Comment.Math.Univ.Carolin. 57,4 (2016) 515–526

An introduction to loopoids

Janusz Grabowski

Abstract. We discuss a concept of *loopoid* as a non-associative generalization of Brandt groupoid. We introduce and study also an interesting class of more general objects which we call *semiloopoids*. A differential version of loopoids is intended as a framework for Lagrangian discrete mechanics.

Keywords: group; Brandt groupoid; Lie group; loop; transversals; discrete mechanics

Classification: Primary 20L05, 20N05, 22A22; Secondary 22E15, 22E60, 58H05

1. Introduction

Compared to the theory of groups, the theory of quasigroups is considerably older, dating back at least to Euler's work on orthogonal Latin squares. But later, the theory of quasigroups was eclipsed by the phenomenal development of the theory of groups and Lie groups. With the initial completion of the classification of the finite simple groups, however, attention is once again becoming more evenly divided between the two theories. Also the theory of smooth quasigroups and loops started to find interesting applications in geometry and physics.

We refer to the books [3], [2], [22], [23] and the survey articles [4], [24], [25] if terms and concepts from non-associative algebra, especially loops, are concerned. However, for completeness and reader's convenience, we recall basic definitions.

Let us recall that a *quasigroup* is an algebraic structure $< G, \cdot >$ with a binary operation (written usually as juxtaposition, $a \cdot b = ab$) such that $r_g : x \mapsto xg$ (the *right translation*) and $l_g : x \mapsto gx$ (the *left translation*) are permutations of G, equivalently, in which the equations $ya = b$ and $ax = b$ are soluble uniquely for x and y, respectively. If we assume only that left (resp., right) translations are permutations, we speak about a *left quasigroup* (resp., *right quasigroup*). A *left loop* is defined to be a left quasigroup with a right identity e, i.e. $xe = x$, while a *right loop* is a right quasigroup with a left identity, $ex = x$. A *loop* is a quasigroup with a two-sided identity element, e, $ex = xe = x$. A loop $< G, \cdot, e >$ with identity e is called an *inverse loop* if to each element a in G there corresponds an element a^{-1} in G such that

$$a^{-1}(ab) = (ba)a^{-1} = b$$

DOI 10.14712/1213-7243.2015.184

Research funded by the Polish National Science Centre grant under the contract number DEC-2012/06/A/ST1/00256.

for all $b \in G$. It can be easily shown that in an inverse loop $< G, \cdot, ^{-1}, e >$ we have, for all $a, b \in G$,

$$aa^{-1} = a^{-1}a = e, \quad (a^{-1})^{-1} = a, \quad \text{and} \quad (ab)^{-1} = b^{-1}a^{-1}.$$

Loops, or more generally left loops, appear naturally as algebraic structures on *transversals* or *sections* of a subgroup of a group. This observation, going back to R. Baer [1] (cf. also [6], [17]), lies at the heart of much current research on loops, also in differential geometry and analysis.

Example 1.1. Let G be a group with the unit e, H be a subgroup, and $S \subset G$ be a left transversal to H in G, i.e. S contains exactly one point from each coset gH in G/H. This means that any element $g \in G$ has a unique decomposition $g = sh$, where $s \in S$ and $h \in H$. This produces an identification $G = S \times H$ of sets. Let $p_S : G \to S$ be the projection on S determined by this identification. If we assume that $e \in S$, then S with the multiplication

$$s \circ s' = p_S(ss')$$

and e as a right unit is a left loop.

Indeed, as $e \circ s = s \circ e = p_S(s) = s$, e is the unit for this multiplication. For $a, b \in S$, there is $h \in H$ such that $p_S(a^{-1}b) = a^{-1}bh$. Hence,

$$p_S(ap_S(a^{-1}b)) = p_S(aa^{-1}bh) = p_S(bh) = b\,,$$

that shows that $p_S(a^{-1}b)$ is a solution of the equation $a \circ x = b$. If c, c' are two such solutions, then $p_S(ac) = p_S(ac')$, so there is $h \in H$ such that $ac = ac'h$, so $c = c'h$ and $c = c'$, since S is transversal to H.

In this paper, we would like to propose a concept of *loopoid*, defined as a nonassociative generalization of a groupoid. Note that here and throughout the paper, by *groupoid* we understand a *Brandt groupoid*, i.e. a small category in which every morphism is an isomorphism, and not an object called in algebra also a *magma*. These are loops which can be considered as nonassociative generalizations of groups. In the case of genuine groupoids, however, the situation is more complicated, because the multiplication is only partially defined, so the axioms of a loop must be reformulated.

A convenient way is to think about groupoids as being defined exactly like groups but with the difference that all objects/maps in the definition are relations, like it has been done by Zakrzewski [28]. In particular, the unity is a relation $\varepsilon : \{e\} -\!\!-\!\!\rhd\ G$, associating to a point e a subset $M = \varepsilon(e) \subset G$, the set of units. Using this idea, we define *semiloopoids*, as well as more specific objects which we will call *loopoids*.

We want to stress that our motivation comes from discrete mechanics, where Lie groupoids have been recently used for a geometric formulation of the Lagrangian formalism [5], [14], [15], [19], [20], [26], [27]. Infinitesimal parts of Lie groupoids are Lie algebroids and the corresponding 'Lie theory' is well established (cf. [18]).

We believe that this can be extended to a differential version of the concept of (semi)loopoid, a *differential (semi)loopoid*.

As the infinitesimal version of associativity is the Jacobi identity, the corresponding 'brackets' will not satisfy the latter. Note that in the literature there are already natural various generalizations of Lie algebroids, e.g. *skew algebroids*, *almost Lie algebroids*, or *Dirac algebroids* [7], [8], [9], [11], [12], where no Jacobi identity is assumed. For instance, the skew algebroid formalism is very useful in describing the geometry of nonholonomic systems [10]. We believe that we can obtain skew and/or almost Lie algebroids as infinitesimal parts of differential loopoids and that standard geometric constructions of the tangent and cotangent groupoid for a given Lie groupoid can be extended to this category. We postpone, however, these questions to a separate paper.

Note finally, that after writing the first version of these notes, we learned that the term *loopoid* has appeared already in a paper by Kinyon [16] in a similar context. The motivating example, however, built as an object 'integrating' the Courant bracket on $\mathsf{T}M \oplus_M \mathsf{T}^* M$, uses the group of diffeomorphisms of the manifold M as integrating the Lie algebra of vector fields on M, not the pair groupoid $M \times M$ as 'integrating' the Lie algebroid $\mathsf{T}M$.

2. Groupoids

Definition 2.1. A *groupoid* over a set M is a set G equipped with source and target mappings $\alpha, \beta : G \to M$, a multiplication map m from $G_2 \stackrel{\text{def}}{=} \{(g,h) \in G \times G |\ \beta(g) = \alpha(h)\}$ to G, an injective units mapping $\epsilon : M \to G$, and an inversion mapping $\iota : G \to G$, satisfying the following properties (where we write gh for $m(g,h)$ and g^{-1} for $\iota(g)$):

- (associativity) $g(hk) = (gh)k$ in the sense that, if one side of the equation is defined, so is the other, and then they are equal;
- (identities) $\epsilon(\alpha(g))g = g = g\epsilon(\beta(g))$;
- (inverses) $gg^{-1} = \epsilon(\alpha(g))$ and $g^{-1}g = \epsilon(\beta(g))$.

The elements of G_2 are sometimes referred to as *composable* (or *admissible*) pairs. A groupoid G over a set M will be denoted $G \rightrightarrows M$. Note that the full information about the groupoid is contained in the *multiplication relation* which is a subset $G_3 \subset G \times G \times G$,

$$G_3 = \{(x,y,z) \in G \times G \times G \mid (x,y) \in G_2 \text{ and } z = xy\}. \tag{1}$$

Example 2.2 (pair groupoid)**.** Let M be a set and $G = M \times M$ and

$$\alpha(u,v) = u\,, \quad \beta(u,v) = v.$$

Then, $M \times M$ is a groupoid over M with the source and target maps α, β, units mapping $\varepsilon(u) = (u,u)$, and the partial composition by $(u,v)(v,z) = (u,z)$. In other words,

$$G_3 = \{(u,v,v,z,u,z) \in G \times G \times G \mid u,v,z \in M\}.$$

Remark 2.3. We can regard M, *via* the embedding ε, as a subset in G, and thus ε as the identity, that simplifies the picture, since α, β become just projections in G. Indeed, in view of associativity,

$$(\varepsilon(\alpha(g)))^2 g = (\varepsilon(\alpha(g)))\varepsilon(\alpha(g))g) = \varepsilon(\alpha(g))g\,,$$

so that $(\varepsilon(\alpha(g)))^2 = \varepsilon(\alpha(g)) = \varepsilon(\alpha(\varepsilon(\alpha(g))))\varepsilon(\alpha(g))$ and, consequently

$$\varepsilon(\alpha(\varepsilon(\alpha(g)))) = \varepsilon(\alpha(g))\,,$$

i.e. $\varepsilon \circ \alpha$ is a projection, $(\varepsilon \circ \alpha)^2 = \varepsilon \circ \alpha$. Similarly, $\varepsilon \circ \beta$ is a projection. We will use this convention in the sequel.

3. Semiloopoids

Following Zakrzewski's idea of obtaining the definition of a groupoid by replacing the objects in the definition of a group by relations, we propose the following.

Definition 3.1. A *semiloopoid over a set M* is a structure consisting of a set G together with projections $\alpha, \beta : G \to M$ onto a subset $M \subset G$ (*set of units*) and a multiplication relation $G_3 \subset G \times G \times G$ such that, for each $g \in G$,

$$(\alpha(g), g, g) \in G_3 \quad \text{and} \quad (g, \beta(g), g) \in G_3\,, \tag{2}$$

and the relations $l_g, r_g \subset G \times G$ defined by

$$(h_1, h_2) \in l_g \quad \Leftrightarrow \quad (g, h_1, h_2) \in G_3\,, \tag{3}$$

$$(h_1, h_2) \in r_g \quad \Leftrightarrow \quad (h_1, g, h_2) \in G_3, \tag{4}$$

are injective. If we forget condition (4) (resp., (3)), then we speak about a *left* (resp. *right*) *semiloopoid*.

A *semiloopoid morphism* between semiloopoids G, H over M, N, respectively, is a pair of maps (Φ, ϕ), where $\Phi : G \to H$ and $\phi : M \to N$, satisfying

$$\Phi_{|M} = \phi\,, \quad \alpha_H \circ \Phi = \phi \circ \alpha_G\,, \quad \beta_H \circ \Phi = \phi \circ \beta_G\,,$$

and such that $(\Phi, \Phi, \Phi) : G \times G \times G \to H \times H \times H$ maps G_3 into H_3. The last condition means that

$$\Phi(gh) = \Phi(g)\Phi(h)\,, \tag{5}$$

provided $(g, h) \in G_2$.

Denote the range of the projection of G_3 onto the first two factors $G \times G$ with G_2. Condition (3) (or (4)) implies that G_3 is actually the graph of a map $m : G_2 \to G$, so we can write $z = m(g, h)$, or simply $z = gh$, instead of $(g, h, z) \in G_3$. In particular, in the above notation, $\alpha(g)g = g$ and $g\beta(g) = g$.

Consequently, we will write $l_g h = z$ and $r_g h = z$ instead of $(h, z) \in l_g$ and $(h, z) \in r_g$, respectively. We can therefore view l_g and r_g as bijections defined on their domains, D^l_g and D^r_g onto their ranges, R^l_g and R^r_g, respectively. The definition of a semiloopoid can be therefore reformulated in a more instructive way as follows.

Definition 3.2 (alternative)**.** A *semiloopoid over a set* M is a structure consisting of a set G including M and equipped with

- a *partial multiplication* $m : G \times G \supset G_2 \to G$, $m(g, h) = gh$, such that, for all $g \in G$,

(6) $$l_g : D^l_g \to R^l_g\,,\ l_g h = gh\,,$$

  is a bijection from $D^l_g = \{h \in G \,|\, (g, h) \in G_2\}$ onto $R^l_g = \{gh \,|(g, h) \in G_2\}$, and

(7) $$r_g : D^r_g \to R^r_g\,,\ r_g h = hg\,,$$

  is a bijection from $D^r_g = \{h \in G \,|\, (h, g) \in G_2\}$ onto $R^r_g = \{hg \,|(h, g) \in G_2\}$;
- a pair of projections $\alpha, \beta : G \to M$ such that, for all $g \in G$,

(8) $$\alpha(g)g = g\,,\ g\beta(g) = g.$$

Remark 3.3. Note that in a semiloopoid all the structural maps, α, β, m, are determined by just G_3 (cf. (2)).

Example 3.4 (trivial semiloopoid over M)**.** On a set G including M let us choose projections $\alpha, \beta : G \to M$ and put

$$G_2 = \{(\alpha(g), g) \,|\, g \in G\} \cup \{(g, \beta(g)) \,|\, g \in G\}.$$

The map $m : G_2 \to G$, given by $m(\alpha(g), g) = g = m(g, \beta(g))$, establishes on G a structure of a semiloopoid over M.

Example 3.5. We can make the above example more complicated, choosing $g_0 \in G\backslash M$, a subset $A \subset G$, $A \cap M = \{\beta(g_0)\}$ and an injective map $l_0 : A \to G\backslash M$ such that $l(\beta(g_0)) = g_0$. Then, we obtain a semiloopoid by putting

$$G_2 = \{(\alpha(g), g) \,|\, g \in G\} \cup \{(g, \beta(g)) \,|\, g \in G\} \cup \{(g_0, h) \,|\, h \in A\}\,,$$

and the partial multiplication $m : G_2 \to G$ which, besides the unity property (8), satisfies

$$m(g_0, h) = l_0(h)\,,\ \text{for } h \in B.$$

Definition 3.6. A semiloopoid will be called a *left inverse semiloopoid* if there is a *left inversion map* $\iota_l : G \to G$ such that for each $(g, h) \in G_2$ also $(\iota_l(g), gh) \in G_2$ and $\iota_l(g)(gh) = h$. A *right inverse semiloopoid* can be defined analogously.

A semiloopoid will be called an *inverse semiloopoid* if there is an *inversion map* $\iota : G \to G$, to be denoted simply by $\iota(g) = g^{-1}$, such that, for each $(g,h),(u,g) \in G_2$, also $(g^{-1},gh),(ug,g^{-1}) \in G_2$ and

$$g^{-1}(gh) = h\,, \quad (ug)g^{-1} = u.$$

Proposition 3.7. *In any inverse semiloopoid the following hold true:*

$$g^{-1}g = \beta(g) = \alpha(g^{-1}), gg^{-1} = \alpha(g) = \beta(g^{-1})\,, \qquad \left(g^{-1}\right)^{-1} = g\,, (gh)^{-1} = h^{-1}g^{-1}\,. \tag{9}$$

The latter condition means that one side of the equality makes sense if and only if the other makes sense (*the elements are composable*) *and they are equal.*

Proof: By definition of the inverse, $g^{-1}g = g^{-1}(g\beta(g)) = \beta(g)$ and, similarly, $gg^{-1} = \alpha(g)$. Now,

$$\left(g^{-1}\right)^{-1}\beta(g) = \left(g^{-1}\right)^{-1}(g^{-1}g) = g\,.$$

But also $g\beta(g) = g$, thus $g = \left(g^{-1}\right)^{-1}$ and

$$\beta(g)g^{-1}g = g^{-1}\left(g^{-1}\right)^{-1} = \alpha(g^{-1})\,.$$

Consequently, $\beta(g^{-1}) = \alpha(g)$ and, finally,

$$(gh)^{-1} = h^{-1}(h(gh)^{-1}) = h^{-1}\left((g^{-1}(gh))(gh)^{-1}\right) = h^{-1}g^{-1}\,. \qquad \square$$

Example 3.8. In Example 3.4 we can use an involutive bijection ι of G, intertwining α and β, for extending the partial multiplication so that we will obtain an inverse semiloopoid.

4. Tranversals

In the context of semiloopoids, we want to define *transversals*, similar but more general than these described in Example 1.1, to produce new examples of semiloopoids.

Definition 4.1. Let G be a semiloopoid. A pair (T,π), where $T \subset G$ and $\pi : G \to T$ is a projection, is called a *transversal* in G if

$$\alpha(T),\beta(T) \subset T\,, \tag{10}$$

and for $t \in T$, the relations $l_t^T, r_t^T \subset T \times G$, reducing the relations l_t, r_t to $T \times G$, i.e.

$$(t',g) \in l_t^T \quad \Leftrightarrow \quad (t,t',g) \in G_3\,, \tag{11}$$

$$(t',g) \in r_t^T \quad \Leftrightarrow \quad (t',t,g) \in G_3\,, \tag{12}$$

are *transversal to* π in the sense that the composition relations $\pi \circ l_t, \pi \circ r_t \subset T \times T$ are injective. In other words, for $t, t', u, u' \in T$, if tt' and uu' are different elements in the same fiber of π, then $t \neq t'$ and $u \neq u'$.

If we forget condition (12) (resp. (11)), then we speak about a *left* (resp. *right*) *transversal*.

Example 4.2. The transversal described in Example 1.1 is an example of a left transversal in G in the above sense. In this case, $T = S$ and $\pi = p_S$. The first condition of Definition 4.1 means that $e \in S$. Condition (11), in turn, means that, for $s, s', s'' \in S$, if $p_S(ss') = p_S(ss'')$, then $s' = s''$. It is clearly satisfied, since $p_S(ss') = p_S(ss'')$ implies that there is $h \in H$ such that $ss'' = ss'h$, thus $s'' = s'h$. From the uniqueness of the decomposition $G = SH$ we get $s'' = s'$.

Proposition 4.3. *If (T, π) is a (left, right) transversal in a semiloopoid G with the source and target maps $\alpha, \beta : G \to M$, then T is a (left, right) semiloopoid itself with the source and target maps*

$$\alpha_{|T}, \beta_{|T} : T \to M_T := \alpha(T) = \beta(T) \subset T$$

and the partial multiplication $m_T : T_2 := G_2 \cap (T \times T) \to T$

$$t \bullet t' = \pi(tt')\,.$$

Proof: Since $\alpha(\beta(T)) = \beta(T) \subset T$, we have trivially $\alpha(T) = \beta(T)$ as the set of units in T. The injectivity of the left and/or right translations follows from the corresponding transversality properties. □

5. Loopoids

As Example 3.5 shows, the maps α, β can be rather pathological, if their general properties are concerned. Let us assume now, that a semiloopoid G over M, with a partial multiplication m and projections $\alpha, \beta : G \to M$, satisfies a very weak associativity condition, hereafter called *unities associativity*:

$$(xy)z = x(yz) \text{ if one of } x, y, z \text{ is a unit (i.e. belongs to } M)\,. \tag{13}$$

The above condition has to be understood as follows: if one side of equation (13) makes sense, the other makes sense and we have equality. The following proposition shows that the condition of unities associativity for a semiloopoid over M implies that the *anchor map* $(\alpha, \beta) : G \to M \times M$ has nice properties, similar to these for groupoids.

Proposition 5.1. *A semiloopoid G over M satisfies the unities associativity condition if and only if*

$$G_2 = \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\} \tag{14}$$

and

$$(\alpha,\beta): G \to M \times M \tag{15}$$

is a semiloopoid morphism into the pair groupoid $M \times M$, i.e.

$$\alpha(gh) = \alpha(g) \text{ and } \beta(gh) = \beta(h)\,. \tag{16}$$

PROOF: If $(g,h) \in G_2$, then according to (13),

$$g(\alpha(h)h) = gh = (g\beta(g))h = g(\beta(g)h)\,,$$

so, in view of injectivity of l_g, we have $h = \alpha(h)h = \beta(g)h$ and, consequently, $\beta(g) = \alpha(h)$. Similarly,

$$\alpha(gh)(gh) = gh = (\alpha(g)g)h = \alpha(g)(gh)\,,$$

so $\alpha(gh) = \alpha(g)$. Analogously we can prove $\beta(gh) = \beta(h)$.

Conversely, let $e \in M$ be such that $e(gh)$ makes sense. Then,

$$e = \beta(e) = \alpha(gh) = \alpha(g)\,.$$

Hence

$$e(gh) = \alpha(gh)(gh) = gh = (\alpha(g)g)h = (eg)h\,. \tag{17}$$

If this is $(eg)h$ that makes sense, then

$$e = \beta(e) = \alpha(g) = \alpha(gh)$$

and we have (17) again. Similarly we prove $(ge)h = g(eh)$ and $(gh)e = g(he)$. □

The inverse images of points under the source and target maps we call α- and *β-fibres*. The fibres through a point g, will be denoted by $\mathcal{F}^\alpha(g)$ and $\mathcal{F}^\beta(g)$, respectively. The unities associativity assumption implies that each element g of G determines the *left and right translation maps*

$$l_g : \mathcal{F}^\alpha(\beta(g)) \to \mathcal{F}^\alpha(\alpha(g))\,, \quad r_g : \mathcal{F}^\beta(\alpha(g)) \to \mathcal{F}^\beta(\beta(g))\,, \tag{18}$$

which are injective.

Definition 5.2. A semiloopoid satisfying the unities associativity assumption and such that the maps (18) are bijective will be called a *loopoid*.

Remark 5.3. In a loop, the multiplication is globally defined, so the unity associativity is always satisfied by properties of the unity element. In this sense, loops are loopoids over one point.

Proposition 5.4. *Let G be a loopoid over M with the source and target maps $\alpha, \beta : G \to M$. Then, for each $u \in M$, the multiplication in G induces a loop structure on the set*

$$G_u = \{g \in G \,|\, \alpha(g) = \beta(g) = u\}$$

.

Proof: G_u is clearly closed with respect to the multiplication and this multiplication is globally defined, $G_u \subset G_2$. As u is the only unit in G_u, we have only one α/β-fiber on which translations act as bijections, thus we deal with a loop. □

The loop G_u above will be called the *isotropy loop* of $u \in M$. A direct consequence of the above proposition is the following.

Corollary 5.5. *A loopoid over a single point is a loop.*

Example 5.6. If the anchor map in a loopoid is *diagonal* (trivial), i.e. $\alpha = \beta$, then the loopoid is a *bundle of loops*. Conversely, any bundle of loops, i.e. a surjection $\tau : G \to M$ whose each fiber carries a loop structure, is canonically a loopoid, with M embedded in G as the section of unit elements in each loop-fiber, $\alpha = \beta = \tau$, and the partial multiplication being the loop multiplication in each fiber. As a more specific example we can take the *unit octonion bundle* $\tau : U\mathbb{O}M \to M$ over a 7-dimensional Riemannian manifold M equipped with a G_2-structure, as defined in [13, Definition.3.11]. It is the bundle of unit vectors in a vector bundle, the *octonion bundle* $\mathsf{T}M \times \mathbb{R}$, whose fibers have the structure of octonions. Since the unit vectors in octonions, geometrically the 7-sphere S^7, form a *Moufang loop*, the unit octonion bundle $U\mathbb{O}M$ is a bundle of loops, so an example of a loopoid.

Example 5.7. Let X be a loop with the unit e and N be a set. On $G = X \times N \times N$ we have an obvious structure of a loopoid as a product structure of the loop X and the pair groupoid $N \times N$ over $M = \{(e, s, s) \,|\, s \in N\} \subset G$. The anchor map is

$$(\alpha, \beta)(x, s, t) = (e, s, t)$$

and the partial multiplication reads

$$(x, s, t) \bullet (y, t, r) = (xy, s, r)\,.$$

If X is an inverse loop, then G is an inverse loopoid with the inverse $\iota(x, s, t) = (x^{-1}, t, s)$. In this example X is the *isotropy loop* G_u at each $u \in M$.

Note that we can consider slightly weaker objects than loopoids.

Definition 5.8. A *left loopoid* (resp., *right loopoid*) is a semiloopoid satisfying $G_2 = \{(g, h) \in G \times G \,|\, \beta(g) = \alpha(h)\}$ and such that, for each $g \in G$, the left translation l_g is a bijection from $\mathcal{F}^\alpha(\beta(g))$ onto $\mathcal{F}^\alpha(\alpha(g))$ (resp., the right translation r_g is a bijection from $\mathcal{F}^\beta(\alpha(g))$ onto $\mathcal{F}^\beta(\beta(g))$).

Example 5.9. Let us finish with an interesting example of a left loopoid which is not a loopoid and can be regarded as a toy example of the intended *left differential loopoid*. Consider the pair groupoid $\mathcal{G} = \mathbb{R}^2 \times \mathbb{R}^2$ with the standard source and target maps $\alpha(u,v) = (u,u)$, $\beta(u,v) = (v,v)$ and composition $(u,v)(v,z) = (u,z)$. For a diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ being an odd function, $\phi(-x) = -\phi(x)$, define a submanifold

$$G = \{((a_1,b_1),(a_2,b_2)) \in \mathcal{G} : a_1 - a_2 = \phi(b_1 - b_2)\} \,. \tag{19}$$

It is a semiloopoid, with the source and target maps inherited from $\mathcal{G}$, and the partial multiplication

$$((a_1,b_1),(a_2,b_2)) \bullet ((a_2,b_2),(a_3,b_3)) = ((a_1,b_1),(a_1 + \phi(b_3 - b_1),b_3)) \,. \tag{20}$$

Indeed, for fixed $(a_1,b_1) \in \mathbb{R}^2$ the map $(a_3,b_3) \mapsto (a_1 + \phi(b_3 - b_1),b_3)$ is a diffeomorphism, so the left translations $l_{(a_1,b_1)}$ are smooth immersions. The right translations are smooth immersions trivially. For G we have $\alpha(gh) = \alpha(g)$, but generally, for non-linear ϕ, we have $\beta(gh) \neq \beta(h)$, so G is not a loopoid.

It is interesting that G has a left inverse

$$\iota_l((a_1,b_1),(a_2,b_2)) = ((a_2,b_2),(a_1,b_1)).$$

Indeed,

$$\begin{aligned}
&((a_2,b_2),(a_1,b_1)) \bullet (((a_1,b_1),(a_2,b_2)) \bullet ((a_2,b_2),(a_3,b_3)))\\
&= ((a_2,b_2),(a_1,b_1)) \bullet ((a_1,b_1),(a_1 + \phi(b_3 - b_1),b_3))\\
&= ((a_2,b_2),(a_2 + \phi(b_3 - b_2),b_3)) = ((a_2,b_2),(a_3,b_3))\,.
\end{aligned}$$

The last equality follows from the fact that $((a_2,b_2),(a_3,b_3)) \in G$, so by (19)

$$a_2 - a_3 = \phi(b_2 - b_3) = -\phi(b_3 - b_2)\,.$$

All this implies that G is a left (inverse) loopoid.

Remark 5.10. Note that G is actually a (Lie) groupoid if ϕ is linear and that G is obtained as a transversal in the groupoid $\mathcal{G}$ with respect to a projection $\pi : \mathcal{G} \to G$ given by

$$\pi\left((a_1,b_1),(a_2,b_2)\right) = ((a_1,b_1)(a_1 + \phi(b_2 - b_1),b_2))\,.$$

Indeed, the subset of units (the diagonal) belongs to G (as $0 = a_1 - a_1 = \phi(b_1 - b_1)$), so $\alpha(G) = \beta(G) \subset G$. Moreover, if $((a_1,b_1),(a_2,b_2))\,,((a_2,b_2),(a_3,b_3)) \in G$, then

$$\begin{aligned}
\pi\left(((a_1,b_1),(a_2,b_2)) \cdot ((a_2,b_2),(a_3,b_3))\right) &= \pi\left((a_1,b_1),(a_3,b_3)\right)\\
&= ((a_1,b_1)(a_1 + \phi(b_3 - b_1),b_3))\,,
\end{aligned}$$

so we recover the multiplication in G.

Acknowledgments. The author thanks Michael Kinyon for his useful comments on an earlier draft of this work.

References

[1] Baer R., *Nets and groups*, Trans. Amer. Math. Soc. **46** (1939), 110–141.

[2] Belousov V.D., *Foundations of the Theory of Quasigroups and Loops*, Nauka, Moscow, 1967 (in Russian).

[3] Bruck R.H., *A Survey of Binary Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 20. Reihe: Gruppentheorie Springer, Berlin-Göttingen-Heidelberg, 1958.

[4] Bruck R.H., *What is a loop*, in Studies in modern algebra, Studies in Mathematics, 2, Prentice-Hall, New Jersey, 1963, pp. 59–99.

[5] Fedorov Y.N., Zenkov D.V., *Discrete nonholonomic LL systems on Lie groups*, Nonlinearity **18** (2005), 2211–2241.

[6] Foguel T., *Groups, transversals, and loops*, Loops'99 (Prague), Comment. Math. Univ. Carolin. **41** (2000), 261–269.

[7] Grabowska K., Grabowski J., *Variational calculus with constraints on general algebroids*, J. Phys. A **41** (2008), 175204, 25pp.

[8] Grabowska K., Grabowski J., *Dirac algebroids in Lagrangian and Hamiltonian mechanics*, J. Geom. Phys. **61** (2011), 2233–2253.

[9] Grabowski J., Jóźwikowski M., *Pontryagin maximum principle on almost Lie algebroids*, SIAM J. Control Optim. **49** (2011), 1306–1357.

[10] Grabowski J., de Leon M., Marrero J.C., Martin de Diego D., *Nonholonomic constraints: A new viewpoint*, J. Math. Phys. **50** (2009), 013520, 17pp.

[11] Grabowski J., Urbański P., *Lie algebroids and Poisson-Nijenhis structures*, Rep. Math. Phys. **40** (1997), 195–208.

[12] Grabowski J., Urbański P., *Algebroids – general differential calculi on vector bundles*, J. Geom. Phys. **31** (1999), 111–141.

[13] Grigorian S., *G_2-structures and octonion bundles*, arXiv:1510.04226.

[14] Iglesias D., Marrero J.C., Martín de Diego D., Martínez E., *Discrete nonholonomic Lagrangian systems on Lie groupoids*, J. Nonlinear Sci. **18** (2008), 221–276.

[15] Iglesias D., Marrero J.C., Martín de Diego D., Padrón E., *Discrete nonholonomic in implicit form*, Discrete Contin. Dyn. Syst. **33** (2013), 1117–1135.

[16] Kinyon M., *The coquecigrue of a Leibniz algebra*, preprint, 2003.

[17] Kinyon M., Weinstein A., *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*, Amer. J. Math. **123** (2001), 525–550.

[18] Mackenzie K., *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series, 213, Cambridge University Press, Cambridge, 2005.

[19] Marrero J.C., Martín de Diego D., Martínez E., *Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids*, Nonlinearity **19** (2006), 1313–1348. Corrigendum: Nonlinearity **19** (2006), 3003–3004.

[20] Marrero J.C., Martín de Diego, Stern A., *Symplectic groupoids and discrete constrained Lagrangian mechanics*, Discrete Contin. Dyn. Syst. **35** (2015), 367–397.

[21] Martínez E., *Lagrangian mechanics on Lie algebroids,* Acta Appl. Math. **67** (2001), 295–320.

[22] Pflugfelder H.O., *Quasigroups and Loops: Introduction*, Heldermann, Berlin, 1990.

[23] Sabinin L.V., *Smooth Quasigroups and Loops*, Kluwer Academic Press, Dordrecht, 1999.

[24] Sabinin L.V., *Smooth quasigroups and loops: forty-five years of incredible growth*, Comment. Math. Univ. Carolin. **41** (2000), 377–400.

[25] Smith J.D.H., *Loops and quasigroups: Aspects of current work and prospects for the future*, Comment. Math. Univ. Carolin. **41** (2000), 415–427.

[26] Stern A., *Discrete Hamilton–Pontryagin mechanics and generating functions on Lie groupoids*, J. Symplectic Geom. **8** (2010), 225–238.

[27] Weinstein A., *Lagrangian mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207–231.

[28] Zakrzewski S., *Quantum and classical pseudogroups I. Union pseudogroups and their quantization*, Comm. Math. Phys. **134** (1990), 347–370.

Polish Academy of Sciences, Institute of Mathematics, Śniadeckich 8,
P.O. Box 21, 00-656 Warsaw, Poland

E-mail: : jagrab@impan.pl

(*Received* October 15, 2015, *revised* June 7, 2016)