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Isotopy invariant quasigroup identities

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Abstract. According to S. Krstić, there are only four quadratic varieties which are closed under isotopy. We give a simple procedure generating quadratic identities and deciding which of the four varieties they define. There are about 37000 such identities with up to five variables.

Keywords: quasigroup; 3-sorted quasigroup; homotopy; isotopy; quadratic identity; gemini identity; coherent identity; variety closed under isotopy (homotopy)

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1. Introduction

The notion of isotopy is a straightforward generalization of the fundamental algebraic notion of isomorphism. However, it does not share some basic properties of isomorphism:

- identities are not necessarily preserved under isotopies, and consequently
- varieties are not necessarily closed under isotopies.

For example, the varieties of all quasigroups and all Moufang loops (in the language $\{\cdot, \backslash, /\}$) are closed under isotopies but varieties of all loops and all groups are not. This suggests the problem of isotopic invariance of quasigroup identities and varieties.

The variety of all *quasigroups* is defined by the usual four identities (Q) (see page 538). V.D. Belousov proved in [1] that the class GI (AI) of all (*Abelian*) *group isotopes* is a variety of quasigroups defined by the additional axiom:

(GI) $$(x(y\backslash z)/u)v \approx x(y\backslash(z/u)v)$$

(AI) $$x\backslash y(u\backslash v) \approx u\backslash y(x\backslash v)$$

The class BI of all *Boolean group isotopes* was defined by either of the following identities (E. Falconer [7]):

(BI) $$xy/z \approx xz/y$$

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$$(x/y)(z\backslash x) \approx zy \tag{1}$$

In the same paper she also gave the first general result:

Theorem 1.1. *A variety V of quasigroups is closed under isotopies iff V is the variety of all isotopes of loops from V.*

The following important result was proved by S. Krstić [15]:

Theorem 1.2. *The only quadratic quasigroup varieties closed under isotopies are:*

- *the variety* Q *of all quasigroups;*
- *the variety* GI *of all group isotopes;*
- *the variety* AI *of all Abelian group isotopes, and*
- *the variety* BI *of all Boolean group isotopes.*

A quasigroup variety is *quadratic* if it is defined by quadratic identities. An identity $s \approx t$ is *quadratic* if every variable from $s \approx t$ occurs there exactly twice. A group is *Boolean* if it is of exponent 2, i.e. if it satisfies $x^2 \approx e$ or equivalently $xx \approx yy$. See Section 2 for more detailed definitions.

Even more important is:

Theorem 1.3 (S. Krstić [14])**.** *A quadratic quasigroup identity is closed under isotopies iff it is coherent.*

A *coherence* is a somewhat involved notion which will be defined in Section 3. It turns out that Krstić was very close to the solution of the problem:

Theorem 1.4 (A.A. Gvaramiya [9])**.** *A quasigroup identity is closed under isotopies iff it is equivalent to a coherent identity.*

This came as one of the results of the series of papers by A.A. Gvaramiya (see also A.A. Gvaramiya, B.I. Plotkin [12]), which connected quasigroups with the notion of $*$-*reversible automata* and 3-*sorted quasigroups*. This will be clarified in Section 3.

2. Quasigroups

We assume that quasigroups are algebras $(S; \cdot, \backslash, /)$ with three binary operations satisfying the following four axioms:

$$\text{(Q)} \qquad \begin{array}{ll} x\backslash xy \approx y & \qquad xy/y \approx x \\ x(x\backslash y) \approx y & \qquad (x/y)y \approx x \end{array}$$

Therefore, the class of all quasigroups is a *variety*. Other important quasigroup varieties we shall use are *loops* (satisfying $x\backslash x \approx y/y$), *commutative quasigroups* (satisfying commutativity $xy \approx yx$), *groups* (satisfying associativity $xy \cdot z \approx x \cdot yz$), *Abelian* i.e. *commutative groups*, *Boolean groups* (satisfying associativity and unipotency $xx \approx yy$), *totally symmetric quasigroups* (satisfying commutativity and left symmetry $x \cdot xy \approx y$) and *totally symmetric* or *Steiner loops*.

Definition 2.1. We can define *dual operations* of $\cdot, \backslash, /$:

$$x * y \approx yx \qquad\qquad x\backslash\!\!\backslash y \approx y\backslash x \qquad\qquad x/\!\!/y \approx y/x$$

They also give rise to quasigroups on S. The operations $\cdot, \backslash, /, *, \backslash\!\!\backslash, /\!\!/$ are *parastrophes* of $\cdot$ (and of each other). In loops, we can also define a 0-ary operation $e \approx x\backslash x$ which is a unit of the given loop (i.e. $ex \approx xe \approx x$).

Definition 2.2. A *quasigroup identity* is an identity in the language $L = \{\cdot, \backslash, /\}$ of the theory of quasigroups. A quasigroup identity $s \approx t$ is *quadratic* if every variable appears exactly twice in $s \approx t$. An identity is *balanced* if every variable appears exactly once in s and once in t.

Let $(S; \cdot, \backslash, /)$ and $(T; \circ, \obslash, \oslash)$ be quasigroups and let $f, g, h : S \longrightarrow T$ be functions such that $h(xy) = f(x) \circ g(y)$. We say that (f, g, h) is a *homotopy* from $(S; \cdot, \backslash, /)$ to $(T; \circ, \obslash, \oslash)$. Homotopy is a generalization of homomorphism. If all three functions f, g, h are bijective, the homotopy is an *isotopy*. If h is the identity mapping the isotopy is *principal*. Isotopy is a generalization of isomorphism. Every quasigroup is isotopic to some loop.

If an identity Eq, true in a quasigroup, is also true in all its isotopic (homotopic) images, we say that Eq is *preserved under isotopies* (*homotopies*). A variety V of quasigroups with the property that every isotope (homotopic image) of a quasigroup from V is also in V, is said to be *closed under isotopies* (*homotopies*). Sometimes, identities preserved and varieties closed under isotopies (homotopies) are said to be *universal* or *isotopy* (*homotopy*) *invariant*.

Definition 2.3 (A. Krapež, M.A. Taylor [13])**.** A quadratic quasigroup identity $s \approx t$ is *gemini* if it is true in all Steiner loops.

The following theorem is our principal tool for handling quadratic identities. Combined with Theorem 1.2 it gives us a way to capture isotopically invariant quadratic identities.

Theorem 2.1 (A. Krapež, M. A. Taylor [13])**.** *A quasigroup satisfying a quadratic but not a gemini identity is isotopic to a group.*

In an unpublished paper (see [3]), M.M. Gluchov recapitulated some known results on identities characterizing (Abelian) group isotopes, i.e. identities equivalent to (GI) ((AI)). He listed (AI) and five other identities (2)–(6):

$$(z/y)(x\backslash w) \approx (w/y)(x\backslash z) \tag{2}$$

$$(xy_1/y_2)y_3 \approx (xy_3/y_2)y_1 \tag{3}$$

$$xy/(u\backslash z) \approx uy/(x\backslash z) \tag{4}$$

$$x_1(x_2\backslash x_3y) \approx x_3(x_2\backslash x_1y) \tag{5}$$

$$(z/y_1)y_2/y_3 \approx (z/y_3)y_2/y_1 \tag{6}$$

The identity (2) was attributed to V.D. Belousov [1], the three identities (3)–(5) to A. Drápal [6] and the identity (6) to A. Tabarov (no reference). To this list

G.B. Belyavskaya added three identities of her own:

$$(z/y)\backslash xv \approx (z/v)\backslash xy \tag{7}$$

$$(z_1/(x\backslash z_2))\backslash z_3 \approx (z_1/(x\backslash z_3))\backslash z_2 \tag{8}$$

$$z_1/((z_2/y)\backslash z_3) \approx z_2/((z_1/y)\backslash z_3) \tag{9}$$

However, (3) and (9) could be already found in S. Krstić [14] and Kh. Shahbazpour [20], respectively.

Belyavskaya also gave two more identities equivalent to (GI):

$$(z/(x\backslash uy))\backslash w \approx x\backslash u((z/y)\backslash w) \tag{10}$$

$$(z/(x\backslash w))y/v \approx z/((xy/v)\backslash w) \tag{11}$$

As far as we know, (1) is the only published identity equivalent to (BI). Our aim is to find new identities with at most five variables which characterize (GI), (AI) or (BI), particularly these which are of the smallest length. For this, it is useful to become familiar with 3-*sorted quasigroups*.

3. 3-sorted quasigroups

Many sorted algebras are generalizations of classical algebras, such that they have several base sets with different types of objects. Typical examples are geometries with points, lines, planes etc.; vector spaces in which we distinguish *scalars* and *vectors*; metric spaces, where we have *objects* and measure *distances* between them (which are real numbers) and, closer to our subject, 3-nets with three separate classes of *parallel lines* and a set of *points* of intersection of some of these lines. There is a growing body of literature on the subject and we mention only a few items: G. Birkhoff, D. Lipson [4], A. Petrescu [18], F.M. Sioson [21]. The logical framework is given for example in M. Manzano [16].

3-sorted quasigroups are many sorted algebras which come in two flavors. As 3-sorted algebras with one operation (multiplication), see among others S. Milić [17], Z. Stojaković [23], V. Satyabhama [19], E. Brožíková [5] and a series of papers by A.A. Gvaramiya ([8]–[11] etc.) where 3-sorted quasigroups were treated in the guise of reversible $*$-automata. Alternatively, they are viewed as 3-sorted algebras with three operations (multiplication and two divisions), see V.D. Belousov [2], A.A. Gvaramiya and B.I. Plotkin [12] etc. We prefer the later type.

A 3-sorted quasigroup is a 3-sorted algebra $(Q_1, Q_2, Q_3; \cdot, \backslash, /)$ where

$$\cdot : Q_1 \times Q_2 \longrightarrow Q_3, \qquad \backslash : Q_1 \times Q_3 \longrightarrow Q_2, \qquad / : Q_3 \times Q_2 \longrightarrow Q_1$$

are three binary functions satisfying the following axioms:

$$x\backslash xy \approx y \qquad\qquad xy/y \approx x$$
$$x(x\backslash z) \approx z \qquad\qquad (z/y)y \approx z$$

Unlike usual quasigroups, in the theory of 3-sorted quasigroups we should respect sorts, so that if x is a variable of sort 1 in some expression, it should remain to be of sort 1 in other occurrences of the same expression as well (as in other expressions in the same context). The term 'coherent' is used to state the intention of keeping sorts separate, i.e. that variables of sort α $(\alpha = 1, 2, 3)$ take values in Q_α only. The formal definition is given below. Observing how we changed quasigroup axioms might help, but a few simple examples should clarify the situation completely.

Example 3.1. The identity $x(u\backslash z) \approx u(x\backslash z)$ is coherent. Namely, x and u are variables of sort 1 and z is a variable of sort 3. It follows that terms $x\backslash z$ and $u\backslash z$ are both of sort 2, so terms $x(u\backslash z)$ and $u(x\backslash z)$ are both of sort 3. This consistency actually defines the coherence of the identity $x(u\backslash z) \approx u(x\backslash z)$.

Example 3.2. The quasigroup term $x \cdot x$ is not well formed in the theory of 3-sorted quasigroups. It forces x to be a variable of both sorts 1 and 2 which is not allowed. Therefore, the question if a 3-sorted quasigroup is unipotent (satisfies $xx \approx yy$) or *idempotent* (satisfies $xx \approx x$) is a meaningless one.

Example 3.3. The same applies to terms $x\backslash x$ and x/x and consequently the notion of (left, right) loop is not well defined in 3-sorted quasigroups.

Example 3.4. Consider the following quasigroup theorems:

$$x/(y\backslash x) \approx y \qquad\qquad (x/y)\backslash x \approx y$$

and their consequence:

$$x/(y\backslash x) \approx (x/y)\backslash x.$$

Neither of the three identities is coherent. We can repair the first two by changing variables:

$$z/(x\backslash z) \approx x \qquad\qquad (z/y)\backslash z \approx y$$

but it does not work in the third case. No change of variables can make it coherent because the left and the right hand side of the identity are α-terms for two different values of α (see below) and that cannot be repaired.

The following general statement gives one of the reasons why the change of variables does not work in the previous example.

Proposition 3.1. *Let $s_1 \circ s_2 \approx t_1 \diamond t_2$ be a coherent identity with $\circ, \diamond \in \{\cdot, \backslash, /\}$. Then $\circ$ and $\diamond$ must be the same operation symbols.*

In the theory of 3-sorted quasigroups we need variables of sort 1 $(x, u, x_1, x_2, \dots)$, variables of sort 2 $(y, v, y_1, y_2, \dots)$ and variables of sort 3 $(z, w, z_1, z_2, \dots)$. The terms are defined by:

- variables of sort α $(\alpha = 1, 2$ or $3)$ are α-terms;
- if t_α $(\alpha = 1, 2$ and $3)$ are α-terms, then $t_1 \cdot t_2$ is a 3-term, $t_1\backslash t_3$ is a 2-term and t_3/t_2 is a 1-term;

- such terms are collectively called coherent terms.

If s and t are both α-terms ($\alpha = 1, 2$ or 3) then $s \approx t$ is an α-identity. An identity is coherent if it is an α-identity for some α. (Strictly speaking, there should be three different equality symbols $\approx_1, \approx_2$ and $\approx_3$ so that an α-identity can be interpreted as an equality in Q_α. But we prefer simplicity over logical purity here.)

Basically, in the above definition the notion of 'coherent terms (identities)' is just a reformulation of 'quasigroup terms (identities) well formed in the theory of 3-sorted quasigroups'.

The class of all 3-sorted quasigroups is a variety. There is also a category **Qtp** with objects 3-sorted quasigroups while morphisms are homotopies between them (see for example A.A. Gvaramiya [11] and G. Voutsadakis [24]). Therefore, we have:

Theorem 3.1. *Coherent identities are preserved under homotopies.*

A particularly important 3-sorted quasigroup is the *monogenic* 3-sorted quasigroup $\mathcal{Z} = (\{1\}, \{2\}, \{3\}; \cdot, \backslash, /)$ with operations defined by:

$$1 \cdot 2 = 3, \qquad 1\backslash 3 = 2, \qquad 3/2 = 1.$$

It is the terminal object of **Qtp** and is interesting because every coherent term evaluates to 1,2 or 3 in $\mathcal{Z}$, while every coherent equality evaluates to $1 = 1, 2 = 2$ or $3 = 3$. Therefore:

Theorem 3.2. *A quasigroup equality is coherent iff it is valid in $\mathcal{Z}$.*

We can use this last property to generate coherent identities. This is the reason why the 3-sorted quasigroup $\mathcal{Z}$ was introduced into the paper.

4. Building up identities

For every quadratic coherent identity J, we can establish a valuation which maps all variables of sort α ($\alpha = 1, 2, 3$) into α. Thus, J is interpreted as a coherent (but not necessarily quadratic) equality Eq on $\mathcal{Z}$. However, each of $1, 2, 3$ appears in Eq an even number of times (possibly zero times). Such equalities on $\mathcal{Z}$ we call *even*. It is easy to see that the converse is also true: every even equality Eq in $\mathcal{Z}$ is an interpretation of some quadratic coherent identity J. The identity J need not be unique. However, for equalities Eq of small length, the number of possible identities is also small and therefore, manageable for handling by computer.

Example 4.1. To see why we insist on even occurrence of every element in Eq, let Eq be the equality $(12/(1\backslash 3))((3/2)\backslash 12) = 3$ which has exactly three occurrences of $1, 2$ and 3 each. Replace any two occurrences of $1, 2, 3$ by x, y, z in that order. We get, for example, $(xy/(1\backslash z))((z/y)\backslash x2) = 3$. Whatever variables we choose to replace the remaining $1, 2, 3$ in that equality, we cannot get a quadratic identity.

The same conclusion holds in general: if any of the elements $1, 2, 3$ appears an odd number of times in Eq, it cannot give rise to a quadratic identity.

An equality Eq, true in $\mathcal{Z}$, is *even* if each of elements $1, 2, 3$ occurs in Eq an even number of times (possibly zero times). All equations we use in the rest of the paper are assumed to be even.

Example 4.2. The equality $1(1\backslash 12) = 12$ is true in $\mathcal{Z}$ and with the constant 1 appearing four times in it while the constant 2 appears just twice. Therefore, the corresponding identities contain two 1-variables x, u and one 2-variable y. The identities depend on the ordering of appearance of the variables x, u (in the identity) which may be $xxuu, xuxu$ or $xuux$ (we use only normalized identities in which the first occurrence of x always comes before the first occurrence of u). So we have three possible identities: $x(x\backslash uy) \approx uy$, $x(u\backslash xy) \approx uy$ and $x(u\backslash uy) \approx xy$. Note that the first and the last identity may be reduced to $uy \approx uy$, $xy \approx xy$ respectively, and eventually to $z \approx z$.

Just as in the example above, in most cases we may be able to reduce identities obtained, using either one of the (3-sorted) quasigroup axioms, or one of the quasigroup theorems:

$$z/(x\backslash z) \approx x \qquad\qquad (z/y)\backslash z \approx y \tag{12}$$

or otherwise one of the following lemmas:

Lemma 4.1. *If $\circ$ is one of the operations $\cdot, \backslash, /$, and p, q, s and t terms of appropriate sorts, we have:*

$$p \circ s \approx p \circ t \Rightarrow s \approx t \qquad\qquad s \circ q \approx t \circ q \Rightarrow s \approx t.$$

Lemma 4.2. *Let a compound term t appear twice in a coherent quadratic identity $Eq[t]$ and let p be a variable not occurring in $Eq[t]$. Then $Eq[t]$ is equivalent to $Eq[p]$.*

Let J be a quadratic coherent identity. There are several possibilities.

- The identity J can be reduced as above to the trivial identity $x \approx x$ (or, equivalently, to $y \approx y$ or $z \approx z$). We say that J is *reducible*. It is also a theorem of the theory of quasigroups.
- The identity J can be reduced (repeatedly if necessary), but neither to $x \approx x, y \approx y$ nor to $z \approx z$. In this case J is also reducible but is not gemini.
- The identity J cannot be reduced by the above methods. In this case we say that J is *irreducible*. In particular, the trivial identities $x \approx x, y \approx y, z \approx z$ are all irreducible.

The above inductive descent suggests:

Lemma 4.3. *Every coherent quadratic identity is equivalent to an irreducible coherent quadratic identity.*

Also, we have

Theorem 4.1. *If a coherent quadratic identity is gemini, then it is a quasigroup theorem.*

It follows that a gemini identity J can be reduced to $x \approx x$ (or $y \approx y$ or $z \approx z$).

By Theorem 2.1, if a quasigroup satisfies an irreducible coherent quadratic identity J which is not gemini, then the quasigroup is (principally) isotopic to a group. Let $+$ be that group, so

$$\begin{cases} x \cdot y &= f(x) + g(y) \\ x \backslash z &= g^{-1}(-f(x) + z) \\ z/y &= f^{-1}(z - g(y)) \end{cases} \tag{13}$$

for some bijections f and g. Utilizing (13), we transform J into a group identity J' which may but need not be true in all groups. If it is, then J is equivalent to (GI). If it is not, by Theorem 1.2 the operation $+$ has to be commutative:

$$x + y = y + x \tag{14}$$

Using (14), we now transform J' into J''. Again, if J'' is true in all Abelian groups then J is equivalent to (AI). Otherwise (by Theorem 1.2 again), the operation $+$ must be unipotent:

$$x + x = 0 \tag{15}$$

where 0 is the unit of the group $+$. The final transformation of the identity J'', using (15), leads to the identity $0 = 0$, while J is equivalent to (BI).

The above algorithm is a decision procedure which determines whether J is equivalent to (Q), (GI), (AI) or (BI).

Building equalities Eq on $\mathcal{Z}$ systematically, and checking corresponding identities J as above, we get the results presented in Tables 1 and 2.

	Number of identities equivalent to:				
n	(Q)	(GI)	(AI)	(BI)	Total
1	3	0	0	0	3
2	0	0	0	0	0
3	0	0	0	6	6
4	0	0	12	15	27
5	0	12	42	381	435

TABLE 1. Irreducible balanced identities

The entry n in both tables is the number of occurrences of variables in corresponding identities. The entries 6* in the Table 2 indicates 6 identities which are not irreducible but are indispensable in reducing all other reducible identities.

n	Number of identities equivalent to: (Q)	(GI)	(AI)	(BI)	Total
1	3	0	0	0	3
2	6*	0	0	0	6*
3	0	0	0	39	39
4	0	0	180	741	921
5	0	192	4440	31491	36123

TABLE 2. Irreducible quadratic identities (including balanced ones)

The four of them are quasigroup axioms; the remaining two are identities (12) included for the sake of symmetry.

Here we give only the list of shortest balanced identities which can serve as additional axioms defining Q, GI, AI and BI. The list of all 37086 irreducible quadratic identities is given at the web address
`http://www.mi.sanu.ac.rs/~bojanm/quasigroups/identities.zip`

n = 1, identities defining Q:

$$x \approx x \qquad y \approx y \qquad z \approx z$$

n = 3, identities defining BI:

$$\begin{array}{rclrclrcl}
z/(x\backslash w) & \approx & w/(x\backslash z) & (z/y)\backslash w & \approx & (w/y)\backslash z & x(u\backslash z) & \approx & u(x\backslash z)\\
xy/v & \approx & xv/y & x\backslash uy & \approx & u\backslash xy & (z/y)v & \approx & (z/v)y
\end{array}$$

n = 4, identities defining AI (for the consistency of notation we use additional variables p, q, r of sorts $1, 2, 3$ respectively):

$$\begin{array}{rclrcl}
z/(x\backslash uy) & \approx & x(u\backslash z)/y & x\backslash u(p\backslash z) & \approx & p\backslash u(x\backslash z)\\
z/((w/y)\backslash r) & \approx & w/((z/y)\backslash r) & x\backslash(z/y)v & \approx & (xy/v)\backslash z\\
xy/(u\backslash z) & \approx & uy/(x\backslash z) & (z/y)\backslash xv & \approx & (z/v)\backslash xy\\
(z/y)v/q & \approx & (z/q)v/y & (z/(x\backslash w))\backslash r & \approx & (z/(x\backslash r))\backslash w\\
x(u\backslash py) & \approx & p(u\backslash xy) & (z/y)(x\backslash w) & \approx & (w/y)(x\backslash z)\\
(z/(x\backslash w))y & \approx & x((w/y)\backslash z) & (xy/v)q & \approx & (xq/v)y
\end{array}$$

n = 5, identities defining GI:

$$\begin{array}{rclrcl}
z/(x\backslash(w/y)v) & \approx & (z/v)y/(x\backslash w) & x\backslash u((z/y)\backslash w) & \approx & (z/(x\backslash uy))\backslash w\\
z/((w/y)\backslash xv) & \approx & (z/v)(x\backslash w)/y & x\backslash(z/y)(u\backslash w) & \approx & (uy/(x\backslash z))\backslash w\\
z/((xy/v)\backslash w) & \approx & (z/(x\backslash w))y/v & x\backslash(z/(u\backslash w))y & \approx & (w/(x\backslash z))\backslash uy\\
xy/((z/v)\backslash w) & \approx & x((w/y)\backslash z)/v & (z/y)\backslash x(u\backslash w) & \approx & (u(x\backslash z)/y)\backslash w\\
x(u\backslash(z/y)v) & \approx & (x(u\backslash z)/y)v & x((z/y)\backslash uv) & \approx & (xy/(u\backslash z))v\\
x((uy/v)\backslash z) & \approx & (xv/y)(u\backslash z) & (z/y)(x\backslash uv) & \approx & (z/(u\backslash xy))v
\end{array}$$

G.B. Belyavskaya mentions in [3] the following unpublished result of M.M. Gluchov:

Theorem 4.2. *There is no balanced identity with three variables characterizing* AI *among quasigroups.*

The theorem is verified by checking lists of irreducible identities with three variables. More generally,

Theorem 4.3. *There is no quadratic identity with less than three* (*four, five*) *variables characterizing* BI (AI, GI) *among quasigroups.*

On the other hand,

Theorem 4.4 (F.N. Sokhatskii [22])**.** *The following nonquadratic and noncoherent identity with four variables:*

$$(x(u\backslash z)/u)v \approx x(u\backslash(z/u)v)$$

is equivalent to (GI).

The following related problem can be formulated:

Problem 4.1. Find axiom system for Q (GI, AI, BI) with the smallest number of axioms.

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