On dicyclic groups as inner mapping groups of finite loops

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Abstract. Let G be a finite group with a dicyclic subgroup H. We show that if there exist H-connected transversals in G, then G is a solvable group. We apply this result to loop theory and show that if the inner mapping group $I(Q)$ of a finite loop Q is dicyclic, then Q is a solvable loop. We also discuss a more general solvability criterion in the case where $I(Q)$ is a certain type of a direct product.

Keywords: solvable loop; inner mapping group; dicyclic group

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1. Introduction

If Q is a loop, then we have two permutations L_a and R_a on Q defined by $L_a(x) = ax$ and $R_a(x) = xa$ for each $a \in Q$. The permutation group $M(Q) = \langle L_a, R_a \mid a \in Q\rangle$ is called the *multiplication group* of the loop Q. The stabilizer of the neutral element of Q is called the *inner mapping group* of Q and denoted by $I(Q)$. A loop Q is *solvable* if it has a series $1 = Q_0 \subseteq \cdots \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and Q_i/Q_{i-1} is an abelian group for each $1 \leq i \leq n$.

In 1996 Vesanen [9] proved the following

Theorem 1.1. *Let Q be a finite loop. If $M(Q)$ is a solvable group, then Q is a solvable loop.*

After this we are naturally interested in those properties of $I(Q)$ which lead to the solvability of $M(Q)$ and hence, in the finite case, of the loop Q itself. The best results include the following: if the inner mapping group $I(Q)$ of a finite loop Q is nilpotent, dihedral or has order pq, where p and q are prime numbers, then $M(Q)$ is solvable, and hence Q is solvable, too.

The purpose of this paper is to show that if $I(Q)$ is a dicyclic group (see Section 3 for the definition), then Q is again a solvable loop. Furthermore, in Section 4, we prove a more general solvability criterion in the case where $I(Q)$ is a certain type of a direct product.

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2. Connected transversals

Let G be a group, $H \leq G$ and let A and B be left transversals to H in G. We say that A and B are H-*connected*, if $[A, B] \leq H$. We denote by H_G the *core* of H in G (i.e. the largest normal subgroup of G contained in H).

If Q is a loop, then $A = \{L_a \mid a \in Q\}$ and $B = \{R_a \mid a \in Q\}$ are $I(Q)$-connected transversals in $M(Q)$, $M(Q) = \langle A, B\rangle$ and $I(Q)_{M(Q)}$, the core of $I(Q)$ in $M(Q)$, is trivial.

The link between loops and connected transversals is given by

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H and H-connected transversals A and B such that $H_G = 1$ and $G = \langle A, B\rangle$.*

For the proof, see [8, Theorem 4.1].

In the following two lemmas, which we need later, we assume that $H \leq G$ and A and B are H-connected transversals in G.

Lemma 2.2. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

Lemma 2.3. *Assume that $H > 1$, $H_G = 1$ and $H \cap H^a = 1$ for some $a \in A$. Then $A = B$ and $a \in Z(\langle A\rangle)$.*

For the proofs, see [8, Lemma 2.5] and [5, Lemma 2.8].

Theorem 2.4. *Let G be a finite group and $H \leq G$, where H is nilpotent, dihedral or a nonabelian group of order pq ($p \neq q$ are prime numbers). If there exist H-connected transversals A and B in G, then G is solvable.*

For the proof, see [6], [7, Theorem 3.1], [2, Corollary 4.7] and [4, Theorem 2.7].

3. The dicyclic case

Definition 3.1. The *dicyclic group Dic_n* of order $4n$ is given by the presentation

$$Dic_n = \langle a, x \mid a^{2n} = 1,\ x^2 = a^n,\ x^{-1}ax = a^{-1}\rangle.$$

The dicyclic group has a unique involution $u = x^2$ which generates the center of the group. Every normal subgroup of a dicyclic group is cyclic or dicyclic and every quotient is cyclic, dicyclic or dihedral. When n is a power of 2, the dicyclic group is isomorphic to the generalized quaternion group. It is clear that all dicyclic groups are solvable.

In the proof of Theorem 3.3 we need the following solvability criterion by J.G. Carr from 1976 [1].

Theorem 3.2. *Let $G = LN$ be a finite group and L and N subgroups of G. If L is abelian and N has a nilpotent subgroup of index at most 2, then G is solvable.*

Theorem 3.3. *Let G be a finite group and $H \leq G$ dicyclic. If there exist H-connected transversals A and B in G, then G is solvable.*

Proof: Let G be a minimal counterexample. If $H_G > 1$, then we consider G/H_G and its subgroup H/H_G, where H/H_G is either dicyclic, cyclic or dihedral. By using induction or Theorem 2.4, it follows that G/H_G is solvable, hence G is solvable.

Thus we may assume that $H_G = 1$. If H is not maximal in G, then there exists a subgroup T such that $H < T < G$. By Lemma 2.2, $T_G > 1$ and we may consider G/T_G and its subgroup $HT_G/T_G = T/T_G$. As $HT_G/T_G \cong H/H \cap T_G$, we may again conclude that G/T_G is solvable. Since T is solvable by induction, we conclude that G is solvable.

We thus assume that H is a maximal subgroup of G. Assume that there exists an element $a \in A$ such that $H \cap H^a = 1$. By Lemma 2.3, we conclude that $A = B$ and $a \in Z(\langle A \rangle)$. If $H^* = \langle A \rangle \cap H$ is nontrivial, then $1 < H^* \leq H \cap H^a = 1$, a contradiction. Thus $\langle A \rangle \cap H = 1$ and hence $A = \langle A \rangle$ is a group. Now $A' = [A, A] \leq A \cap H = 1$, thus A is an abelian group and $G = AH$. By Theorem 3.2, we conclude that G is solvable.

Now $D = H \cap H^a > 1$ and $G = \langle H, H^a \rangle$ for every $a \in A \setminus H$. If an odd prime p divides $|D|$, then a subgroup $P \leq D$ of order p is normal in H and in H^a, hence $P \trianglelefteq G$, which is not possible as $H_G = 1$. Thus $|D|$ is even, and the unique involution u of H is in D. But then $u \in Z(\langle H, H^a \rangle) = Z(G)$, hence $\langle u \rangle \trianglelefteq G$, which gives us our final contradiction. $\square$

By combining Theorem 3.3 with Theorem 2.1 and Theorem 1.1, we get the following loop theoretical consequence.

Corollary 3.4. *Let Q be a finite loop. If $I(Q)$ is dicyclic, then $M(Q)$ is a solvable group and Q is a solvable loop.*

In 2002 Drápal [2] proved that generalized quaternion groups never occur as inner mapping groups of loops.

Open problem 3.5. Are there loops with dicyclic inner mapping groups?

4. A more general solvability criterion

In the proof of the main result of this section we need the following theorem by Wielandt (see [3, Satz 5.8, p. 285]).

Theorem 4.1. *Let G be a finite group with a nilpotent Hall ω-subgroup H. Then every ω-subgroup of G is contained in a conjugate of H.*

We define a class of finite groups

Definition 4.2. We say that $H \in S^*$ if the following holds: if K is isomorphic to any quotient of H and G is a finite group with K-connected transversals A and B in G, then G is solvable.

Remark. It follows that if $H \in S^*$, then H is solvable.

Theorem 4.3. *Let G be a finite group, $H \leq G$ and $H = S \times L$, where $S \in S^*$, L is abelian and $(|S|, |L|) = 1$. If there exist H-connected transversals A and B in G, then G is solvable.*

Proof: Let G be a minimal counterexample. If $H_G > 1$, then we consider G/H_G and its subgroup H/H_G. By induction, G/H_G is solvable. As $S \in S^*$ is solvable, we conclude that G is solvable, too.

Thus we may assume that $H_G = 1$. If H is not maximal in G, then there exists a subgroup T such that $H < T < G$. By Lemma 2.2, $T_G > 1$ and we may consider G/T_G and its subgroup $T/T_G = HT_G/T_G$. By induction, G/T_G is solvable. Now T is solvable by induction, hence G is solvable.

We thus assume that H is a maximal subgroup of G. We may assume that L is nontrivial. Let P be a Sylow p-subgroup of L for a prime number p. As $H_G = 1$, we conclude that P is a Sylow p-subgroup of G. Thus it follows that L is a Hall subgroup of G. Clearly, $N_G(P) = H = C_G(P)$ and by Burnside normal p-complement theorem, there is a normal p-complement in G for each prime divisor p of $|L|$. It follows that $G = LK$, where K is normal in G and $(|L|, |K|) = 1$.

If $1 \neq a \in A$, then $a = lk$, where $l \in L$ and $k \in K$. Then $aK = lK$ and $(aK)^d = K$, where d divides $|L|$. Thus $a^d \in K$, hence $(a^d)^t = 1$, where t divides $|K|$. It follows that $(a^t)^d = 1$, hence $|a^t|$ divides $|L|$. Since L is an abelian Hall subgroup of G, we may apply Theorem 4.1 and it follows that $a^t \in L^g$ for some $g \in G$. As L^g is abelian, $\langle a^t \rangle$ is normal in $\langle H^g, a \rangle = G$. As $H_G = 1$, we conclude that $a^t = 1$. Since $(d, t) = 1$, there exist integers m and n such that $md + nt = 1$. Thus $a = a^{md+nt} = (a^d)^m (a^t)^n = (a^d)^m \in K$.

We may conclude that $A \cup B \subseteq K$. Clearly $S \leq K$ and thus $K = AS = BS$. By the definition of S^*, K is a solvable group. It follows that $G = LK$ is solvable, too. □

From Theorems 2.4 and 3.3 we know that dihedral and dicyclic groups and groups of order pq ($p \neq q$ are prime numbers) all belong to S^*. Thus we get

Corollary 4.4. *Let G be a finite group, $H \leq G$ and $H = S \times L$, where S is dihedral, dicyclic or a group of order pq ($p \neq q$ are prime numbers), L is abelian and $(|S|, |L|) = 1$. If there exist H-connected transversals A and B in G, then G is solvable.*

By applying Theorems 2.1 and 1.1 we obtain the loop theoretical consequence

Corollary 4.5. *Let Q be a finite loop. If $I(Q) = S \times L$, where S is dihedral, dicyclic or a group of order pq ($p \neq q$ are prime numbers), L is abelian and $(|S|, |L|) = 1$, then Q is solvable.*

References

[1] Carr J.G., *A solubility criterion for factorized groups*, Arch. Math. **27** (1976), 225–231.
[2] Drápal A., *Orbits of inner mapping groups*, Monatsh. Math. **134** (2002), 191–206.
[3] Huppert B., *Endliche Gruppen I*, Springer, Berlin-Heidelberg, 1967.

[4] Leppälä E., Niemenmaa M., *On finite loops whose inner mapping groups are direct products of dihedral groups and abelian groups*, Quasigroups Related Systems **20** (2012), no. 2, 257–260.
[5] Leppälä E., Niemenmaa M., *On finite commutative loops which are centrally nilpotent*, Comment. Math. Univ. Carolin. **56** (2015), no. 2, 139–143.
[6] Mazur M., *Connected transversals to nilpotent groups*, J. Group Theory **10** (2007), 195–203.
[7] Niemenmaa M., *Finite loops with dihedral inner mapping groups are solvable*, J. Algebra **273** (2004), 288–294.
[8] Niemenmaa M., Kepka T., *On multiplication groups of loops*, J. Algebra **135** (1990), 112–122.
[9] Vesanen A., *Solvable loops and groups*, J. Algebra **180** (1996), 862–876.

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