\mathcal{T} = \{A(J) : J \in \Upsilon\}$. Evidently $\mathcal{T}$ contains $\kappa^{\aleph_0}$ pairwise disjoint clopen subsets of $\beta_0 X \setminus \upsilon_0 X$. This implies that $c(\beta_0 X \setminus \upsilon_0 X) \geq \kappa^{\aleph_0}$. □

By applying Proposition 2.9 and Theorem 3.12, the following corollary is immediate.

Corollary 3.13. *Let X be a zero-dimensional, locally compact space which is not pseudocompact. Then the cellularity of each of the subspaces $\beta_0 X \setminus \upsilon_0 X$ and $\beta_0 X \setminus X$ of $\beta_0 X$ are at least $2^{\aleph_0}$.*

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