The commingling of commutativity and associativity in Bol loops

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Abstract. Commutative Moufang loops were amongst the first (nonassociative) loops to be investigated; a great deal is known about their structure. More generally, the interplay of commutativity and associativity in (not necessarily commutative) Moufang loops is well known, e.g., the many associator identities and inner mapping identities involving commutant elements, especially those involving the exponent three. Here, we investigate all of this in the variety of Bol loops.

Keywords: Bol; Moufang; loop; commutant; associator

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1. Introduction and a note on Prover9

A *loop* is a set with a single binary operation such that in $x \cdot y = z$, knowledge of any two of x, y, and z specifies the third uniquely, and with a unique two-sided identity element, denoted by 1. We usually write xy instead of $x \cdot y$, and reserve $\cdot$ to have lower priority than juxtaposition among factors to be multiplied; for instance, $x \cdot yz$ stands for $x(yz)$. A *left Bol loop* is a loop satisfying the identity $x(y \cdot xz) = (x \cdot yx)z$; *right Bol loops* satisfy the mirror identity. A left Bol loop that is also a right Bol loop is a *Moufang loop*. There are many well known equivalent loop identities that axiomatize the variety of Moufang loops. We use these without mention in this paper. We refer to left Bol loops simply as *Bol loops* for the balance of the paper. We use the notation x^{-1} to denote the unique 2-sided inverse of x, whose existence is guaranteed in Bol loops.

The *commutant*, $C(L)$, of a loop L is the set of those elements which commute with each element in the loop. That is, $C(L) = \{c : \forall x \in L, cx = xc\}$; it need not be a subloop, even in Bol loops [3]. We define the *associator*, (x, y, z) of x, y, and z, as follows: $xy \cdot z = (x \cdot yz)(x, y, z)$. We say that the set $\{a, b, c\}$ *associates* if the elements a, b, and c associate in any order, that is, if each of the six associators $(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b)$, and (c, b, a) vanishes.

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In Moufang loops there are many well known identities involving associators and commutant elements. We compile some of these identities in this paper. We then investigate settings in which these identities hold in the variety of Bol loops.

We use the standard notation for the right and left translations: $xR(y) = yL(x) = xy$. The *multiplication group*, $\mathrm{Mlt}(L)$, of a loop L is the subgroup of the group of all bijections on L generated by right and left translations. Clearly $\mathrm{Mlt}(L)$ acts as a permutation group on L. The subgroup of $\mathrm{Mlt}(L)$ which fixes 1 is called the *inner mapping group*, is denoted by $\mathrm{I}(L)$, and is generated by the following three families of mappings [1]:

$$T(x) = L(x)^{-1}R(x)$$

$$R(x,y) = R(x)R(y)R(xy)^{-1}$$

$$L(x,y) = L(x)L(y)L(yx)^{-1}.$$

If L is Moufang, then $\mathrm{I}(L)$ is generated by the first two of these three families [1].

Our investigations were aided by the automated reasoning tool Prover9 [4] and by the finite model builder Mace4 [5]. Many authors simply use the Prover9 output file as the proof of a theorem; it is common practice to publish untranslated Prover9 proofs [6]. This is mathematically sound since the program can be made to output a simple *proof object*, which can be independently verified by a short `lisp` program. You may find Prover9 output files (proofs) for each of the theorems that appears without proof in this paper, here:
`http://euclid.nmu.edu/ jophilli/paper-supplements.html`.

2. Commutativity and Moufang loops

We record the following well known, fundamental facts about commutative Moufang loops.

Theorem 2.1.

(1) *The left semi-medial law, $xx \cdot yz = xy \cdot xz$, axiomatizes — in the variety of loops — the (sub)variety of commutative Moufang loops.*
(2) *Let L be a commutative Moufang loop. For all x, y and $z \in L$:*
  (a) *if any one of x, y or z is a cube, then $\{x, y, z\}$ associates,*
  (b) *$(x, y, z)^3$ vanishes,*
  (c) *$R(x, y)^3 = 1$, and*
  (d) *$R(x, y)$ is an automorphism.*

The aim of this paper is to generalize this theorem to the variety of Bol loops. This will require "localizing" both the Moufang law and the commutative law. A careful investigation, in the variety of Bol loops, of the local versions of the Moufang laws will lead us to a useful definition of "Moufang subset" (given in the final section). The local version of the commutative law gives the commutant, which in turn gives the following generalization of Theorem 2.1 (we believe that some of these identities have not yet appeared in the literature).

Theorem 2.2. *Let L be a Moufang loop. Then $\forall x, y \in L$,*

(1) *for $c \in L$ the following are equivalent:*
  (a) $c \in C(L)$,
  (b) $c^2 \cdot xy = cx \cdot cy$,
  (c) $x^2 \cdot cy = xc \cdot xy$.

(2) *if $c \in C(L)$, then*
  (a) $\{c^3, x, y\}$ *and* $\{c, x^3, y\}$ *associate,*
  (b) *each of* $(c, x, y)^3, (x, c, y)^3$, *and* $(x, y, c)^3$ *vanishes,*
  (c) $cR(x, y)^3 = c$ *and* $R(x, c)^3 = R(c, x)^3 = 1$, *and*
  (d) $R(x, c)$ *and* $R(c, x)$ *are automorphisms.*

Proof: The equivalence of (1a) and (1b) is straightforward. For (1a) implies (1c) we note that if c is in $C(L)$ then $x^2 \cdot cy = cx^2c^{-1} \cdot yc = c(x^2c^{-1} \cdot y)c = c(xc^{-1}x \cdot y)c = c(x[c^{-1} \cdot xy])c = (cx)([c^{-1} \cdot xy] \cdot c) = xc \cdot xy$. For (1c) implies (1a), take $y = 1$.

(2a) and (2b) are widely known. The first equality in (2c) is (probably) implicit in Chapter 7 in [1]; a proof also may be found at the website listed above. Proofs of the remaining two equalities in (2c) may be found at the website listed above.

(2d) is trivial (since each of $R(x, c)$ and $R(c, x)$ is a pseudoautomorphism with trivial companion (see [1])). $\square$

The parallel statement to (1c) in Theorem 2.2 if c is in "the third slot" does not hold. To see this, let L be an arbitrary noncommutative Moufang loop, let c be an arbitrary commutant element, and assume that $x^2 \cdot yc = xy \cdot xc$ for all $x, y \in L$. In this case we would have $x^2 \cdot yc = xy \cdot cx = x(yc \cdot x)$, that is, $x \cdot yc = yc \cdot x$. Replacing y with yc^{-1}, we would have $xy = yx$; that is, L would be commutative.

3. Bol loops

Most of Theorem 2.2 does not hold for Bol loops. It is straightforward to construct examples. For instance, the following example is of a 16 element Bol loop in which 1 is a commutant element, but in which none of the following six associators vanishes: $(1^3, 4, 8), (4, 1^3, 8), (4, 1, 8^3), (8^3, 1, 4), (1, 4, 8^3)$, and $(1, 8^3, 4)$.

Example 3.1.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	0	2	5	7	4	6	9	11	8	10	14	12	15	13
2	0	3	1	6	4	7	5	10	8	11	9	13	15	12	14
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	14	13	15	8	10	9	11
5	7	4	6	1	3	0	2	13	12	15	14	10	11	8	9
6	4	7	5	2	0	3	1	14	15	12	13	9	8	11	10
7	6	5	4	3	2	1	0	15	13	14	12	11	9	10	8
8	9	10	11	12	14	13	15	0	1	2	3	4	6	5	7
9	11	8	10	13	12	15	14	1	3	0	2	6	7	4	5
10	8	11	9	14	15	12	13	2	0	3	1	5	4	7	6
11	10	9	8	15	13	14	12	3	2	1	0	7	5	6	4
12	14	13	15	8	9	10	11	4	5	6	7	0	2	1	3
13	12	15	14	9	11	8	10	5	7	4	6	2	3	0	1
14	15	12	13	10	8	11	9	6	4	7	5	1	0	3	2
15	13	14	12	11	10	9	8	7	6	5	4	3	1	2	0

We do not know what happens with the final three cases of (2a) in Theorem 2.2: (x, y, c^3), (x^3, y, c), and (x, y^3, c). Since, in a Bol loop, L, for $c \in C(L)$ and $\forall x, y \in L$, an easy Prover9 check shows that $(x, y, c^3) = 1$ if and only if $(x^3, y, c) = 1$ if and only if $(x, y^3, c) = 1$, we really have only one unresolved case, which we state as an open problem.

Problem 3.2. *If L is a Bol loop and c is a commutant element, must (x, y, c^3) vanish for each x and y in L?*

A lengthy Mace4 search shows that a counterexample, should one exist, must have order at least 42.

Now, consider the following eight element Bol loop in which 1 is a commutant element, but $(1, 2, 3)^3 = 6 = (2, 1, 3)^3$. Thus, (2b) in Theorem 2.2 fails for Bol loops. This counterexample is of minimal order.

Example 3.3.

0	1	2	3	4	5	6	7
1	0	3	2	7	6	5	4
2	3	0	1	5	4	7	6
3	2	5	6	0	7	4	1
4	7	1	0	6	2	3	5
5	6	4	7	2	0	1	3
6	5	7	4	3	1	0	2
7	4	6	5	1	3	2	0

Problem 3.4. *If L is a Bol loop and c is a commutant element, must $(x, y, c)^3$ vanish for each x and y in L?*

A Mace4 search shows that a counterexample, should one exist, must have order at least 28.

The next lemma and example together show that only part of (1) from Theorem 2.2 pushes through to Bol loops.

Lemma 3.5. *Let L be a Bol loop. For $c \in L$, the following are equivalent:*

(1) $c^2 \cdot xy = cx \cdot cy$, $\forall x, y \in L$,
(2) $x^2 \cdot cy = xc \cdot xy$, $\forall x, y \in L$.

Both conditions imply that $c \in C(L)$.

We note here that the first condition in Lemma 3.5 characterizes the so-called *Moufang center* of an arbitrary loop; that is, it describes the set of all those elements that are both in the commutant and Moufang elements (see next section for definition). Thus, each of the equivalent conditions in Lemma 3.5 characterizes the Moufang center in Bol loops.

The following example, of a Bol loop of order 8, shows that both conditions in Lemma 3.5 are stronger than the condition that c be in the commutant. In this Bol loop, 1 is a commutant element, but $(1 \cdot 1) \cdot (2 \cdot 3) \neq (1 \cdot 2) \cdot (1 \cdot 3)$. We note that this example is of minimal size.

Example 3.6.

0	1	2	3	4	5	6	7
1	0	3	2	7	6	5	4
2	3	0	1	5	4	7	6
3	2	5	6	0	7	4	1
4	7	1	0	6	2	3	5
5	6	4	7	2	0	1	3
6	5	7	4	3	1	0	2
7	4	6	5	1	3	2	0

4. Moufang elements in Bol loops

We begin by noting that in the variety of loops, each of the following four identities implies the other three:

$$(A): \quad z(xy \cdot z) = zx \cdot yz \qquad\qquad (C): \quad z(x \cdot zy) = (zx \cdot z)y$$
$$(B): \quad (z \cdot xy)z = zx \cdot yz \qquad\qquad (D): \quad (xz \cdot y)z = x(z \cdot yz)$$

A loop that satisfies any one (hence, all four) of these identities is called a *Moufang loop*. Thus, there are (at least) 12 possible ways to "localize" the Moufang laws:

$$\begin{array}{rlrl}
(A2): & a(xy \cdot a) = ax \cdot ya & (C2): & a(x \cdot ay) = (ax \cdot a)y \\
(A1x): & z(ay \cdot z) = za \cdot yz & (C1x): & z(a \cdot zy) = (za \cdot z)y \\
(A1y): & z(xa \cdot z) = zx \cdot az & (C1y): & z(x \cdot za) = (zx \cdot z)a \\
\\
(B2): & (a \cdot xy)a = ax \cdot ya & (D2): & (xa \cdot y)a = x(a \cdot ya) \\
(B1x): & (z \cdot ay)z = za \cdot yz & (D1x): & (az \cdot y)z = a(z \cdot yz) \\
(B1y): & (z \cdot xa)z = zx \cdot az & (D1y): & (xz \cdot a)z = x(z \cdot az)
\end{array}$$

We will use $(A1x)_L$ to denote the set of elements in a given loop, L, that satisfy *A1x*, and analogously for the other 11 identities. Thus, for a given loop, these 12 different "local Moufang laws" axiomatize 12 different "Moufang subsets" (and indeed none of them has to be a subloop). See [7] for a full account of these in arbitrary loops. In a Bol loop, however, there is more structure.

Theorem 4.1. *Let L be a Bol loop. Then*

(1) $(A1y)_L, (B1x)_L, (C1y)_L, (D1x)_L$ *and* $(D1y)_L$ *coincide, and*
(2) $(A2)_L, (B2)_L$ *and* $(D2)_L$ *coincide.*

Thus, there are at most six different subsets of L defined by one of the 12 localized Moufang laws: $(A2)_L, (A1x)_L, (A1y)_L, (B1y)_L, (C2)_L$ and $(C1x)_L$.

Theorem 4.2. $(A1y)_L$ *is either empty or it is all of L. The former case occurs precisely when L is a proper (i.e., nonMoufang) Bol loop; the latter case occurs precisely when L is Moufang.*

PROOF: If the identity element is in $(A1y)_L$ then clearly L is flexible, hence Moufang. That the identity element is in $(A1y)_L$ precisely when $(A1y)_L$ is nonempty, follows from the proofs on the website listed above. □

Thus, we consider $(A1y)_L$ to be a "trivial" subset, and so in the balance of this section, we consider the five nontrivial Moufang subsets of L: $(A2)_L$, $(A1x)_L$, $(B1y)_L$, $(C2)_L$ and $(C1x)_L$.

Theorem 4.3. *Let L be a Bol loop. Then $(A2)_L$ and $(A1x)_L$ are subloops.*

In the following example, of a Bol loop of order 12, $(C2)_L$ is not a subloop, since 1 and 2 are $(C2)_L$-elements, but $(1 \cdot 2) \cdot (6 \cdot ((1 \cdot 2) \cdot 0)) = 3 \cdot (6 \cdot (3 \cdot 0)) = 3 \cdot (6 \cdot 3) = 3 \cdot 9 = 10 \neq 6 = 9 \cdot 3 = (3 \cdot 6) \cdot 3 = (((1 \cdot 2) \cdot 6) \cdot (1 \cdot 2)) \cdot 0$; that is $1 \cdot 2$ is not a $(C2)_L$-element. This example is of minimal order.

Example 4.4.

0	1	2	3	4	5	6	7	8	9	10	11
1	0	3	2	5	4	7	6	9	8	11	10
2	3	4	5	0	1	8	9	10	11	6	7
3	2	5	4	1	0	9	8	11	10	7	6
4	5	0	1	2	3	10	11	6	7	8	9
5	4	1	0	3	2	11	10	7	6	9	8
6	7	8	9	10	11	0	1	2	3	4	5
7	6	11	10	9	8	1	0	5	4	3	2
8	9	10	11	6	7	2	3	4	5	0	1
9	8	7	6	11	10	3	2	1	0	5	4
10	11	6	7	8	9	4	5	0	1	2	3
11	10	9	8	7	6	5	4	3	2	1	0

In the following example, of a Bol loop of order 16, $(C1x)_L$ is not a subloop, since 1 and 2 are $(C1x)_L$-elements, but $((4\cdot(1\cdot 2))\cdot 4)\cdot 0 = (4\cdot 3)\cdot 4 = 7\cdot 4 = 3 \neq 15 = 4\cdot 7 = 4\cdot(3\cdot 4) = 4\cdot((1\cdot 2)\cdot(4\cdot 0))$; that is, $1\cdot 2$ is not a $(C1x)_L$-element This example is of minimal order.

Example 4.5.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	12	13	14	15	3	2	1	0	11	10	9	8
5	4	8	9	13	12	3	2	14	15	0	1	10	11	7	6
6	8	4	10	14	3	12	1	13	0	15	2	9	7	11	5
7	9	10	4	3	14	13	0	12	1	2	15	8	6	5	11
8	6	5	11	15	2	1	12	0	13	14	3	7	9	10	4
9	7	11	5	2	15	0	13	1	12	3	14	6	8	4	10
10	11	7	6	1	0	15	14	2	3	12	13	5	4	8	9
11	10	9	8	0	1	2	3	15	14	13	12	4	5	6	7
12	13	14	15	11	10	9	8	7	6	5	4	0	1	2	3
13	12	15	14	10	11	8	9	6	7	4	5	1	0	3	2
14	15	12	13	9	8	11	10	5	4	7	6	2	3	0	1
15	14	13	12	8	9	10	11	4	5	6	7	3	2	1	0

Problem 4.6. *If L is a Bol loop, must $(B1y)_L$ be a subloop?*

The remaining theorems and examples in this section address the containment relationships between and among the five nontrivial Moufang subsets in a Bol loop.

The following example, of a Bol loop of order 8, shows that $(C2)$ is the weakest of the five nontrivial, localized Moufang laws (again, in the variety of Bol loops).

In this example, 1 is in $(C2)_L$, but it is not in $(A2)_L, (A1x)_L, (B1y)_L$ or $(C1x)_L$. This example is of minimal order. (Note that an element a in a Bol loop is in $(C2)_L$ if and only if it satisfies the following localized flexible law for all x in L: $ax \cdot a = a \cdot xa$.)

Example 4.7.

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	4	0	6	1	7	3	5
3	5	6	0	7	1	2	4
4	2	1	7	0	6	5	3
5	3	7	1	6	0	4	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

Theorem 4.8. *If L is a Bol loop, then $(A2)_L$ is contained in $(A1x)_L$, and $(A1x)_L$ is contained in both $(C2)_L$ and $(C1x)_L$.*

Theorem 4.9. *If L is a Bol loop, then $(B1y)_L$ is contained in $(C1x)_L$ and $(C2)_L$.*

In the Bol loop in the next example, 1 is in $(A1x)_L$ but is not in $(A2)_L$ or $(B1y)_L$.

Example 4.10.

0	1	2	3	4	5	6	7	8	9	10	11
1	2	0	4	5	3	7	8	6	10	11	9
2	0	1	5	3	4	8	6	7	11	9	10
3	4	5	0	1	2	9	10	11	6	7	8
4	5	3	1	2	0	10	11	9	7	8	6
5	3	4	2	0	1	11	9	10	8	6	7
6	7	8	9	10	11	0	1	2	3	4	5
7	8	6	10	11	9	1	2	0	4	5	3
8	6	7	11	9	10	2	0	1	5	3	4
9	11	10	6	8	7	3	5	4	0	2	1
10	9	11	7	6	8	4	3	5	1	0	2
11	10	9	8	7	6	5	4	3	2	1	0

In the Bol loop in the next example, 1 is in $(C1x)_L$ but is not in $(A1x)_L, (A2)_L$ or $(B1y)_L$.

Example 4.11.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	12	13	14	15	3	2	1	0	11	10	9	8
5	4	8	9	13	12	3	2	14	15	0	1	10	11	7	6
6	7	4	5	2	3	0	1	13	12	15	14	9	8	11	10
7	6	5	4	3	2	1	0	12	13	14	15	8	9	10	11
8	9	10	11	15	14	13	12	0	1	2	3	7	6	5	4
9	8	11	10	14	15	12	13	1	0	3	2	6	7	4	5
10	11	7	6	1	0	15	14	2	3	12	13	5	4	8	9
11	10	9	8	0	1	2	3	15	14	13	12	4	5	6	7
12	13	14	15	11	10	9	8	7	6	5	4	0	1	2	3
13	12	15	14	10	11	8	9	6	7	4	5	1	0	3	2
14	15	12	13	9	8	11	10	5	4	7	6	2	3	0	1
15	14	13	12	8	9	10	11	4	5	6	7	3	2	1	0

There are four unresolved cases, which we state as open problems.

Problem 4.12. *If L is a Bol loop, then:*

(1) *is $(B1y)_L$ is contained in $(A1x)_L$?*
(2) *is $(B1y)_L$ is contained in $(A2)_L$?*
(3) *is $(A2)_L$ contained in $(B1y)_L$?*
(4) *is $(C1x)_L$ contained in $(C2)_L$?*

Regarding this problem, note two things.

(1) In a Bruck loop, the answer to each of these four questions is "yes." (A *Bruck loop* is a Bol loop in which the inverse mapping is an automorphism.) Hence, in Bruck loops, there are exactly five Moufang subsets, and they form a chain.
(2) Four Mace4 searches show that counterexamples, should they exist, for each of the four questions in the previous problem, must have orders at least 24, 24, 24 and 20, respectively.

5. (A2)-elements in Bol loops

An element a in a loop L is called a *Moufang element* if the following two equations are satisfied for each x and y in L: $ax \cdot ya = a(xy \cdot a)$ and $a(x \cdot ay) = (ax \cdot a)y$ [7]. In Bol loops, these two equations are equivalent to each other [7], and so in a Bol loop, an element a is a Moufang element precisely when it is an (A2)-element; this coincides with Florja's earlier definition of Moufang element [2]. We have thus found a local condition that gives us the Bol version of the first part of (2a) in Theorem 2.2:

Theorem 5.1. *Let L be a Bol loop, and let c be in $C(L)$. If any one of a, b or c is a Moufang element, then $\{c^3, a, b\}$ associates.*

The next theorem generalizes the second part of (2a) in Theorem 2.2.

Theorem 5.2. *Let L be a Bol loop, and let c be in $C(L)$. If either of a or c is a Moufang element, then $\{c, a^3, b\}$ associates.*

Since Moufang loops have the inverse property, if $cL(a,b)^3 = c$, then we also have $cR(a,b)^3 = c$. The next theorem, then, is a Bol version of (2c) and (2d) in Theorem 2.2 (thus, it is necessary to check both right and left inner mappings).

Theorem 5.3. *Let L be a Bol loop, and let c be in $C(L)$. If any one of a, b or c is a Moufang element, then*

(1) $cL(a,b)^3 = cR(a,b)^3 = c$ *and* $R(a,c)^3 = R(c,a)^3 = L(a,c)^3 = L(c,a)^3 = 1$,
(2) $R(a,c), R(c,a), L(c,a)$ *and* $L(a,c)$ *are automorphisms.*

The next theorem generalizes (1) in Theorem 2.2; also, cf: Lemma 3.5.

Theorem 5.4. *Let L be a Bol loop, and let c be a Moufang element in L. Then the following are equivalent:*

(1) $c \in C(L)$,
(2) $c^2 \cdot xy = cx \cdot xy \; \forall x, y \in L$,
(3) $x^2 \cdot cy = xc \cdot xy \; \forall x, y \in L$.

There are two unresolved cases, which we state as open problems, in our attempt to generalize Theorem 2.2 to Bol loops.

Problem 5.5. *If L is a Bol loop, if c is a commutant element, and if any one of a, b and c is a Moufang element, then must each of $(c,a,b)^3$, $(a,c,b)^3$, and $(a,b,c)^3$ vanish?*

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