Spaces with star countable extent

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Abstract. For a topological property P, we say that a space X is star P if for every open cover $\mathcal{U}$ of the space X there exists $A \subset X$ such that $st(A,\mathcal{U}) = X$. We consider space with star countable extent establishing the relations between the star countable extent property and the properties star Lindelöf and feebly Lindelöf. We describe some classes of spaces in which the star countable extent property is equivalent to either the Lindelöf property or separability. An example is given of a Tychonoff star Lindelöf space with a point countable base which is not star countable.

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1. Spaces with star countable extent

If X is a topological space and $\mathcal{U}$ is a family of subsets of X, then the star of a subset $A \subset X$ with respect to $\mathcal{U}$ is the set

$$st\,(A,\mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}\,.$$

Definition 1. Let P be a topological property. A space X is said to be star P if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subset X$ with the property P, such that $st(A,\mathcal{U}) = X$. The set A will be called a star kernel of the cover $\mathcal{U}$.

The term star P was coined in 2007 [9], but certain star properties have been studied by several authors ([5], [6], [8]).

In this paper we shall concentrate on the property star countable extent. The goal is to develop the concept of star countable extent and study the relationships between this property and others star properties such as that of being star countable and star Lindelöf. In the remainder of this paper, spaces are assumed to be Hausdorff spaces and to have at least two points.

Definition 2. The extent $e(X)$ for a space X is defined as

$$e(X) = \sup\{|M| : M \subset X \text{ is a closed and discrete subspace of } X\}\,.$$

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Hereafter we refer to the star countable extent property as SCE property and we call a space with star countable extent an SCE-space.There are many results about star P spaces which are quite easy to prove. For example, since every dense subset of a space X is a star kernel of every open cover $\mathcal{U}$, then we have the following result.

Proposition 3. *Let P be a topological property. If a space X contains a dense subspace $D \subset X$ with the property P, then X is star P.*

Another result easy to prove has to do with the union of star P spaces.

Proposition 4. *For a cardinal number κ, let P be a topological property that is preserved under unions of size κ. If $X = \bigcup_{\alpha<\kappa} Y_\alpha$ and each Y_α is a star P space, then X is star P.*

In particular, since the countable extent property is preserved under countable unions then the star countable extent property is also preserved under countable unions.

In [1] and [2] the authors offer a very deep and comprehensive study of the star P property for "$P =$ *countable*, σ-*compact* or *Lindelöf*" (see also [12], [13]) and they also deal with the *feebly Lindelöf* concept. For these properties we have the well-known implications:

$$\textit{star countable} \Rightarrow \textit{star } \sigma\textit{-compact} \Rightarrow \textit{star Lindelöf},$$
$$\textit{star Lindelöf} \Rightarrow \textit{feebly Lindelöf}.$$

In general, none of the implications can be reversed ([1]). Other important implications are

$$\textit{separable} \Rightarrow \textit{star countable},$$
$$\textit{Lindelöf} \Rightarrow \textit{countable extent} \Rightarrow \textit{star countable},$$
$$\textit{star countable} \Leftrightarrow \textit{star separable}.$$

With the exception of the last, the reverse implications are not valid.

To begin with the study of the SCE property, let us notice that since every Lindelöf space has countable extent then every star-Lindelöf space is an SCE-space. Now we will see that every SCE-space is a feebly Lindelöf space.

Definition 5. A topological space X is called feebly Lindelöf if every locally finite family of non-empty open sets in X is countable.

The following result is a well-known fact.

Proposition 6. *Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a locally finite family of non-empty open sets in a space X. For each $\alpha < \kappa$ take $x_\alpha \in U_\alpha$. Then $D = \{x_\alpha : \alpha < \kappa\}$ is a closed and discrete subspace of X.*

Recall that a pairwise disjoint collection of non-empty open sets in a space X is called a cellular family. As an easy consequence of Proposition 6 we have the next result.

Corollary 7. *If a space X has an uncountable locally finite cellular family, then X is not an SCE-space.*

This last corollary together with the fact that every locally finite collection of non-empty open sets induces an uncountable locally finite cellular family implies the following result.

Corollary 8. *If X is an SCE-space then X is feebly Lindelöf.*

So far we have that the concept of SCE lies between the classes of star Lindelöf spaces and the feebly Lindelöf spaces. Now we will see examples to show that the reversed implications do not hold.

The following space was presented in [1] as an example of a feebly Lindelöf space but not star Lindelöf. Now we show that this space is also an SCE-space.

Proposition 9. *There is a Tychonoff space which is an SCE-space but not star Lindelöf.*

Proof: Let $S = \{\alpha+1 : \alpha < \omega_1\}$ and let $X = (\omega_1 \times \omega) \cup (S \times \{\omega\})$ be considered as a subspace of $\omega_1 \times (\omega + 1)$. Call $Z = \omega_1 \times \omega$ and notice that Z is dense in X. Since Z is σ-countably compact, it follows that $e(Z) \leq \omega$, and therefore X is an SCE-space.

Now let us check that X is not star Lindelöf. Consider the open cover

$$\mathcal{U} = \{\omega_1 \times \omega\} \cup \{\{\alpha\} \times (\omega + 1) : \alpha \in S\}.$$

If $L \subset X$ is a Lindelöf subspace, then L must be bounded on its first coordinates, i.e., there exists $\alpha_0 < \omega_1$ such that if $(\alpha, n) \in L$ then $\alpha \leq \alpha_0$. As a consequence we have that L is disjoint from $\{\alpha_0 + 1\} \times (\omega + 1)$, thus $(\alpha_0 + 1, \omega) \notin st(L, \mathcal{U})$, which proves that X cannot be star Lindelöf. □

The existence of a dense subset with countable extent in the example given in Proposition 9 suggests that the concepts of star Lindelöf and SCE maintain some distance from each other. However, as we will see, there are several classes of spaces in which these concepts coincide.

The last example also allows us to establish the fact that we cannot change the hypothesis in Proposition 3 related to the existence of a dense subspace with the property P for the existence of a dense star P subspace. At least in the case of P being Lindelöf, according to the space constructed in Proposition 9, we have a non star Lindelöf space with a star Lindelöf dense subspace.

In Proposition 39, an example is given of a feebly Lindelöf Moore space that is not an SCE-space. We have the following result.

Proposition 10. *There is a feebly Lindelöf Tychonoff space which is an SCE-space.*

2. Subspaces of star countable extent spaces

In this section we study what kind of subspaces inherit the property star countable extent.

Example 11. If $X = \psi(\mathcal{A})$ is a Mrówka space for some MAD family $\mathcal{A}$ in ω, then X is an SCE-space because X is separable. However, $\mathcal{A}$ is an uncountable closed discrete subspace; therefore it cannot be an SCE-space. This proves that the SCE property is not necessarily inherited either by closed sets or closed G_δ-sets or zero sets.

Example 12. If X is the one-point compactification of an uncountable discrete space D, then X is an SCE-space but D is not. This proves that the SCE property is not necessarily inherited either by open sets or dense subspaces, nor even by an open dense subspace.

Now we give a positive result.

Proposition 13. *Every open F_σ-subset of an SCE-space is an SCE-space.*

Proof: Let X be an SCE-space and $Y \subset X$ be an open F_σ-subset, say $Y = \bigcup_{n<\omega} F_n$, where each F_n is closed in X. Let $\mathcal{U}$ be an open cover of Y. Being Y an open set in X it follows that every member of $\mathcal{U}$ is open in X. For each $n < \omega$ consider the family $\mathcal{U}_n = \mathcal{U} \cup \{X \setminus F_n\}$. $\mathcal{U}_n$ is an open cover of X. By hypothesis there exists a countable extent kernel $M_n \subset X$ of $\mathcal{U}_n$. Since for every $n < \omega$ we have that $F_n \subset st(M_n \cap Y, \mathcal{U})$, then $Y = st(M, \mathcal{U})$, where $M = \bigcup_{n<\omega}(M_n \cap Y)$.

We claim that $e(M) \leq \omega$. To this end it is enough to show that each $M_n \cap Y$ has countable extent. The set $M_n \cap Y$ is equal to $\bigcup_{m<\omega}(M_n \cap F_m)$. Since F_m is closed in X and $e(M_n) \leq \omega$, we have that $e(M_n \cap F_m) \leq \omega$, and thus $e(M_n \cap Y) \leq \omega$. This proves that Y is star countable extent. $\square$

Corollary 14. *A clopen subspace of an SCE-space is an SCE-space.*

Corollary 15. *A cozero subspace of an SCE-space is an SCE-space.*

Other classes of subspaces to consider are the regular open and the regular closed sets. In both cases the answer is negative.

Example 16. Let L be the one-point Lindelöfication of an uncountable discrete space and let S be a convergent sequence together with its limit. Let X be the quotient space obtained from $L \oplus S$ by identifying the non-isolated points of L and S. Being a continuous image of a Lindelöf space, we have that X is a Lindelöf space. On the other hand, it is easy to check that the isolated points of L form a regular open subset of X which is not an SCE-space. Therefore the SCE property is not inherited by regular open sets.

Proposition 17. *There is a Tychonoff star countable space with a regular closed subset that is not an SCE-space.*

Proof: Recursively we are going to build a MAD family of countable subsets of $\omega \cdot \omega_1$ of size 2^ω.

Let $\mathcal{A}_0$ be any MAD family in $\omega \cdot 1$ with $|\mathcal{A}_0| = 2^\omega$ and $\bigcup \mathcal{A}_0 = \omega \cdot 1$. Assume that for every $\alpha < \beta$ we have constructed a MAD family $\mathcal{A}_\alpha$ in $\omega \cdot \alpha$ such that $\bigcup \mathcal{A}_\alpha = \omega \cdot \alpha$ and if $\delta < \alpha$, then $\mathcal{A}_\delta \subset \mathcal{A}_\alpha$. Since $\bigcup_{\alpha<\beta} \mathcal{A}_\alpha$ is almost disjoint in $\omega \cdot \beta$, we can extend the family into a MAD family $\mathcal{A}_\beta$ in $\omega \cdot \beta$ such that $\bigcup \mathcal{A}_\beta = \omega \cdot \beta$. The fact that for every $\alpha < \omega_1$, $\mathcal{A}_\alpha$ is a MAD family in $\omega \cdot \alpha$ such that $\bigcup \mathcal{A}_\alpha = \omega \cdot \alpha$ implies that if $\alpha < \beta < \omega_1$ then $\mathcal{A}_\beta \setminus \mathcal{A}_\alpha$ is uncountable.

Let $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. The collection $\mathcal{A}$ is an almost disjoint family in $\omega \cdot \omega_1$. We claim that $\mathcal{A}$ is also maximal. Let $A \subset \omega \cdot \omega_1$ with $|A| = \omega$. Take $\alpha < \omega_1$ such that $A \subset \omega \cdot \alpha$. Since $\mathcal{A}_\alpha$ is maximal in $\omega \cdot \alpha$, there exists $B \in \mathcal{A}_\alpha \subset \mathcal{A}$ such that $|A \cap B| = \omega$, which proves that $\mathcal{A}$ is maximal. Notice that since $|\mathcal{A}_0| = 2^\omega$, we have $|\mathcal{A}| = 2^\omega$.

Let $X_1 = \mathcal{A} \cup \omega \cdot \omega_1$ be the Mrówka space determined by $\mathcal{A}$, i.e., each point in $\omega \cdot \omega_1$ is isolated and an open neighborhood of $A \in \mathcal{A}$ takes the form $\{A\} \cup A \setminus F$ where $F \subset A$ is finite. Now we show that X_1 is not an SCE-space.

First notice that if $A \in \mathcal{A}_{\alpha+1} \setminus \mathcal{A}_\alpha$, then $|A \cap \omega \cdot \alpha| < \omega$. If this is not true then, by maximality of $\mathcal{A}_\alpha$, there exists $B \in \mathcal{A}_\alpha \subset \mathcal{A}_{\alpha+1}$ such that $|A \cap B| = \omega$, but this is not possible because $\mathcal{A}_{\alpha+1}$ is almost disjoint. Therefore, the set $\{A\} \cup (A \setminus \omega \cdot \alpha)$ is open in X_1.

For each $A \in \mathcal{A}$, we define $U_A = \{A\} \cup (A \setminus \omega \cdot \alpha)$ if $A \in \mathcal{A}_{\alpha+1} \setminus \mathcal{A}_\alpha$ for some $\alpha < \omega_1$, and define $U_A = \{A\} \cup A$ otherwise. Let

$$\mathcal{U} = \{U_A : A \in \mathcal{A}\} \cup \{\{\alpha\} : \alpha \in \omega \cdot \omega_1\}.$$

Let $N \subset X_1$ be any subspace with $e(N) \leq \omega$. Notice that $N \cap \mathcal{A}$ is countable and since $N \setminus \bigcup_{A \in N \cap \mathcal{A}} U_A \subset \omega \cdot \omega_1$ is closed and discrete in N, we have that N is actually countable. Take $\alpha < \omega_1$ such that $N \cap \omega \cdot \omega_1 \subset \omega \cdot \alpha$. Since $\mathcal{A}_{\alpha+1} \setminus \mathcal{A}_\alpha$ is uncountable, we can take $A \in \mathcal{A}_{\alpha+1} \setminus (\mathcal{A}_\alpha \cup (N \cap \mathcal{A}))$. By construction, $U_A \cap N = \emptyset$, thus $A \notin st(N, \mathcal{U})$. Therefore $\mathcal{U}$ cannot have a star kernel with countable extent.

Now let $\mathcal{A}'$ be a MAD family in ω with $|\mathcal{A}'| = 2^\omega$ and let $X_2 = \mathcal{A}' \cup \omega$ be the Mrówka space associated to $\mathcal{A}'$. Take $\varphi : \mathcal{A} \to \mathcal{A}'$ a bijection and let X be the quotient image of $X_1 \oplus X_2$ by identifying each $A \in \mathcal{A}$ with $\varphi(A) \in \mathcal{A}'$, say

$$X = \omega \cdot \omega_1 \cup \{(A, \varphi(A)) : A \in \mathcal{A}\} \cup \omega.$$

Let $q : X_1 \oplus X_2 \to X$ be the quotient map and call $Z = q[X_1]$. We claim that X is a star countable space and Z is a regular closed subset of X homeomorphic to X_1.

It is easy to check that $Z = cl_X(\omega \cdot \omega_1)$, hence Z is a regular closed subset of X.

Clearly $q\restriction_{X_1} : X_1 \to Z$ is a continuous bijection. To see that $q\restriction_{X_1}$ is also open, take $A \in \mathcal{A}$ and $F \subset A$ finite. Define

$$W = \{(A, \varphi(A))\} \cup (A \setminus F) \cup \varphi(A).$$

Then W is open in X and $q[\{A\} \cup (A \setminus F)] = W \cap Z$. Thus Z is homeomorphic to X_1.

Finally we show that X is star countable. Let $\mathcal{U}$ be an open cover of X. Since $q[X_2]$ is separable, there exists a countable subset $M \subset q[X_2]$ such that $q[X_2] \subset st(M, \mathcal{U})$; in particular $\{(A, \varphi(A)) : A \in \mathcal{A}\} \subset st(M, \mathcal{U})$. Call $N = Z \setminus st(M, \mathcal{U})$

and notice that N is finite. Otherwise we can take $B \subset Z \setminus st(M,\mathcal{U}) \subset \omega \cdot \omega_1$ with $|B| = \omega$, and by maximality of $\mathcal{A}$ there is $A \in \mathcal{A}$ such that $|A \cap B| = \omega$. Since $(A, \varphi(A)) \in st(M,\mathcal{U})$, there exists $F \subset A$ finite such that $A \setminus F \subset st(M,\mathcal{U})$, but this implies that $B \cap st(M,\mathcal{U}) \neq \emptyset$ which is not possible. Therefore $M \cup N$ is the countable kernel of $\mathcal{U}$. Thus X is star countable. □

3. Invariance of the star countable extent property

In what follows we shall see what happens with the images and the inverse images of the SCE-spaces.

The following result appears in [9] as Proposition 32 and has a simple proof.

Proposition 18. *If P is a topological property preserved under continuous images, then the property star P is also preserved under continuous images.*

Since the countable extent is preserved under continuous images we have the following result.

Corollary 19. *The SCE property is preserved under continuous images.*

Recall that a closed function with compact fibers is called a perfect map.

Lemma 20. *The countable extent property is an inverse invariant of perfect mappings.*

Proof: Let $f : X \to Y$ be a perfect map, where Y is a space such that $e(Y) \leq \omega$. Fix a closed and discrete subspace $M \subset X$. First we check that $f[M]$ is discrete in Y. Given $p \in f[M]$, the set $A = M \setminus f^{-1}[\{p\}]$ is closed in X because M is closed and discrete in X. Therefore $Y \setminus f[A]$ is an open set in Y and clearly $(Y \setminus f[A]) \cap f[M] = \{p\}$. With this we have that $f[M]$ is discrete in Y, but also $f[M]$ is closed because f is perfect. Hence $f[M]$ is countable. On the other hand, since f is perfect and M closed, then $f\restriction_M : M \to f[M]$ is perfect. So for every $p \in f[M]$, $(f\restriction_M)^{-1}[\{p\}]$ is compact in M and thus finite. Finally, since $M = \bigcup\{(f\restriction_M)^{-1}[\{p\}] : p \in f[M]\}$, we get that $|M| \leq \omega$. This proves that $e(X) \leq \omega$. □

Proposition 21. *If $f : X \to Y$ is an open perfect surjection and Y is an SCE-space, then X is an SCE-space.*

Proof: Let $\mathcal{U}$ be an open cover of X. For each $y \in Y$ there exists a finite subset $\mathcal{U}_y \subset \mathcal{U}$ such that $f^{-1}[\{y\}] \subset \bigcup\mathcal{U}_y$. We can assume that for every $U \in \mathcal{U}_y$, $U \cap f^{-1}[\{y\}] \neq \emptyset$. Define $V_y = (Y \setminus f[X \setminus \bigcup\mathcal{U}_y]) \cap \bigcap\{f[U] : U \in \mathcal{U}_y\}$. V_y is an open set because $\mathcal{U}_y$ is finite and f is an open-closed map. The condition $U \cap f^{-1}[\{y\}] \neq \emptyset$ together with the fact that $f^{-1}[\{y\}] \subset \bigcup\mathcal{U}_y$ imply that $y \in V_y$. This shows that the family $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover for Y. By hypothesis we can take a kernel $N \subset Y$ of $\mathcal{V}$ with $e(N) \leq \omega$. Call $M = f^{-1}[N]$. Notice that Lemma 20 is valid because $f\restriction_M : M \to N$ is a perfect surjection and hence $e(M) \leq \omega$. Now we show that M is a kernel of $\mathcal{U}$. Given $x \in X$, there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap N \neq \emptyset$. Since $V_y \subset Y \setminus f[X \setminus \bigcup\mathcal{U}_y]$,

$f^{-1}[V_y] \subset \bigcup \mathcal{U}_y$, then $x \in U$ for some $U \in \mathcal{U}_y$. Since $V_y \subset f[U]$ and $V_y \cap N \neq \emptyset$, we have $U \cap M \neq \emptyset$, and thus $x \in st(M, \mathcal{U})$. Therefore X is an SCE-space. □

It is worth mentioning that in Proposition 21 we do not need the continuity of the function. However the hypothesis of f being perfect and open seems to be quite strong. We shall see an example which shows us that we cannot drop the open mapping condition.

The following lemma has a very simple proof and is left to the reader.

Lemma 22. *Let X be a topological space and $A(X) = X \times \{0, 1\}$ the Alexandroff duplicate of X. Then the function $f : A(X) \to X$ defined as $f(x, i) = x$, for $i = 0, 1$, is a continuous perfect surjection.*

The next example was proposed in [6] to exhibit that the star Lindelöf property is not an inverse invariant of perfect mappings. Now we shall see that this same space allows us to show that we cannot drop the hypothesis of open map in Proposition 21.

Example 23. Let S^2 be the product space of the Sorgenfrey line S. Since S^2 is separable, then it is also an SCE-space. By Lemma 22, $f : A(S^2) \to S^2$ defined as $f(p, i) = p$, $i = 0, 1$, is a continuous perfect surjection. Consider the subset $L = \{(x, -x) \in S^2 : x \in S\}$; it is well-known that L is closed and discrete in S^2. For $p \in S^2 \setminus L$ define $U(p) = U \times \{0, 1\}$, where $U \subset S^2$ is an open set such that $p \in U$ and $U \cap L = \emptyset$. Similarly for $p \in L$ define $U(p) = (U \times \{0, 1\}) \setminus \{(p, 1)\}$, where $U \subset S^2$ is an open set such that $p \in U$ and $U \cap L = \{p\}$. In both cases $U(p) \cap (L \times \{1\}) = \emptyset$. This implies in particular that $L \times \{1\}$ is closed in $A(S^2)$, but all the points of $L \times \{1\}$ are isolated. Hence $L \times \{1\}$ is an uncountable clopen discrete subspace of $A(S^2)$, thus is not an SCE-space. Since the SCE property is hereditary in clopen sets, we can conclude that $A(S^2)$ cannot be an SCE-space.

The key argument to show that $A(S^2)$ is not an SCE-space can be generalized in the following result.

Proposition 24. *If $A(X)$ is an SCE-space, then $e(X) \leq \omega$.*

Of course it is natural to ask whether the reverse implication in Proposition 24 is also valid. The answer is affirmative, in fact we can achieve a stronger result.

Proposition 25. *If $e(X) \leq \omega$ then $e(A(X)) \leq \omega$.*

PROOF: The proof is an immediate consequence of the fact that $A(X)$ is a perfect inverse image of X and the countable extent property is a perfect inverse invariant. □

Combining these last two propositions, we see that in the class of spaces which are Alexandroff duplicates of topological spaces, the concepts countable extent and SCE are equivalent (therefore they also coincide with any intermediate concept).

Corollary 26. *For any topological space X, $e(A(X)) \leq \omega$ if and only if $A(X)$ is an SCE-space.*

4. Finite products

In this section we shall see that the SCE property is fragile when we take finite products, even between spaces with a stronger covering property than the SCE.

As a consequence of Proposition 21 we have the following result.

Proposition 27. *If X is an SCE-space and K is compact, then $X \times K$ is an SCE-space.*

From this proposition we obtain a slightly better result.

Corollary 28. *If X is an SCE-space and Y is σ-compact, then $X \times Y$ is an SCE-space.*

Proof: Suppose $Y = \bigcup_{n<\omega} K_n$, where each K_n is a compact subspace. Then $X \times Y = \bigcup_{n<\omega}(X \times K_n)$ and each $X \times K_n$ is an SCE-space. Since the SCE property is preserved under countable unions then $X \times Y$ is an SCE-space. $\square$

Next we present an example that shows us that in Proposition 27 we cannot replace σ-compact by Lindelöf nor by countably compact; moreover, this same example shows us that the product of a countably compact space with a Lindelöf space is not necessarily an SCE-space.

Proposition 29. *There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not an SCE-space.*

Proof: Let $Z = \omega_1 \times L$ where L is the one-point Lindelöfication of ω_1. Let

$$\mathcal{U} = \{(\alpha, \omega_1) \times \{\alpha\} : \alpha < \omega_1\}.$$

It is easy to see that the set $U = \{(\alpha, \beta) \in Z : \beta \geq \alpha\}$ is an open subset of X. Since $\mathcal{V} = \mathcal{U} \cup \{U\}$ is a partition of X in open subsets, then it is also a partition in clopen subsets, so $\mathcal{V}$ is an uncountable discrete family. By Corollary 7, X is not an SCE-space. $\square$

The following example shows us that the SCE property is not preserved if we take the product of two Lindelöf spaces. To construct the example we use the concept of a totally imperfect set [7].

Definition 30. A subset $A \subset \mathbb{R}$ is totally imperfect if A does not contain a copy of the Cantor set.

An important remark is that if $A \subset \mathbb{R}$ is a totally imperfect set and $F \subset \mathbb{R}$ is an uncountable closed set then $F \not\subseteq A$. The next theorem is due to Bernstein, a proof can be found in [7].

Theorem 31 (Bernstein Theorem)**.** *There exists $A \subset \mathbb{R}$ such that $|A| = |\mathbb{R} \setminus A| = 2^\omega$ and both A and $\mathbb{R} \setminus A$ are totally imperfect sets.*

Proposition 32. *There exist two Lindelöf spaces X and Y such that $X \times Y$ is not an SCE-space.*

Proof: Let $A \subset \mathbb{R}$ be as in the Bernstein Theorem. Take the Alexandroff duplicate of $\mathbb{R}$, $A(\mathbb{R}) = \mathbb{R} \times \{0,1\}$. Let

$$X_1 = (A \times \{0\}) \cup (\mathbb{R} \times \{1\}) \text{ and } X_2 = ((\mathbb{R} \setminus A) \times \{0\}) \cup (\mathbb{R} \times \{1\})$$

subspaces of $A(\mathbb{R})$.

Let $\mathcal{B}$ be the collection of open intervals in $\mathbb{R}$ and let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$. The collection $\mathcal{A}_1 = \{U_x \times \{0,1\} \setminus \{(x,1)\} : x \in A \text{ and } U_x \in \mathcal{B}_x\} \cup \{\{(x,1)\} : x \in \mathbb{R}\}$ is a base for X_1; and $\mathcal{A}_2 = \{U_x \times \{0,1\} \setminus \{(x,1)\} : x \in \mathbb{R} \setminus A \text{ and } U_x \in \mathcal{B}_x\} \cup \{\{(x,1)\} : x \in \mathbb{R}\}$ is a base for X_2.

We shall see that X_1 and X_2 are Lindelöf. Let $\mathcal{U}$ be an open cover of X_1 by sets belonging to $\mathcal{A}_1$. Assume that

$$\mathcal{U} = \{(U_x \times \{0,1\}) \setminus \{(x,1)\} : x \in F\} \cup \{\{(x,1)\} : x \in B\}$$

for some subsets $F, B \subset \mathbb{R}$. Notice that $\{U_x : x \in F\}$ is an open cover of A. Being A a subspace of $\mathbb{R}$, there exists a countable subset $F_0 \subset F$ such that $A \subset \bigcup_{x \in F_0} U_x$. Define $U = \bigcup_{x \in F_0} U_x$. Since $\mathbb{R} \setminus U$ is a closed set contained in $\mathbb{R} \setminus A$ and this is a totally imperfect set, it follows that $\mathbb{R} \setminus U$ is countable. For each $x \in F_0 \cup (\mathbb{R} \setminus U)$ fix $W_x \in \mathcal{U}$ such that $(x,1) \in W_x$. Let

$$\mathcal{U}_0 = \{(U_x \times \{0,1\}) \setminus \{(x,1)\} : x \in F_0\} \cup \{W_x : x \in F_0 \cup (\mathbb{R} \setminus U)\}.$$

$\mathcal{U}_0$ is a countable subfamily of $\mathcal{U}$, let us see that it is also a subcover. By the choice of F_0 we only need to show that $\mathbb{R} \times \{1\}$ is covered by $\mathcal{U}_0$. Take $(t,1) \in \mathbb{R} \times \{1\}$ and assume that $t \notin F_0 \cup \mathbb{R} \setminus U$, then $t \in U = \bigcup_{x \in F_0} U_x$, thus there exists $x \in F_0$ such that $t \in U_x$. Since $t \neq x$ we have that $(t,1) \in (U_x \times \{0,1\}) \setminus \{(x,1)\} \in \mathcal{U}_0$. Therefore X_1 is Lindelöf. Similarly X_2 is Lindelöf.

Now we show that $X_1 \times X_2$ is not an SCE-space. Consider the diagonal set $\Delta = \{(p,p) : p \in A(\mathbb{R})\}$ of $A(\mathbb{R}) \times A(\mathbb{R})$. Since Δ is closed in $A(\mathbb{R}) \times A(\mathbb{R})$, we have that $D = \Delta \cap (X_1 \times X_2)$ is closed in $X_1 \times X_2$. Notice that $D = \{(p,p) : p \in \mathbb{R} \times \{1\}\}$; so each element of D is isolated in $X_1 \times X_2$, which gives us that D is actually an uncountable clopen discrete subspace and therefore not an SCE-space. Since the SCE is clopen hereditary, $X_1 \times X_2$ cannot be an SCE-space. □

5. Relations between star P spaces

To finish this study of the SCE property, we shall analyze several classes of spaces in which the SCE property is equivalent to any of the following three properties: separability, Lindelöf or countable extent (remember that each of these implies the SCE property). We will also consider those classes in which SCE is similar to either star countable or star Lindelöf.

In Corollary 26 we already gave a first result on this matter because we have shown that the Alexandroff duplicate of a space has countable extent if and only if it is an SCE-space. Considering that every space is continuous image of its

Alexandroff duplicate, it is clear that in Corollary 26 neither $l(A(X)) \leq \omega$ nor $d(A(X)) \leq \omega$ can be substituted for $e(A(X)) \leq \omega$.

Recall that a space in which every G_δ-set is open is called a P-space. In [2] the following result is shown.

Proposition 33. *If X is a normal P-space, then $e(X) \leq \omega$ if and only if X is feebly Lindelöf.*

A natural question to ask is whether the normality can be weakened to Tychonoff in Proposition 33. In [2] a consistent example of a feebly Lindelöf Tychonoff P-space which is not star Lindelöf is given. We now see that this space is not an SCE-space.

A family $\mathcal{A}$ of subsets of ω_1 is almost disjoint if every element of $\mathcal{A}$ is uncountable and for each pair of distinct sets $A, B \in \mathcal{A}$, $|A \cap B| \leq \omega$. Using a construction similar to that of a Mrówka space, we may define a topology on $\mathcal{A} \cup \omega_1$ by declaring that each $d \in \omega_1$ is isolated and an open neighborhood of $A \in \mathcal{A}$ takes the form $\{A\} \cup A \setminus C$, where $C \subset A$ is countable. It turns out that this space is Hausdorff and 0-dimensional; we will call this space a Mrówka space on ω_1 (determined by $\mathcal{A}$).

The proof of the following lemma is entirely analogous to the proof that a Mrówka Ψ-space is feebly compact and is left to the reader.

Lemma 34. *If $\mathcal{A}$ is an almost disjoint family on ω_1, then the Mrówka space $\mathcal{A} \cup \omega_1$ is a P-space. Moreover, if $\mathcal{A}$ is maximal then $\mathcal{A} \cup \omega_1$ is feebly Lindelöf.*

It is well-known that the existence of a MAD family $\mathcal{A}$ on ω_1 such that $|\mathcal{A}|^\omega = |\mathcal{A}|$ is independent of ZFC.

Proposition 35. *If there exists a MAD family $\mathcal{A}$ on ω_1 such that $|\mathcal{A}|^\omega = |\mathcal{A}|$, then there is a feebly Lindelöf P-space which is not an SCE-space.*

Proof: Let $X = \mathcal{A} \cup \omega_1$ be the Mrówka space related to $\mathcal{A}$, and assume that $\bigcup \mathcal{A} = \omega_1$. By Lemma 34 we already know that X is a feebly Lindelöf P-space.

In [1] it is shown that this space is not star-Lindelöf. Since a subspace M of X has countable extent if and only if M is Lindelöf, we have that X cannot be an SCE-space. □

This space shows us that at least consistently, normal cannot be replaced by Tychonoff in Proposition 33. We have the question open whether it is possible to find in ZFC a Tychonoff feebly Lindelöf P-space which is not an SCE-space.

Another important class of spaces are the Moore spaces. Recall that a development for a space X is a sequence of open covers $\{\mathcal{U}_n : n < \omega\}$ such that for each $x \in X$, the family $\{st(x, \mathcal{U}_n) : n < \omega\}$ is a local base at x. A Moore space is a regular space with a development.

In [2] the following is shown.

Theorem 36. *If X is a Moore space, then X is separable if and only if X is star Lindelöf.*

It is well-known that every Moore space is semistratifiable [4], which in particular implies that every Moore space is a D-space and therefore the Lindelöf degree and the extent of the space match. Thus we have a slight generalization of Theorem 36.

Corollary 37. *If X is a Moore space, then X is separable if and only if X is an SCE-space.*

We may now ask if the following implications are true in the class of Moore spaces:

$$SCE \Rightarrow \textit{countable extent}\,?$$
$$\textit{feebly Lindelöf} \Rightarrow SCE\ ?$$

The answer to both questions is negative. The Niemytzki plane is a separable Moore space with uncountable extent. To show that the second implication is not valid, we use the following lemma. The proof is a simple exercise and is left to the reader.

Lemma 38. *If X is a ccc space, then X is feebly Lindelöf.*

In the following example we employ the Pixley-Roy topology on the hyperspace of all non-empty finite subsets of $\mathbb{R}$ (see [10] and [5]).

Proposition 39. *There is a feebly Lindelöf Moore space which is not an SCE-space.*

Proof: Let $X = [\mathbb{R}]^{<\omega} \setminus \{\emptyset\}$ be endowed with the Pixley-Roy topology. Recall that for each $F \in X$ a canonical open neighborhood of F takes the form

$$[F, U] = \{A \in X : F \subset A \subset U\}$$

where $U \subset \mathbb{R}$ is an Euclidean open set such that $F \subset U$. This construction was first described in [10], and X with this topology is a Moore space.

To see that X is feebly Lindelöf it is enough to show that X is *ccc*. Take $\mathcal{B}$ as any countable base of $\mathbb{R}$ and define

$$\mathcal{B}_0 = \left\{ \bigcup \mathcal{U} : \mathcal{U} \in [\mathcal{B}]^{<\omega} \right\}.$$

Notice that for any open set $[F, U]$, there exists $B \in \mathcal{B}_0$ such that $[F, B] \subset [F, U]$. This is because if for each $x \in F$ we take $V_x \in \mathcal{B}$ such that $x \in V_x \subset U$, then $[F, B] \subset [F, U]$, where $B = \bigcup\{V_x : x \in F\}$.

If $\mathcal{C} = \{[F_\alpha, U_\alpha] : \alpha < \kappa\}$ is a cellular family of basic open sets, then for each $\alpha < \kappa$ take $B_\alpha \in \mathcal{B}_0$ such that $[F_\alpha, B_\alpha] \subset [F_\alpha, U_\alpha]$; the countability of $\mathcal{B}_0$ now implies the countability of $\mathcal{C}$.

Finally we show that X is not an SCE-space. By Theorem 36 it suffices to show that X is not star countable. Let $\mathcal{U} = \{[\{t\}, \mathbb{R}] : t \in \mathbb{R}\}$. Consider any countable subset of X, $Y = \{F_n \in X : n < \omega\}$. Then $A = \bigcup Y \subset \mathbb{R}$ is countable and thus we can take $s \in \mathbb{R} \setminus A$. The only member of $\mathcal{U}$ that contains the point $\{s\} \in X$ is $[\{s\}, \mathbb{R}]$; however, if there is an $F_n \in [\{s\}, \mathbb{R}]$, then $s \in F_n \subset A$, which is impossible. Therefore $\{s\} \notin st(A, \mathcal{U})$ and consequently X cannot be star countable. □

It is worth mentioning that in [2] a first countable feebly Lindelöf space which is not star Lindelöf is given. The example given in Proposition 39, being a Moore space, represents an improvement on the first countable condition. The example given in [2] is due to Shakhmatov, who in [11] constructed a pseudocompact space with a point-countable base and with extent larger than 2^ω; in [2] the authors proved that this space cannot be star Lindelöf. To achieve this, the authors of [2] proved and made use of the following result.

Proposition 40. *If X is a regular star Lindelöf space with a point-countable base, then $l(X) \leq 2^\omega$.*

This proposition, as well as Shakhmatov's space, raises several questions.

(i) Is a pseudocompact Tychonoff space an SCE-space?

(ii) Can the bound $l(X) \leq 2^\omega$ in Proposition 40 be reached?

(iii) In the class of spaces with a point-countable base, is every star Lindelöf space star countable?

As we shall see, the answers to (i) and (iii) are negative while (ii) has a positive answer. To see that the answer to (i) is negative we will show that Shakhmatov's example mentioned above cannot be an SCE-space.

In [3] it was shown that every space with a point-countable base is a D-space, with this in mind we have as an immediate consequence the following result.

Proposition 41. *If X has a point-countable base, then X is star Lindelöf if and only if X is an SCE-space.*

Corollary 42. *There exists a Tychonoff pseudocompact space which is not an SCE-space.*

Proof: Shakhmatov's space [11] is a pseudocompact Tychonoff space with a point-countable base which is not star Lindelöf. By Proposition 41 this space cannot be an SCE-space either. □

The next example simultaneously proves that questions (ii) and (iii) have positive and negative answers, respectively. By responding negatively to (ii) we also solve (in a stronger way) the question posed in [1]: is a first countable star Lindelöf space star countable? Note that the space constructed in Proposition 9 is also related to the question posed in [1], because it is an example of a first countable SCE-space which is not star Lindelöf.

To construct the required space we are going to use again the totally imperfect sets of $\mathbb{R}$.

Example 43. Let $A \subset \mathbb{R}$ be a totally imperfect set as in the Bernstein Theorem. Consider $Y = A \cup (\mathbb{R} \setminus A)$ with the topology generated by the collection

$\mathcal{B} = \{\{x\} : x \in \mathbb{R} \setminus A\} \cup \{(x-\epsilon, x+\epsilon) : x \in A \text{ and } \epsilon > 0\}$ as a base. The space Y endowed with this topology is Lindelöf as we now show: if $\mathcal{U} \subseteq \mathcal{B}$ is a basic open cover, then there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}$ consisting of Euclidean open sets such that $A \subset \bigcup \mathcal{V}$. Hence $Y \setminus \bigcup \mathcal{V}$ is a Euclidean closed set disjoint from A. By the Bernstein Theorem we have that $Y \setminus \bigcup \mathcal{V}$ is countable. Therefore $\mathcal{U}$ has a countable subcover of Y. Let $X = \mathbb{R} \setminus A$ with the subspace topology of $\mathbb{R}$ and let $M = X \times Y$.

Proposition 44. *M has a point-countable base.*

PROOF: Since X and $A \subset Y$ have the Euclidean topology, both are separable. Fix countable dense sets $D_1 \subset X$ and $D_2 \subset A$. For $x \in \mathbb{R}$ and $r > 0$, denote

$$B(x,r) = \{t \in \mathbb{R} : |x - t| < r\}$$

and in case $x \in X$ let $B_X(x,r) = B(x,r) \cap X$. Let

$$\mathcal{B}_1 = \left\{ B_X\left(x, \frac{1}{n}\right) \times B\left(a, \frac{1}{n}\right) : x \in D_1, a \in D_2, n \in \mathbb{N} \right\}$$

and

$$\mathcal{B}_2 = \left\{ B_X\left(x, \frac{1}{n}\right) \times \{y\} : x \in D_1, y \in Y \setminus A, n \in \mathbb{N} \right\},$$

then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a base for M. Let $(x,t) \in M$. If $t \in A$, then (x,t) does not belong to any element of $\mathcal{B}_2$. Since $\mathcal{B}_1$ is countable it follows that $\mathcal{B}$ is point countable at (x,t). On the other hand if $t \in Y \setminus A$, then (x,t) belongs to elements of $\mathcal{B}_2$ only of the form $B_X(a, \frac{1}{n}) \times \{t\}$ (assuming that $x \in B_X(a, \frac{1}{n})$). Using again that $\mathcal{B}_1$ is countable we have that $\mathcal{B}$ is also point-countable at (x,t). $\square$

Proposition 45. *M is not star countable.*

PROOF: Let $D = \{(x,x) : x \in \mathbb{R} \setminus A\}$. Since for every $x \in \mathbb{R} \setminus A$

$$D \cap (X \times \{x\}) = \{(x,x)\},$$

D is discrete. On the other hand, if $(x,t) \in M \setminus D$, we can take $r = \frac{1}{2}|x - t| > 0$ and it follows that

$$(B_X(x,r) \times B(t,r)) \cap D = \emptyset.$$

Hence D is also closed.

Consider the open cover $\mathcal{U} = \{M \setminus D\} \cup \{X \times \{y\} : y \in Y \setminus A\}$. $\mathcal{U}$ witnesses that M cannot be star countable. If $L \subset M$ is any kernel of $\mathcal{U}$, then L must be such that for every $y \in Y \setminus A$, $L \cap (X \times \{y\}) \neq \emptyset$. This is because $X \times \{y\}$ is the only member of $\mathcal{U}$ that contains the point (y,y). Since $\{X \times \{y\} : y \in Y \setminus A\}$ is a cellular family, it follows that $|L| = 2^\omega$. Therefore M is not star countable. $\square$

The open cover considered in Proposition 45 also witnesses that the Lindelöf degree of M is 2^ω. Once we demonstrate that M is star Lindelöf we will have shown that question (ii) indeed has a positive answer.

Proposition 46. *M is star Lindelöf.*

PROOF: First notice that for each $p \in X$, the space $\{p\} \times Y$ is Lindelöf, since this subspace of M is homeomorphic to Y. Thus if $D = \{d_n : n < \omega\}$ is a dense subset of X, then $\bigcup\{\{d_n\} \times Y : n < \omega\}$ is a dense Lindelöf subspace of M, therefore M is star-Lindelöf. □

Corollary 47. *There exists a Tychonoff space with a point-countable base, star Lindelöf and with Lindelöf degree equal to 2^ω which is not star countable.*

We have already mentioned that every Moore space is semistratifiable (and thus a D-space). We need to check if Theorem 36 is also valid in semistratifiable spaces. Knowing that every space with a point countable base is a D-space and since the space M in Example 43 is star Lindelöf and has a point-countable base but is not star countable, we have that Theorem 36 is no longer valid for D-spaces in general. For the semistratifiable case the authors do not know the answer. However, at least in this class of spaces the properties SCE and star countable are equivalent. This is actually a consequence of the following theorem (see [4]).

Theorem 48. *Let X be a semistratifiable space. The following are equivalent:*

(i) *X is Lindelöf;*
(ii) *X is hereditarily separable;*
(iii) *X has countable extent.*

Corollary 49. *If X is semistratifiable, then X is star countable if and only if X is an SCE-space.*

PROOF: Assume that X is an SCE-space. The property of being semistratifiable is hereditary, thus every subspace $M \subset X$ with countable extent is separable and therefore X is star separable. But the star separable property is the same as the star countable property. Consequently X is star countable. □

6. Open questions

1. Is the product of an SCE-space and a separable space an SCE-space?
2. Is the product of an SCE-space and a second countable space an SCE-space?
3. Is every SCE-space P-space, star Lindelöf (star countable)?
4. Is every SCE-space and semistratifiable space separable?
5. Is every SCE-space and semimetric space separable?
6. Is every SCE-space and semidevelopable space separable?
7. Is every SCE-space and σ-paralindelöf space star Lindelöf?
8. Is every SCE-space and pseudocompact Tychonoff space star Lindelöf?

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