Construction of Mendelsohn designs by using quasigroups of (2,q)-varieties

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Abstract. Let q be a positive integer. An algebra is said to have the property $(2,q)$ if all of its subalgebras generated by two distinct elements have exactly q elements. A variety $\mathcal{V}$ of algebras is a variety with the property $(2,q)$ if every member of $\mathcal{V}$ has the property $(2,q)$. Such varieties exist only in the case of q prime power. By taking the universes of the subalgebras of any finite algebra of a variety with the property $(2,q)$, $2<q$, blocks of Steiner system of type $(2,q)$ are obtained.

The stated correspondence between Steiner systems of type $(2,3)$ and the finite algebras of the varieties with the property $(2,3)$ is a folklore. There are also more general and significant results on $(2,q)$-varieties which can be considered as a part of an "algebraic theory of Steiner systems". Here we discuss another connection between the universal algebra and the theory of combinatorial designs, and that is the relationship between the finite algebras of such varieties and Mendelsohn designs. We prove that these algebras can be used not only as models of Steiner systems, but for construction of Mendelsohn designs, as well.

For any two elements a and b of a groupoid, we define a sequence generated by the pair (a,b) in the following way: $w_0=a$, $w_1=b$, and $w_k=w_{k-2}\cdot w_{k-1}$ for $k\geq 2$. If there is an integer $p>0$ such that $w_p=a$ and $w_{p+1}=b$, then for the least number with this property we say that it is the period of the sequence generated by the pair (a,b). Then the sequence can be represented by the cycle $(w_0,w_1,\ldots,w_{p-1})$. The main purpose of this paper is to show that all of the sequences generated by pairs of distinct elements in arbitrary finite algebra of a variety with the property $(2,q)$ have the same periods (we say it is the period of the variety), and they contain unique appearance of each ordered pair of distinct elements. Thus, the cycles with period p obtained by a finite quasigroup of a variety with the property $(2,q)$ are the blocks (all of them of order p) of a Mendelsohn design.

Keywords: Mendelsohn design; quasigroup; $(2,q)$-variety; t-design

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1. Introduction

Steiner system $S(k,n,v)$, $2\leq k<n<v$, is a pair $(V,\mathcal{B})$ of a v-element set V and a collection $\mathcal{B}$ of n-element subsets of V (called blocks), such that every k-element subset of V is contained precisely in one of the blocks of $\mathcal{B}$. We say that

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such Steiner system is of type (k,n). Steiner systems of type $(2,3)$ and $(3,4)$ are known as Steiner triple systems and Steiner quadruple systems, respectively ([3]).

The pair $(V,\mathcal{B})$ is said to be a $(v,K,1)$-*Mendelsohn design* if V is a v-element set (of points) and $\mathcal{B}$ is a collection of cyclically ordered subsets of V (also called blocks) whose orders belong to K and with the property that every ordered pair of points subsequently appears exactly in one of the blocks. We write simply (v,k)-MD to denote a $(v,\{k\},1)$-Mendelsohn design ([3]). A block $(a_1,a_2,\ldots,a_n)$ of a Mendelsohn design can be considered as the set $\{(a_1,a_2),(a_2,a_3),\ldots,(a_{n-1},a_n),(a_n,a_1)\}$.

An algebra A is said to *have the property* (k,n) if every k-element subset of A generates an n-element subalgebra. Here we will use the notion (k,n)-algebra to denote such a structure. A variety $\mathcal{V}$ of algebras is a *variety with the property* (k,n) (or (k,n)-variety) if every algebra of $\mathcal{V}$ is a (k,n)-algebra. The (k,n)-varieties have some interesting characteristics due to the restrictive defining property of its algebras. For instance, an important property is that the free algebra on k-element free base in a (k,n)-variety is finite and it has n elements. In fact, in a (k,n)-variety, all of the algebras generated by k distinct elements are isomorphic, and hence free in the variety. We emphasize that in the case of a $(2,q)$-variety of groupoids, all of its elements are idempotent quasigroups ([5]).

It is clear that by taking the universe of a finite (k,n)-algebra and the set of the universes of all of its subalgebras generated by k-element subsets, one can obtain a Steiner system of type (k,n). So, every v-element algebra with the property $(2,q)$ can be taken as a model of a Steiner system $S(2,q,v)$. But we want to consider classes of Steiner systems of certain type instead of a single Steiner system. Therefore, we deal with classes of algebras, and since the varieties are most suitable for that purpose (they are closed under the forming of direct products, homomorphic images, and subalgebras), we focus our attention to $(2,q)$-varieties. The main issue connected to this is whether such varieties contain every Steiner system of type $(2,q)$, or not. In the affirmative case, we say that the variety is $(2,q)$-coordinatizing. It is proved in [4] that there is a $(2,q)$-coordinatizing variety for every prime power q. Moreover, for every prime power q, there is so called $(2,q)$-Micado coordinatizing variety, possessing two-variable equational base and having the additional property that all of its fundamental operations are binary.

Varieties with the property $(2,q)$ have also some interesting properties from the algebraic point of view. They have permutable, regular and normal congruences, the strong amalgamation property, the finite embedding property and thus a solvable word problem, and a nice equational base ([4]). In the same paper, it is shown that there is $(2,q)$-Micado variety of idempotent quasigroups coordinatizing all Steiner systems of type $(2,q)$. In order to clarify the preceding statement, we will present a brief description of the way of obtaining such varieties, also given in the stated paper [4].

The basis of the main result of our paper is the fact that the free algebra with two generators in a $(2,q)$-variety has a sharply doubly transitive automorphism group. Grätzer ([7]) proved that there is a strong correspondence between sharply doubly transitive groups and the so called G-fields.

If G is a sharply doubly transitive group on a set N, then one can define an algebraic structure $(N; -, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying $(G1)$–$(G5)$:

$(G1)$ $(N, \cdot, 0)$ is a semigroup with zero 0,
$(G2)$ $(N \setminus \{0\}, \cdot, 1)$ is a group,
$(G3)$ $a - 0 = a$,
$(G4)$ $a(b - c) = ab - ac$,
$(G5)$ $a - (b - c) = \begin{cases} c, & \text{if } a = b, \\ (a - b)(1 - (b - a)^{-1}c), & \text{if } a \neq b. \end{cases}$

Such a structure is called a G-field. All finite fields and near fields are G-fields. On the other hand, if $(N; -, \cdot, 0, 1)$ is a G-field, then the mappings $\alpha_{a,b} : N \to N$, defined by $\alpha_{a,b}(x) = b - ax$, $a, b \in N$, $a \neq 0$, form a sharply doubly transitive group on N.

Let $\mathcal{V}$ be any $(2, q)$-variety, $q \geq 3$, and let $(M; -, \cdot, 0, 1)$ be the G-field determined by $\mathcal{V}$. Let $\bar{\mathcal{V}}$ be the variety whose fundamental operations are the binary ones f_a, $a \in M$, satisfying the equations $(E1)$–$(E4)$:

$(E1)$ $(\forall a \in M)\, f_a(x, x) = x$,
$(E2)$ $f_0(x, y) = x$,
$(E3)$ $(\forall a \in M)\, f_a(y, x) = f_{1-a}(x, y)$,
$(E4)$ $(\forall a, b, c \in M)\, f_a(f_b(x, y), f_c(x, y)) = f_{b-(b-c)a}(x, y)$.

The free two-generated algebras of $\mathcal{V}$ and $\bar{\mathcal{V}}$ are rationally equivalent, so $\bar{\mathcal{V}}$ is a $(2, q)$-variety. The next result proves that $\bar{\mathcal{V}}$ is also $(2, q)$-coordinatizing variety.

Let $\bar{\mathcal{V}}$ be a $(2, q)$-Micado variety and let $(S, \mathcal{B})$ be a Steiner system of type $(2, q)$. Choose for any block $b \in \mathcal{B}$ a bijection $\alpha_b : b \mapsto M$. For $a \in M$ and $x, y \in S$, $x \neq y$, define $f_a(x, x) = x$ and $f_a(x, y) = \alpha_b^{-1}(\alpha_b(x) - (\alpha_b(x) - \alpha_b(y))a)$, where $x, y \in b \in \mathcal{B}$. Then the algebra $(S; (f_a)_{a \in M})$ belongs to $\bar{\mathcal{V}}$ and its two-generated subalgebras are exactly the blocks $b \in \mathcal{B}$.

The latest result shows that there are many different ways to "coordinatize" a Steiner system of a given type. The coordinatization is unique only when $q = 3$.

Hence, any G-field of order q defines a variety $\bar{\mathcal{V}}$ with q binary operations and the equations $(E1)$ – $(E4)$. In the special case of the Galois field $GF(q)$, one of the binary operations generates all the others. Consider a primitive element a of $GF(q)$ and the operation $x \circ y = ax + (1 - a)y$ on $GF(q)$. If we use the notation $[x, y]^0 = x$ and $[x, y]^{n+1} = [x, y]^n \circ y$, then the following equations hold:

$(F1)$ $x \circ x = x$;
$(F2)$ $[x, y]^{q-1} = x$;
$(F3)$ $[x, y]^n = [y, x]^m$, whenever $1 \leq n, m \leq q - 2$ such that $a^n + a^m = 1$.

Every variety of groupoids satisfying $(F1)$–$(F3)$ is a $(2, q)$-Micado variety. Note that the equations of such variety depend on the choice of the primitive $a \in GF(q)$.

In this paper we prove that finite quasigroups of $(2, q)$-varieties can also be used to construct Mendelsohn designs. Despite of the fact that there are several constructions of Mendelsohn designs, mainly by using fields, here we present a very simple method, based only on the restrictive properties of the algebras in $(2,q)$-varieties.

2. Sequences in groupoids

Given a binary operation f, we define inductively a sequence of binary operations $(w_0, w_1, w_2, w_3, \dots)$ generated by f, as follows: $w_0(x,y) = x$ and $w_1(x,y) = y$ are the first and the second projection, and

$$w_i(x,y) = f(w_{i-2}(x,y), w_{i-1}(x,y))$$

for every $i \geq 2$.

Let $(G,\cdot)$ be a groupoid and let $(w_0, w_1, w_2, w_3, \dots)$ be generated by the binary operation "$\cdot$". Then, for $a,b \in G$, the preceding sequence of binary operations produces the sequence $(w_0(a,b), w_1(a,b), w_2(a,b), w_3(a,b), \dots)$ of elements of G, called *the sequence generated by the pair* (a,b). So, we have

$$\begin{aligned} &w_0(a,b) = a,\\ &w_1(a,b) = b,\\ &w_i(a,b) = w_{i-2}(a,b) \cdot w_{i-1}(a,b) \text{ for } i \geq 2. \end{aligned}$$

It is clear that $w_i(a,b) = w_2(w_{i-2}(a,b), w_{i-1}(a,b))$ for every $i \geq 2$.

For example, consider the semigroup $(\mathbb{N}, +)$. Then the sequence generated by $(1,1)$ is $1,1,2,3,5,8,\dots$, i.e., it is the sequence of Fibonacci numbers.

The least integer $p > 0$ such that $w_p(a,b) = a$ and $w_{p+1}(a,b) = b$ is said to be the *period* of the sequence $(w_0(a,b), w_1(a,b), w_2(a,b), \dots)$ and we denote $per(a,b) = p$. If there is no such integer p, then we say that the period of the sequence generated by (a,b) is infinite and we write $per(a,b) = \infty$.

A sequence with a finite period $p = per(a,b)$ is completely determined by its first p members (since they are periodically repeated, i.e., $w_{kp+i}(a,b) = w_i(a,b)$ for every $k \in \mathbb{N}$ and each i, $0 \leq i < p$). Therefore, the sequence generated by the pair (a,b) with period p can be represented by the cycle $(w_0(a,b), w_1(a,b), w_2(a,b), \dots, w_{p-1}(a,b))$ of length p.

Lemma 1. *Let $(G,\cdot)$ be a groupoid and $u,v \in G$. Then*

$$w_k(u,v) = w_{k-1}(w_1(u,v), w_2(u,v)),$$

for every $k > 0$.

Proof: We have $w_1(u,v) = w_0(w_1(u,v), w_2(u,v))$ and $w_2(u,v) = w_1(w_1(u,v), w_2(u,v))$. By the definition of w_k and the inductive hypothesis, for $k > 1$, we have:

$$\begin{aligned} &w_k(w_1(u,v), w_2(u,v))\\ &\quad = w_{k-2}(w_1(u,v), w_2(u,v)) \cdot w_{k-1}(w_1(u,v), w_2(u,v))\\ &\quad = w_{k-1}(u,v) \cdot w_k(u,v) = w_{k+1}(u,v). \end{aligned}$$

□

Lemma 2. *Let $(G, \cdot)$ be a groupoid and $u, v \in G$. Then for every $k \geq 0$ and each $j \in \{0, 1, 2, \ldots, k\}$,*

$$w_k(u, v) = w_{k-j}(w_j(u, v), w_{j+1}(u, v)).$$

Proof: We use induction on j. The case $j = 0$ is trivial. For $j = 1$, by Lemma 1, $w_k(u, v) = w_{k-1}(w_1(u, v), w_2(u, v))$. Then for $j > 1$, again by Lemma 1, we obtain:

$$\begin{aligned} &w_{k-j}(w_j(u, v), w_{j+1}(u, v)) \\ &\quad = w_{k-j}(w_1(w_{j-1}(u, v), w_j(u, v)), w_2(w_{j-1}(u, v), w_j(u, v))) \\ &\quad = w_{k-(j-1)}(w_{j-1}(u, v), w_j(u, v)) = w_k(u, v). \end{aligned}$$ □

For a given groupoid, we identify the sequences generated by the pairs (a, b) and (c, d) if the following is satisfied: $per(a, b) = per(c, d) = p \in \mathbb{N}$ and there is a j, $0 \leq j < p$, such that $w_j(a, b) = c, w_{j+1}(a, b) = d$. In that case we have $w_k(c, d) = w_k(w_j(a, b), w_{j+1}(a, b)) = w_{k+j}(a, b)$ for every $k > 1$, as well. In other words, $(w_0(a, b), w_1(a, b), \ldots, w_{p-1}(a, b))$ represents a cyclic permutation of the sequence $(w_0(c, d), w_1(c, d), \ldots, w_{p-1}(c, d))$, i.e., we have different representations of the same cyclically ordered set. Therefore, we identify such cycles.

Trivially, $per(a, a) = 1$, for every idempotent groupoid G and each element $a \in G$. Since the members of $(2, q)$-varieties of quasigroups are idempotent, this holds in such varieties.

3. Period of a variety

An extensive survey on the properties of (k, n)-algebras can be found in [5]. Here we prove one of the statements which we will use below.

Lemma 3. *Let $\mathcal{V}$ be a (k, n)-variety, let U be its free algebra on k-element free base B and let A be arbitrary n-element algebra of $\mathcal{V}$. Then every one-to-one mapping from B to A can be extended to an isomorphism from U into A.*

Proof: Let $B = \{b_1, b_2, \ldots, b_k\}$ and let $a_1, a_2, \ldots, a_k$ be mutually distinct elements of A. Because of the universal mapping property, the correspondence $b_i \mapsto a_i$, $i = 1, 2, \ldots k$, can be extended to a homomorphism $\phi : U \to A$. The algebra A is generated by the set $\{a_1, a_2, \ldots, a_k\}$, and therefore $\phi(U) = A$. Since $|U| = |A| = n$, we get that ϕ is an isomorphism. □

Corollary 1. *Let $\mathcal{V}$ be a $(2, q)$-variety. Then all of its algebras generated by two distinct elements are isomorphic.* □

Corollary 2. *Every algebra generated by two distinct elements is free in a $(2, q)$-variety.* □

In what follows, we will establish some characteristics of the sequences generated by a pair of distinct elements in the quasigroups of a $(2,q)$-variety. First we will prove that all sequences have finite periods that do not exceed the value of q, and that the elements of the corresponding cycles are distinct. Afterwards, we will show that the cycles actually have the same length.

Theorem 1. *Let $(Q,\cdot)$ be a non-trivial quasigroup of a $(2,q)$-variety, let $a,b\in Q$ and $a\neq b$. Then $per(a,b)\leq q$ and $w_i(a,b)\neq w_j(a,b)$ for all $i\neq j\leq per(a,b)$.*

PROOF: Let $w_m(a,b)=w_i(a,b)$ for some $i<m$, and let all of the elements $w_0(a,b),w_1(a,b),\dots,w_{m-1}(a,b)$ be distinct. All of the members of the sequence generated by the pair (a,b) are elements of the subquasigroup generated by a and b, which is isomorphic to the free quasigroup on two generators. Hence, the equations on these two elements represent the corresponding identities in the variety. Therefore, by Lemma 2, we have $w_{m-i}(w_i(a,b),w_{i+1}(a,b))=w_m(a,b)=w_i(a,b)=w_0(w_i(a,b),w_{i+1}(a,b))$, and hence we obtain that $w_{m-i}(x,y)\approx w_0(x,y)$ is an identity in $(Q,\cdot)$. Then $w_{m-i}(a,b)=w_0(a,b)$ and, since $m-i>0$, according to the condition $w_j(a,b)\neq w_k(a,b)$ for all $j,k<m$, $j\neq k$, we get $i=0$, that is $w_m(a,b)=w_0(a,b)=a$. By the identity and Lemma 1, we also get that $w_{m+1}(a,b)=w_m(w_1(a,b),w_2(a,b))=w_0(w_1(a,b),w_2(a,b))=w_1(a,b)=b$, implying $per(a,b)=m$. Also, $m\leq q$ since all the elements in the sequence $w_0(a,b),w_1(a,b),\dots,w_{m-1}(a,b)$ are distinct and they belong to the subquasigroup generated by a and b which has exactly q elements. □

Note that the period p of the sequence generated by (a,b), $a\neq b$, of a quasigroup in a $(2,q)$-variety is not less than 3. We have that $p\neq 1$ since $w_1(a,b)=b\neq a=w_0(a,b)$. Beside this, $p\neq 2$ since $w_2(a,b)=a\cdot b\neq a=w_0(a,b)$. Namely, the quasigroups of a $(2,q)$-variety are idempotent ones and the cancellation laws hold, therefore $a\cdot b=a\Rightarrow a\cdot b=a\cdot a\Rightarrow b=a$.

Theorem 2. *Let $\mathcal{V}$ be a $(2,q)$-variety of quasigroups. Then there is a $p\in\mathbb{N}$, $3\leq p\leq q$, such that $p=per(a,b)$, for every quasigroup Q of $\mathcal{V}$ and arbitrary elements $a\neq b$ of Q.*

PROOF: Let Q_1 and Q_2 be $\mathcal{V}$-quasigroups, and let a and b be distinct elements of Q_1, and c and d be distinct elements of Q_2. Let $per(a,b)=p$, $per(c,d)=s$, and assume that $p<s$. By the argument stated in the preceding proof, $x\approx w_p(x,y)$ and $x\approx w_q(x,y)$ are identities in $\mathcal{V}$. But then, for the cycle $(w_0(c,d),w_1(c,d),\dots,w_{p-1}(c,d),\dots,w_{s-1}(c,d))$ we have that $c=w_p(c,d)$, which is impossible, since all of its elements differ from each other. □

As a result of Theorem 2, we define a notion *period of a variety* $\mathcal{V}$, referring to the period of any of the quasigroups in $\mathcal{V}$.

4. Main result

Let $(Q,\cdot)$ be a quasigroup of a $(2,q)$-variety $\mathcal{V}$, and consider the set $\mathcal{B}$ of all the sequences which are generated by the pairs of distinct elements of Q. If we summarize the results contained in Theorems 1 and 2, we obtain that $\mathcal{B}$ is the set of cycles of the same length (the period of $\mathcal{V}$), and it contains a unique appearance of every ordered pair of distinct elements as a member of some of its cycles. Hence, we have the following theorem and its combinatorial corollary.

Theorem 3. *Let $\mathcal{V}$ be a $(2,q)$-variety of quasigroups with period p, $(Q,\cdot)$ be in $\mathcal{V}$ and $\mathcal{B} = \{(x_0, x_1, \ldots, x_{p-1}) \,|\, x_0, x_1 \in Q, x_0 \neq x_1, x_{i+1} = x_{i-1} \cdot x_i, 0 < i < p-1\}$. Then for arbitrary pair of elements (a,b) of Q, $a \neq b$, there exists unique element of $\mathcal{B}$ containing the pair (a,b).* □

Corollary 3. *Let $\mathcal{V}$ be a $(2,q)$-variety of quasigroups with period p, let $(Q,\cdot)$ be a v-element quasigroup of $\mathcal{V}$ and $\mathcal{B} = \{(x_0, x_1, \ldots, x_{p-1}) \,|\, x_0, x_1 \in Q, x_0 \neq x_1, x_{i+1} = x_{i-1} \cdot x_i, 0 < i < p-1\}$. Then the pair $(Q,\mathcal{B})$ is a (v,p)-MD.* □

5. Other designs related to $(2,q)$-quasigroups

The class of Steiner systems is a subclass of a larger class of combinatorial designs. A pair $(V,\mathcal{B})$ of v-element set (of points) V and a collection $\mathcal{B}$ of k-element subsets of V (blocks), with the property that every t-element subset of V is contained in exactly λ of the blocks of $\mathcal{B}$, is said to be $t-(v,k,\lambda)$ *design*. Steiner system $S(t,k,v)$ is a $t-(v,k,1)$ design.

Let $(V,\mathcal{B})$ be $(v,p,1)$-Mendelsohn design arising from a quasigroup in a $(2,q)$-variety with period p. If $a \in V$ is an arbitrary element, then it is contained precisely in $v-1$ of the blocks of $\mathcal{B}$ (as the first member of the pair (a,b) with each of the other elements $b \in V$, and having in mind that the appearance of an element in a block is unique). If we exclude a from these blocks, then there are totally $(p-1)(v-1)$ elements which are uniformly distributed in these blocks (i.e., every element of $V \setminus \{a\}$ is contained in exactly $p-1$ of the blocks). We have to elaborate the last statement.

Assume that the opposite holds, which means that the distribution of the elements of $V \setminus \{a\}$ in the blocks containing a,

$$\begin{array}{l}(a, x_{12}, x_{13}, \ldots, x_{1p}),\\ (a, x_{22}, x_{23}, \ldots, x_{2p}),\\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\\ (a, x_{v-1,2}, x_{v-1,3}, \ldots, x_{v-1,p}),\end{array}$$

is not uniform. This implies that there is an element $b \neq a$ appearing at least p times in the above blocks. Then we have two blocks $(a, x_{i2}, x_{i3}, \ldots, x_{ip})$ and $(a, x_{j2}, x_{j3}, \ldots, x_{jp})$ such that $x_{ik} = x_{jk} = b$, for some $2 \leq k \leq p$. Put for simplicity $x = x_{i2}$, $y = x_{j2}$. Then we have $x_{i3} = ax, x_{i4} = x(ax), x_{i5} = (ax)(x(ax)), \ldots, x_{ik} = b$ and $x_{j3} = ay, x_{j4} = y(ay), x_{j5} = (ay)(y(ay)), \ldots, x_{jk} = b$. Since x_{ik} is a product of a and x and x_{jk} is a product of a and y, from the equality $x_{ik} = x_{jk}$ in the free quasigroup, we get an identity $x_{ik} \approx x_{jk}$ in the variety. (For instance, if $k = 5$, we have the identity $(ax)(x(ax)) \approx (ay)(y(ay))$.) Hence, for each i, $1 \leq i \leq v-1$, we have $x_{ik} = b$. Since every ordered pair (x,b), $x \neq b$, is contained exactly in one of the blocks of $\mathcal{B}$, we have that the pair

$(x_{i,k-1}, b)$, $i = 1, 2, \dots, v-1$, in the above blocks are all distinct. Therefore, there is $j \in \{1, 2, \dots, v-1\}$ such that $(x_{j,k-1}, b) = (a, b)$. But, the element a is contained in all of these blocks, and since the appearance of any element in any block is unique, according to our representation of the blocks ("starting" with the element a), the only possibility is that $k-1 = 1$. But this leads to the obvious contradiction that all of the stated blocks start with the pair (a, b) implying that they are equal.

Now, if we consider the blocks as (unordered) sets, then for arbitrary quasigroup element $b \neq a$, the two-element subset $\{a, b\}$ is contained in $p-1$ of the blocks. Hence, we have the following relationship between $(2, q)$-varieties of quasigroups and 2-designs.

Corollary 4. *Let $\mathcal{V}$ be a $(2, q)$-variety of quasigroups with period p, let $(Q, \cdot)$ be v-element quasigroup of $\mathcal{V}$ and $\mathcal{B} = \{(x_0, x_1, \dots, x_{p-1}) \mid x_0, x_1 \in Q, x_0 \neq x_1, x_{i+1} = x_{i-1} \cdot x_i, 0 < i < p-1\}$. If $\mathcal{C} = \{\{x_0, x_1, \dots, x_{p-1}\} \mid (x_0, x_1, \dots, x_{p-1}) \in \mathcal{B}\}$, then the pair $(Q, \mathcal{C})$ is a $2-(v, p, p-1)$ design.* □

6. Values of periods

A period p of $(2, q)$-varieties of quasigroups does not strictly depend on q. There are varieties whose period is q or $q-1$ but this is not a general rule. A variety $\mathcal{V}$ with the property (k, n) is said to be *maximal* if every variety with the same property containing $\mathcal{V}$ is equal to $\mathcal{V}$, see [5]. For example, there are three maximal $(2, 5)$-varieties of quasigroups. (The proof of this result is given in Appendix.) One of them has period 5, and the other two have period 4. On the other hand, the $(2, 9)$-variety which is constructed in [6] is defined by the identities $x \cdot xy \approx yx$, $xy \cdot (y \cdot xy) \approx x$, and its period is 4. Nevertheless, having in mind that there are $\frac{q(q-1)}{p}$ cycles which are produced by one subquasigroup on two generators, we have that $q(q-1) \equiv 0 \pmod{p}$.

The problem of determination of more precise expression representing a value of the period of a $(2, q)$-variety remains open.

7. Example

In this section we present an example of a quasigroup of order 16 of the $(2, 4)$-variety $\mathcal{V}_4$ (see Table 1) and the corresponding combinatorial structures. The unique variety of quasigroups with the property $(2, 4)$ is determined by the identities $x \cdot xy \approx yx$, $xy \cdot yx \approx x$ ([5], [10]). The variety $\mathcal{V}_4$ has period 3, and the sequences which are generated by the ordered pairs of distinct elements of the quasigroup (which match the blocks of the corresponding $(16, 3)$-MD) are given in Table 2. Note that they also represent a decomposition of the complete directed graph of order 16, and this holds in general. (In the Appendix the decompositions of all maximal $(2, q)$-varieties for $q = 3, 4, 5$ are described.) Moreover, the replacement of each of these cycles with the set consisting of the same elements results with the blocks of a $2-(16, 3, 2)$ design. The Steiner system $S(2, 4, 16)$, obtained by the quasigroup given in Table 1, is presented in Table 3.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	3	4	2	6	7	5	9	10	8	12	13	11	15	16	14
2	4	2	1	3	8	12	9	11	15	13	5	14	16	6	7	10
3	2	4	3	1	13	8	10	16	11	12	14	7	15	9	5	6
4	3	1	2	4	10	9	11	12	13	14	16	15	6	5	8	7
5	7	11	15	14	5	1	6	2	12	4	8	16	3	10	13	9
6	5	14	16	13	7	6	1	3	4	11	15	2	9	12	10	8
7	6	15	12	16	1	5	7	13	2	3	4	10	14	8	9	11
8	10	5	6	15	11	16	14	8	1	9	2	4	7	13	12	3
9	8	7	14	6	16	13	15	10	9	1	3	5	4	11	2	12
10	9	16	7	5	14	15	12	1	8	10	6	3	2	4	11	13
11	13	8	9	7	2	10	16	5	14	15	11	1	12	3	6	4
12	11	6	10	8	9	14	3	15	16	7	13	12	1	2	4	5
13	12	10	5	9	15	4	8	14	6	16	1	11	13	7	3	2
14	16	12	11	10	4	2	13	7	3	5	9	6	8	14	1	15
15	14	9	13	12	3	11	2	4	7	6	10	8	5	16	15	1
16	15	13	8	11	12	3	4	6	5	2	7	9	10	1	14	16

Table 1. $\mathcal{V}_4$-quasigroup of order 16

(1,2,3)	(1,3,4)	(1,4,2)	(2,4,3)	(1,5,6)	(1,6,7)	(1,7,5)
(5,7,6)	(1,8,9)	(1,9,10)	(1,10,8)	(8,10,9)	(1,11,12)	(1,12,13)
(1,13,11)	(11,13,12)	(1,16,14)	(14,16,15)	(2,5,8)	(2,8,11)	(2,11,5)
(5,11,8)	(2,6,12)	(2,12,14)	(2,14,6)	(6,14,12)	(2,7,9)	(2,9,15)
(2,15,7)	(7,15,9)	(2,10,13)	(2,13,16)	(2,16,10)	(10,16,13)	(3,5,13)
(3,13,15)	(3,15,5)	(5,15,13)	(3,6,8)	(3,8,16)	(3,16,6)	(6,16,8)
(3,7,10)	(3,10,12)	(3,12,7)	(7,12,10)	(3,9,11)	(3,11,14)	(3,14,9)
(9,14,11)	(4,5,10)	(4,10,14)	(4,14,5)	(5,14,10)	(4,6,9)	(4,9,13)
(4,13,6)	(6,13,9)	(4,7,11)	(4,11,16)	(4,16,7)	(7,16,11)	(4,8,12)
(4,12,15)	(4,15,8)	(8,15,12)	(5,9,12)	(5,12,16)	(5,16,9)	(9,16,12)
(6,10,11)	(6,11,15)	(6,15,10)	(10,15,11)	(7,8,13)	(7,13,14)	(7,14,8)
(8,14,13)	(1,14,15)	(1,15,16)				

Table 2. $(16, 3)$-MD obtained by the quasigroup given in Table 1

Appendix

Maximal varieties with property (2,5)

Recall that a variety $\mathcal{V}$ with the property (k, n) is maximal if every variety with the same property containing $\mathcal{V}$ is equal to $\mathcal{V}$. Here we will show that there are exactly three maximal $(2, 5)$-varieties of quasigroups, one with period 5, and the others with period 4. Two of these varieties are found by R. Padmanabhan ([10]) and the third by L. Goracinova-Ilieva ([5]).

{1,2,3,4}	{1,5,6,7}	{1,8,9,10}	{1,11,12,13}	{1,14,15,16}
{2,5,8,11}	{2,6,12,14}	{2,7,9,15}	{2,10,13,16}	{3,5,13,15}
{3,6,8,16}	{3,7,10,12}	{3,9,11,14}	{4,5,10,14}	{4,6,9,13}
{4,7,11,16}	{4,8,12,15}	{5,9,12,16}	{6,10,11,15}	{7,8,13,14}

Table 3. Steiner system $S(2,4,16)$ obtained by the quasigroup given in Table 1

Theorem 4. *The variety of groupoids $\mathcal{V}_5$ defined by the identities*

$$xy \approx yx \qquad (1)$$
$$x \cdot (xy \cdot y) \approx y \qquad (2)$$
$$x \cdot (xy \cdot x) \approx xy \cdot y \qquad (3)$$

is a $(2,5)$-variety of commutative quasigroups.

Proof: Let $(G,\cdot)$ be any groupoid satisfying the identities (1)–(3) of Theorem 4. It follows from (1) and (2) that for each $a,b \in G$ the equations $ax = b$ and $ya = b$ have solutions $x = y = ab \cdot b$. By the implications $ax = az \Longrightarrow ax \cdot a = az \cdot a \Longrightarrow a(ax \cdot a) = a(az \cdot a) \stackrel{(3)}{\Longrightarrow} ax \cdot x = az \cdot z \Longrightarrow a(ax \cdot x) = a(az \cdot z) \stackrel{(2)}{\Longrightarrow} x = z$ we have that the groupoid is cancellative, i.e., it is a quasigroup.

We will show that the subquasigroup generated by elements $x,\ y,\ x \neq y$, of any nontrivial $\mathcal{V}_5$-quasigroup has the structure as in Table 4.

	x	y	xy	$xy \cdot x$	$xy \cdot y$
x	x	xy	$xy \cdot x$	$xy \cdot y$	y
y	xy	y	$xy \cdot y$	x	$xy \cdot x$
xy	$xy \cdot x$	$xy \cdot y$	xy	y	x
$xy \cdot x$	$xy \cdot y$	x	y	$xy \cdot x$	xy
$xy \cdot y$	y	$xy \cdot x$	x	xy	$xy \cdot y$

Table 4. $\mathcal{V}_5$-quasigroup generated by $\{x,y\}$

Namely, besides the identities (1)–(3), the next identities also hold.

$$xx \approx x \qquad (4)$$
$$xy \cdot (xy \cdot x) \approx y \qquad (5)$$
$$xy \cdot (xy \cdot y) \approx x \qquad (6)$$
$$(xy \cdot x)(xy \cdot y) \approx xy \qquad (7).$$

(4): By (1) and (3) we have $x \cdot (xx \cdot x) = x \cdot xx$ and, after cancellation, $xx = x$;
(5): $xy \cdot (xy \cdot x) \stackrel{(1)}{=} xy \cdot (yx \cdot x) \stackrel{(3)}{=} xy \cdot y(yx \cdot y) \stackrel{(1)}{=} xy \cdot y(xy \cdot y) \stackrel{(1)}{=} xy \cdot (xy \cdot y)y \stackrel{(2)}{=} y$;
(6): $xy \cdot (xy \cdot y) \stackrel{(3)}{=} xy \cdot x(xy \cdot x) \stackrel{(1)}{=} xy \cdot (xy \cdot x)x \stackrel{(2)}{=} x$;
(7): $(xy \cdot x)(xy \cdot y) \stackrel{(1)}{=} (xy \cdot x)(y \cdot xy) \stackrel{(5)}{=} (xy \cdot x)((xy \cdot (xy \cdot x)) \cdot xy) \stackrel{(1)}{=}$

$\overset{(1)}{=} (xy \cdot x)(((xy \cdot x) \cdot xy) \cdot xy) \overset{(2)}{=} xy.$

To complete the proof, we have to show that the elements x, y, xy, $xy \cdot x$ and $xy{\cdot}y$ are different when $x \neq y$. Let us suppose that $x = xy$. Then, by (4), $xx = xy$, leading to $x = y$ after cancellation. If $x = xy{\cdot}x$ then $xx = xy{\cdot}x$, i.e., $x = xy$, which is not possible (as we have already seen). If $x = xy{\cdot}y$ then $xx = x(xy{\cdot}y)$, implying $x = y$ by (2) and (4). The assumption $xy = xy \cdot x$, together with the idempotency and the cancellation property, gives again $xy = x$. The same arguments show that the equality $xy = xy \cdot y$ is not true. The assumption $xy \cdot x = xy \cdot y$ immediately gives $x = y$. □

Theorem 5. *The variety of groupoids $\mathcal{V}_5'$ defined by the identities*

$$x \cdot xy \approx y,$$
$$xy \cdot y \approx yx,$$

is a $(2,5)$*-variety of anti-commutative quasigroups.* □

The proof of Theorem 5 follows the steps of the proof of Theorem 4, and we are not stating the details. Table 5 shows only the subquasigroup generated by elements $x,\ y$, $x \neq y$, of any nontrivial $\mathcal{V}_5'$-quasigroup.

	x	y	xy	yx	$x\cdot yx$
x	x	xy	y	$x\cdot yx$	yx
y	yx	y	$x\cdot yx$	x	xy
xy	$x\cdot yx$	yx	xy	y	x
yx	xy	$x\cdot yx$	x	yx	y
$x\cdot yx$	y	x	yx	xy	$x\cdot yx$

Table 5. $\mathcal{V}_5'$-quasigroup generated by $\{x, y\}$

Theorem 6. *The variety of groupoids $\mathcal{V}_5''$, defined by the identities*

$$x \cdot xy \approx yx,$$
$$xy \cdot y \approx x,$$

is a $(2,5)$*-variety of anti-commutative quasigroups.* □

Instead of complete proof, Table 6 shows only the subquasigroup generated by elements $x,\ y$, $x \neq y$, of any nontrivial $\mathcal{V}_5''$-quasigroup.

We will show that any $(2,5)$-variety of groupoids is contained in one of the varieties $\mathcal{V}_5$, $\mathcal{V}_5'$ or $\mathcal{V}_5''$. Also, any one of these varieties is not a subvariety of some of the other two.

Theorem 7. *The varieties $\mathcal{V}_5$, $\mathcal{V}_5'$ and $\mathcal{V}_5''$ are the only maximal* $(2,5)$*-varieties of groupoids.*

Proof: Let $\mathcal{V}$ be a $(2,5)$-variety of groupoids and let $\mathbf{U}$ be its free groupoid with free 2-elements base $\{x, y\}$. Then, as we stated earlier, $\mathbf{U}$ is an idempotent quasigroup.

	x	y	xy	yx	$x\cdot yx$
x	x	xy	yx	$x\cdot yx$	y
y	yx	y	$x\cdot yx$	xy	x
xy	$x\cdot yx$	x	xy	y	yx
yx	y	$x\cdot yx$	x	yx	xy
$x\cdot yx$	xy	yx	y	x	$x\cdot yx$

TABLE 6. $\mathcal{V}_5^{''}$-quasigroup generated by $\{x,y\}$

At first, suppose that **U** is commutative. Then $x\cdot xy\notin\{x,xy\}$, since in the opposite case we will get $x=y$. The same is true for the element $y\cdot xy$, and it is different from $x\cdot xy$, too. Thus, $U=\{x,y,xy,x\cdot xy,y\cdot xy\}$. The product of the element x with the elements $x\cdot xy$ and $y\cdot xy$ must be defined by $x\cdot(x\cdot xy)=y\cdot xy$ and $x\cdot(y\cdot xy)=y$, since **U** is a quasigroup. The last two equalities give the identities (3) and (2) for the variety $\mathcal{V}_5$ and, by the commutativity, we have that $\mathcal{V}$ is a subvariety of $\mathcal{V}_5$.

Let us consider now the non-commutative case, $xy\neq yx$.

If we suppose that $x\cdot xy=y$, then $x\cdot yx$ must be a new element different from x, y, xy and yx. Since **U** is a quasigroup, we have $x\cdot(x\cdot yx)=yx$, $y\cdot xy=x\cdot yx$, $y\cdot(x\cdot yx)=xy$, $xy\cdot yx=y$. Then $xy\cdot y=xy\cdot(xy\cdot yx)=yx$, meaning that $\mathcal{V}$ is a subvariety of $\mathcal{V}_5^{'}$.

Now take the opposite, i.e., suppose that **U** is not commutative and $x\cdot xy\neq y$. We have two cases to consider.

(i) Let $x\cdot xy=yx$. Then $x\cdot yx$ is a new element different from x,y,xy,yx, and the equality $x\cdot(x\cdot yx)=y$ holds true. Then $yx\cdot x=x\cdot(x\cdot yx)=y$, implying that $\mathcal{V}$ is contained in $\mathcal{V}_5^{''}$.

(ii) The last remaining case is when $x\cdot xy\notin\{x,y,xy,yx\}$ is a new element. Then, in order the multiplication table of **U** to be completed to idempotent quasigroup, we must have the following equalities: $x\cdot yx=y$, $x\cdot(x\cdot xy)=yx$, $y\cdot xy=x$, $y\cdot yx=x\cdot xy$, $y(x\cdot xy)=xy$, $xy\cdot(x\cdot xy)=x$, $yx\cdot(x\cdot xy)=y$. The partly fulfilled multiplicative table of **U** is presented in Table 7. It can be seen from Table 7 that we cannot define the product $(yx)(xy)$ such that **U** is a quasigroup.

	x	y	xy	yx	$x\cdot xy$
x	x	xy	$x\cdot xy$	y	yx
y	yx	y	x	$x\cdot xy$	xy
xy			xy		x
yx				yx	y
$x\cdot xy$					$x\cdot xy$

TABLE 7. Partly fulfilled multiplicative table of a $(2,5)$-groupoid where $x\cdot xy\neq x,y,xy,yx$

Hence, we have shown that $\mathcal{V}$ is a subvariety of some of the varieties $\mathcal{V}_5$, $\mathcal{V}_5^{'}$ or $\mathcal{V}_5^{''}$.

To complete the proof, we have to show that none of the varieties $\mathcal{V}_5$, $\mathcal{V}_5^{'}$ or $\mathcal{V}_5^{''}$ is contained into some of the others. By the commutativity of $\mathcal{V}_5$ and the anti-commutativity of $\mathcal{V}_5^{'}$ and $\mathcal{V}_5^{''}$, we have that $\mathcal{V}_5$ is not a subvariety of neither $\mathcal{V}_5^{'}$ nor $\mathcal{V}_5^{''}$, and vice versa. On the other hand, for arbitrary groupoid $(G,\cdot)$ belonging to $\mathcal{V}_5^{'}$ and $\mathcal{V}_5^{''}$, and for any $x,y \in G$ we have $x = xx = x(y \cdot yx) = x \cdot xy = y$ (since $x \cdot xy \approx y$ is an identity of $\mathcal{V}_5^{'}$, and $xy \approx y \cdot yx$ is an identity of $\mathcal{V}_5^{''}$), consequently $(G,\cdot)$ is the trivial groupoid. □

At the end, we present the cyclic decompositions of the complete directed graphs which correspond to the basic structures (the free quasigroups on two generators) of all maximal $(2,q)$-varieties of quasigroups for $q = 3,4,5$.

The $(2,3)$-variety $\mathcal{V}_3$, defined by the identities $xx \approx x$, $xy \approx yx$ and $x(xy) \approx y$, has the decomposition (x,y,xy), (x,xy,y).

The $(2,4)$-variety $\mathcal{V}_4$, defined by the identities $x(xy) \approx yx$ and $(xy)(yx) \approx x$, has the decomposition (x,y,xy), (x,xy,yx), (x,yx,y), (y,yx,xy).

The decompositions of the $(2,5)$-varieties are as follows.

$\mathcal{V}_5$: $(x,y,xy,(xy)y)$, $(x,xy,(xy)x,y)$, $(x,(xy)x,(xy)y,xy)$, $(x,(xy)y,y,(xy)x)$, $(y,(xy)y,(xy)x,xy)$.

$\mathcal{V}_5^{'}$: $(x,y,xy,x(yx))$, (x,xy,y,yx), $(x,yx,x(yx),y)$, $(x,x(yx),yx,xy)$, $(y,x(yx),xy,yx)$.

$\mathcal{V}_5^{''}$: $(x,y,xy,x(yx),yx)$, $(x,xy,yx,y,x(yx))$, $(x,yx,x(yx),xy,y)$, $(x,x(yx),y,yx,xy)$.

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