Quantum idempotence, distributivity, and the Yang-Baxter equation

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Abstract. Quantum quasigroups and loops are self-dual objects that provide a general framework for the nonassociative extension of quantum group techniques. They also have one-sided analogues, which are not self-dual. In this paper, natural quantum versions of idempotence and distributivity are specified for these and related structures. Quantum distributive structures furnish solutions to the quantum Yang-Baxter equation.

Keywords: Hopf algebra; quantum group; quasigroup; loop; quantum Yang-Baxter equation; distributive

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1. Introduction

Hopf algebras (or "quantum groups") have been developed over the last few decades as an important extension of the concept of a group, from the category of sets with the cartesian product to a more general symmetric, monoidal category $\mathbf{V}$ [3], [13]. Over the same time period, there has been an intensive parallel development of the theory of quasigroups and loops [15]. Some work has also been done on extending Hopf algebras to non-associative products [1], [8], [9], [12].

Recently, the self-dual concepts of quantum quasigroup and loop have been introduced as a far-reaching unification of Hopf algebras (along with their non-associative extensions) and quasigroups [17]. Consider a *bimagma* (A, ∇, Δ), an object of $\mathbf{V}$ with $\mathbf{V}$-morphisms giving a *magma* structure $\nabla\colon A \otimes A \to A$, and a *comagma* structure $\Delta\colon A \to A \otimes A$, such that Δ is a magma homomorphism. The self-dual definition of a quantum quasigroup requires the invertibility of two dual morphisms: the *left composite*

$$\mathsf{G}\colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A$$

and the *right composite*

$$\reflectbox{G}\colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A\ .$$

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Quantum quasigroups and loops also have one-sided analogues [16]. The definition of a left quantum quasigroup requires only the invertibility of the left composite. Dually, the definition of a right quantum quasigroup requires only the invertibility of the right composite. On the other hand, within the left Hopf algebras of Taft *et al.* [6], [11], [14], the left composite is a section, while the right composite is a retract. (Right Hopf algebras are dual.)

The primary goal of the present work is to initiate investigation of the connections between these structures and the well-known quantum Yang-Baxter equation (QYBE)

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{1.1}$$

[3, §2.2C]. The QYBE applies to an endomorphism

$$R\colon A \otimes A \to A \otimes A$$

of the tensor square of an object A in a symmetric, monoidal category. For a given integer $n > 1$, the notation R^{ij}, for $1 \leq i < j \leq n$, means applying R to the i-th and j-th factors in the n-th tensor power of A. Since the left and right composite morphisms are also endomorphisms of tensor squares, it is natural to seek conditions under which they satisfy the QYBE. Then, as anticipated by B.B. Venkov working in the category of sets with cartesian product [5, §9], the QYBE corresponds generally to various distributivity conditions on the products $\nabla\colon A \otimes A \to A$ appearing in the left and right composites. Indeed, it transpires that distributive counital left and right quantum quasigroups, along with commutative Moufang loops of exponent three, yield solutions to the quantum Yang-Baxter equation. If the left (or right) composite of a bimagma satisfies the QYBE, then the bimagma is said to possess the property of left (or right) quantum distributivity.

In the theory of quasigroups, distributivity and idempotence are closely related. For example, one has the implications of identities

$$x \cdot yz = xy \cdot xz \quad \Rightarrow \quad x \cdot xx = xx \cdot xx \quad \Rightarrow \quad x = xx$$

in a left distributive right quasigroup. Thus a secondary goal of the paper is a study of quantum idempotence in a bimagma (A, ∇, Δ), defined by the requirement that the diagram

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\nabla"] & \\ A \arrow[ur, "\Delta"] \arrow[rr, "1_A"'] & & A \end{tikzcd}$$

be commutative, i.e., the requirement that the comultiplication is a section for the multiplication (compare [7]).

The layout of the paper is as follows. Section 2 recalls the basic definitions of one- and two-sided quasigroups and loops. Section 3 reviews symmetric monoidal categories, and various structures (from magmas through to Hopf algebras) that appear within them. A discussion of quantum quasigroups and loops, along with their one-sided analogues, is given in Section 4. Then quantum idempotence and quantum distributivity are presented in Section 5 and Section 6 respectively.

For algebraic concepts and conventions that are not otherwise discussed in this paper, readers are referred to [18]. In particular, algebraic notation is used throughout the paper, with functions to the right of, or as superfixes to, their arguments. Thus compositions are read from left to right. These conventions serve to minimize the proliferation of brackets.

2. Quasigroups and loops

2.1 Combinatorial or equational quasigroups. Quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. A *loop* is a quasigroup Q with an *identity* element e such that $e \cdot x = x = x \cdot e$ for all x in Q.

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll} \text{(SL)} & x \cdot (x\backslash z) = z\,; & \text{(SR)} & z = (z/x) \cdot x\,; \\ \text{(IL)} & x\backslash(x \cdot z) = z\,; & \text{(IR)} & z = (z \cdot x)/x. \end{array} \tag{2.1}$$

If x and y are elements of a group $(Q, \cdot)$, the left division is given by $x\backslash y = x^{-1}y$, with $x/y = xy^{-1}$ as right division. For an abelian group considered as a combinatorial quasigroup under subtraction, the right division is addition, while the left division is subtraction.

2.2 Equational or combinatorial one-sided quasigroups. Equationally, a *left quasigroup* $(Q, \cdot, \backslash)$ is a set Q equipped with a multiplication and left division satisfying the identities (SL) and (IL) of (2.1). Dually, a *right quasigroup* $(Q, \cdot, /)$ is a set Q equipped with a multiplication and right division satisfying the identities (SR) and (IR) of (2.1). A *left loop* is a left quasigroup with an identity element. Dually, a *right loop* is a right quasigroup with an identity element.

Combinatorially, a left quasigroup $(Q, \cdot)$ is a set Q with a multiplication such that in the equation $a \cdot x = b$, specification of a and b determines x uniquely. In equational terms, the unique solution is $x = a\backslash b$. The combinatorial definition of right quasigroups is dual. If Q is a set, the right projection product $xy = y$ yields a left quasigroup structure on Q, while the left projection product $xy = x$ yields a right quasigroup structure.

3. Structures in symmetric monoidal categories

The general setting for the algebras studied in this paper is a symmetric monoidal category (or "symmetric tensor category" — compare [19, Chapter 11]) $(\mathbf{V}, \otimes, \mathbf{1})$. The standard example is provided by the category $\underline{\underline{K}}$ of vector spaces over a field K, under the usual tensor product. More general concrete examples are provided by varieties $\mathbf{V}$ of entropic (universal) algebras, algebras on which each (fundamental and derived) operation is a homomorphism (compare [4]). These include the category **Set** of sets (under the cartesian product), the category of pointed sets, the category $\underline{\underline{R}}$ of (right) modules over a commutative, unital ring R, the category of commutative monoids, and the category of semilattices.

In a monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, there is an object $\mathbf{1}$ known as the *unit object*. For example, the unit object of $\underline{\underline{K}}$ is the vector space K, while the unit object of **Set** under the cartesian product is a terminal object $\top$, a singleton. For objects A and B in a monoidal category, a *tensor product* object $A \otimes B$ is defined. For example, if U and V are vector spaces over K with respective bases X and Y, then $U \otimes V$ is the vector space with basis $X \times Y$, written as $\{x \otimes y \mid x \in X,\ y \in Y\}$. There are natural isomorphisms with components

$$\alpha_{A,B;C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C),\ \ \rho_A \colon A \otimes \mathbf{1} \to A,\ \ \lambda_A \colon \mathbf{1} \otimes A \to A$$

satisfying certain *coherence* conditions guaranteeing that one may as well regard these isomorphisms as identities [19, p. 67]. Thus the bracketing of repeated tensor products is suppressed in this paper, although the natural isomorphisms ρ and λ are retained for clarity in cases such as the unitality diagram (3.1) below. In the vector space example, adding a third space W with basis Z, one has

$$\alpha_{U,V;W} \colon (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$$

for $z \in Z$, along with $\rho_U \colon x \otimes 1 \mapsto x$ and $\lambda_U \colon 1 \otimes x \mapsto x$ for $x \in X$.

A monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is *symmetric* if there is a given natural isomorphism with *twist* components $\tau_{A,B} \colon A \otimes B \to B \otimes A$ such that $\tau_{A,B}\tau_{B,A} = 1_{A \otimes B}$ [19, pp. 67–8]. One uses $\tau_{U,V} \colon x \otimes y \mapsto y \otimes x$ with $x \in X$ and $y \in Y$ in the vector space example.

3.1 Diagrams. Let A be an object in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Consider the respective *associativity* and *unitality diagrams*

(3.1)
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1_A \otimes \nabla"] \arrow[d, "\nabla \otimes 1_A"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[dr, "\nabla"] & A \otimes \mathbf{1} \arrow[l, "1_A \otimes \eta"'] \arrow[d, "\rho_A"] \\ \mathbf{1} \otimes A \arrow[u, "\eta \otimes 1_A"] \arrow[r, "\lambda_A"'] & A \end{tikzcd}$$

in the category $\mathbf{V}$, the respective dual *coassociativity* and *counitality diagrams*

(3.2)
$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "1_A \otimes \Delta"'] \\ A \otimes A \arrow[u, "\Delta \otimes 1_A"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "1_A \otimes \varepsilon"] \arrow[d, "\varepsilon \otimes 1_A"'] & A \otimes \mathbf{1} \\ \mathbf{1} \otimes A & A \arrow[l, "\lambda_A^{-1}"] \arrow[ul, "\Delta"'] \arrow[u, "\rho_A^{-1}"'] \end{tikzcd}$$

in the category $\mathbf{V}$, the *bimagma diagram*

(3.3)
$$\begin{tikzcd}
 & A \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[ur, "\nabla"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[drr, dashed] & & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "1_A\otimes\tau\otimes 1_A"'] \arrow[urr, dashed] & & A \otimes A \otimes A \otimes A \arrow[u, "\nabla\otimes\nabla"']
\end{tikzcd}$$

in the category $\mathbf{V}$, the *biunital diagram*

(3.4)
$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\nabla"] & \mathbf{1} & & \mathbf{1} \arrow[ll, "1"'] \arrow[dl, "\eta"] \arrow[r, "\Delta"] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\eta\otimes\eta"] \\
A \otimes A \arrow[u, "\varepsilon\otimes\varepsilon"] \arrow[rr, "\nabla"] & & A \arrow[ul, "\varepsilon"'] \arrow[rr, "\Delta"] & & A \otimes A
\end{tikzcd}$$

in the category $\mathbf{V}$, and the *antipode diagram*

(3.5)
$$\begin{tikzcd}
 & A \otimes A \arrow[r, "S\otimes 1_A"] & A \otimes A \arrow[dr, "\nabla"] & \\
A \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbf{1} \arrow[rr, "\eta"'] & & A \\
 & A \otimes A \arrow[r, "1_A\otimes S"'] & A \otimes A \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

in the category $\mathbf{V}$, all of which are commutative diagrams. The arrow across the bottom of the bimagma diagram (3.3) makes use of the twist isomorphism $\tau_{A,A}$ or $\tau\colon A \otimes A \to A \otimes A$.

3.2 Magmas and bimagmas. This paragraph and its successor collect a number of basic definitions of various structures and homomorphisms between them.

Definition 3.1. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A *magma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\nabla\colon A \otimes A \to A$$

known as *multiplication*.

(a.2) Let A and B be magmas in $\mathbf{V}$. Then a *magma homomorphism* $f\colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram

$$\begin{tikzcd} A & A \otimes A \arrow[l, "\nabla"'] \arrow[d, "f \otimes f"] \\ B & B \otimes B \arrow[l, "\nabla"] \end{tikzcd}$$

with $A \arrow[d, "f"'] B$ on the left,

commutes.

(b.1) A *comagma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\Delta\colon A \to A \otimes A$$

known as *comultiplication*.

(b.2) Let A and B be comagmas in $\mathbf{V}$. A *comagma homomorphism* $f\colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\Delta"] & B \otimes B \\ A \arrow[u, "f"] \arrow[r, "\Delta"'] & A \otimes A \arrow[u, "f \otimes f"'] \end{tikzcd}$$

commutes.

(c) A *bimagma* (A, ∇, Δ) in $\mathbf{V}$ is a magma (A, ∇) and comagma (A, Δ) in $\mathbf{V}$ such that the bimagma diagram (3.3) commutes.

Remark 3.2. (a) Commuting of the bimagma diagram (3.3) in a bimagma (A, ∇, Δ) means that

$$\Delta\colon (A, \nabla) \to \big(A \otimes A, (1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla)\big)$$

is a magma homomorphism (commuting of the upper-left solid and dotted quadrilateral), or equivalently, that

$$\nabla\colon \big(A \otimes A, (\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)\big) \to (A, \Delta)$$

is a comagma homomorphism (commuting of the upper-right solid and dotted quadrilateral).

(b) If $\mathbf{V}$ is an entropic variety of universal algebras, the comultiplication of a comagma in $\mathbf{V}$ may be written as

$$\Delta\colon A \to A \otimes A; a \mapsto \big((a^{L_1} \otimes a^{R_1}) \dots (a^{L_{n_a}} \otimes a^{R_{n_a}})\big)w_a \tag{3.6}$$

in a universal-algebraic version of the well-known *Sweedler notation*. In (3.6), the *tensor rank* of the image of a (or any such general element of $A \otimes A$) is the smallest arity n_a of the derived word w_a expressing the image (or general element) in terms of elements of the generating set $\{b \otimes c \mid b, c \in A\}$ for $A \otimes A$. A more compact but rather less explicit version of Sweedler notation, generally appropriate within any concrete monoidal category $\mathbf{V}$, is $a\Delta = a^L \otimes a^R$, with the understanding that the tensor rank of the image is not implied to be 1.

(c) Magma multiplications on an object A of a concrete monoidal category are often denoted by juxtaposition, namely $(a \otimes b)\nabla = ab$, or with $a \cdot b$ as an infix notation, for elements a, b of A.

(d) With the notations of (b) and (c), commuting of the bimagma diagram (3.3) in a concrete bimagma (A, ∇, Δ) amounts to

$$a^L b^L \otimes a^R b^R = (ab)^L \otimes (ab)^R \tag{3.7}$$

for a, b in A.

Definition 3.3. Suppose that A is an object in a symmetric monoidal category $\mathbf{V}$.

(a) A magma (A, ∇) is *commutative* if $\tau\nabla = \nabla$. Thus if $\mathbf{V}$ is concrete, this may be written in the usual form $ba = ab$ for $a, b \in A$.
(b) A comagma (A, Δ) is *cocommutative* if $\Delta\tau = \Delta$. This condition takes the form $a^R \otimes a^L = a^L \otimes a^R$ in Sweedler notation for $a \in A$.
(c) A magma (A, ∇) is *associative* if the associativity diagram (3.1) commutes. In the concrete case, one often writes $ab \cdot c = a \cdot bc$, with infix $\cdot$ binding less strongly than juxtaposition, for a, b, c in A.
(d) A comagma (A, Δ) is *coassociative* if the coassociativity diagram (3.2) commutes. Coassociativity takes the form

$$a^{LL} \otimes a^{LR} \otimes a^R = a^L \otimes a^{RL} \otimes a^{RR} \tag{3.8}$$

when written in Sweedler notation for $a \in A$.

Remark 3.4. (a) In a bimagma (A, ∇, Δ), the concepts of Definition 3.3 may be applied to the respective magma and comagma reducts of A.

(b) Note that the usual Sweedler notation $a\Delta = a_{(1)} \otimes a_{(2)}$, which merely records the linear order of the tensor factors, cannot be used reliably for noncoassociative comultiplications. Indeed, it renders both sides of (3.8) as $a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$.

3.3 Unital structures and Hopf algebras.

Definition 3.5. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A magma (A, ∇) in $\mathbf{V}$ is *unital* if it has a $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ such that the unitality diagram (3.1) commutes.

(a.2) Let A and B be unital magmas in $\mathbf{V}$. Then a *unital magma homomorphism* $f\colon A \to B$ is a magma homomorphism such that the diagram

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathbf{1} \arrow[l, "\eta"'] \arrow[d, "1"] \\ B & \mathbf{1} \arrow[l, "\eta"] \end{tikzcd}$$

commutes.

(b.1) A comagma (A, Δ) in $\mathbf{V}$ is *counital* if it has a $\mathbf{V}$-morphism $\varepsilon\colon A \to \mathbf{1}$ such that the counitality diagram (3.2) commutes.

(b.2) Let A and B be comagmas in $\mathbf{V}$. Then a *counital comagma homomorphism* $f\colon A \to B$ is a comagma homomorphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varepsilon"] & \mathbf{1} \\ A \arrow[u, "f"] \arrow[r, "\varepsilon"'] & \mathbf{1} \arrow[u, "1"'] \end{tikzcd}$$

commutes.

(c) A *biunital bimagma* $(A, \nabla, \Delta, \eta, \varepsilon)$ is a unital magma (A, ∇, η) and counital comagma (A, Δ, ε) such that (A, ∇, Δ) is a bimagma, and the biunital diagram (3.4) commutes.

Remark 3.6. (a) The joint commuting of the bimagma diagram (3.3) and biunital diagram (3.4) in a biunital bimagma $(A, \nabla, \Delta, \eta, \varepsilon)$ means that the comultiplication $\Delta\colon A \to A \otimes A$ is a unital magma homomorphism, or equivalently, that $\nabla\colon A \otimes A \to A$ is a counital comagma homomorphism.

(b) If $\mathbf{V}$ is an entropic variety of universal algebras, where the algebra $\mathbf{1}$ is free on a generating singleton $\{x\}$, then the image of the generator x under the $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ of a unital magma (A, ∇, η) is often written as the element 1 of A.

Definition 3.7. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A *monoid* in $\mathbf{V}$ is an associative unital magma in $\mathbf{V}$.

(b) A *comonoid* in $\mathbf{V}$ is a coassociative counital comagma in $\mathbf{V}$.

(c) A *bimonoid* in $\mathbf{V}$ is defined as an associative, coassociative, and biunital bimagma.

(d) A *Hopf algebra* in $\mathbf{V}$ is a bimonoid A in $\mathbf{V}$ that is equipped with a $\mathbf{V}$-morphism $S\colon A \to A$ known as the *antipode*, such that the antipode diagram (3.5) commutes.

4. Quantum quasigroups and loops

4.1 Basic definitions.

Definition 4.1. Let (A, ∇, Δ) be a bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) On (A, ∇, Δ), the endomorphism

$$\mathsf{G}\colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \tag{4.1}$$

of $A \otimes A$ is known as the *left composite* morphism.

(b) On (A, ∇, Δ), the endomorphism

$$\reflectbox{G}\colon A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A \tag{4.2}$$

of $A \otimes A$ is known as the *right composite* morphism.

Definition 4.2. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) A *left quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the left composite morphism G is invertible.
(b) A *right quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the right composite morphism $\reflectbox{$\mathsf{G}$}$ is invertible.
(c) A *quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ where both G and $\reflectbox{$\mathsf{G}$}$ are invertible.

Definition 4.3. Let $(A, \nabla, \Delta, \eta, \varepsilon)$ be a biunital bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) Suppose that (A, ∇, Δ) is a left quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *left quantum loop*.
(b) Suppose that (A, ∇, Δ) is a right quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *right quantum loop*.
(c) If (A, ∇, Δ) is a quantum quasigroup in $\mathbf{V}$, then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *quantum loop*.

Since these basic definitions are expressed entirely within the structure of a symmetric, monoidal category, their concepts are maintained under the symmetric, monoidal functors which preserve that structure. A typical example of such a functor is the free monoid functor from sets under cartesian products to the category of modules over a commutative ring, with the usual tensor product.

Proposition 4.4. *Suppose that $(\mathbf{V}, \otimes, \mathbf{1_V})$ and $(\mathbf{W}, \otimes, \mathbf{1_W})$ are symmetric monoidal categories. Let $F\colon \mathbf{V} \to \mathbf{W}$ be a symmetric monoidal functor.*

(a) *If (A, ∇, Δ) is a left, right, or two-sided quantum quasigroup in $\mathbf{V}$, then the structure (AF, ∇^F, Δ^F) is a respective left, right, or two-sided quantum quasigroup in $\mathbf{W}$.*

(b) *Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a left, right, or two-sided quantum loop in* $\mathbf{V}$. *Then* $(AF, \nabla^F, \Delta^F, \eta^F, \varepsilon^F)$ *is a respective left, right, or two-sided quantum loop in* $\mathbf{W}$.

Corollary 4.5. *Within the context of Proposition* 4.4, *validity of any one of the commutativity, cocommutativity, associativity, or coassociativity conditions for the left, right, or two-sided quantum quasigroup* (A, ∇, Δ) *implies validity of the corresponding condition for* (AF, ∇^F, Δ^F).

4.2 Relations with other structures. Both quantum quasigroups and quantum loops are self-dual structures. A Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$ includes a bimagma reduct $(A, \nabla, \Delta, \eta, \varepsilon)$ that is a quantum loop. A similar situation holds for various nonassociative generalizations of Hopf algebras considered by various authors [17]. So-called *left Hopf algebras* [6], [11], [14] satisfy all of the requirements for a Hopf algebra listed in Definition 3.7(d), except for the commuting of the lower pentagon in the antipode diagram (3.5). In this situation, the $\mathbf{V}$-morphism S is known as a *left antipode*. In a left Hopf algebra, the left composite is a section, while the right composite is a retract.

Lemma 4.6. *Suppose that* (A, Δ, ε) *is a counital comagma in* $(\mathbf{Set}, \times, \top)$. *Then the comultiplication is the diagonal embedding* $\Delta\colon a \mapsto a \otimes a$. *On the other hand, the diagonal embedding on each set* A *yields a cocommutative, coassociative counital comagma* (A, Δ, ε) *in* $(\mathbf{Set}, \times, \top)$.

Proof: If $\top = \{x\}$, then $a\varepsilon = x$ for each element a of A. Now consider the counitality diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "1_A \otimes \varepsilon"] \arrow[d, "\varepsilon \otimes 1_A"'] & A \otimes \mathbf{1} \\ \mathbf{1} \otimes A & A \arrow[l, "\lambda_A^{-1}"] \arrow[u, "\rho_A^{-1}"'] \arrow[ul, "\Delta"'] \end{tikzcd}$$

Suppose $a\Delta$ is actually $a^L \otimes a^R$ for an element a of A. Then $x \otimes a^R = a^L\varepsilon \otimes a^R = a\Delta(\varepsilon \otimes 1_A) = a\lambda_A^{-1} = x \otimes a$, so $a^R = a$, and similarly $a^L = a$. Thus $a\Delta = a \otimes a$, as required. $\square$

Corollary 4.7. *Left quantum loops and counital left quantum quasigroups in* $(\mathbf{Set}, \times, \top)$ *are cocommutative and coassociative.*

Theorem 4.8 ([16], [17])**.** *Consider the category* $\mathbf{Set}$ *of sets and functions, with the symmetric monoidal category structure* $(\mathbf{Set}, \times, \top)$.

(a) *Counital left or two-sided quantum quasigroups in* $(\mathbf{Set}, \times, \top)$ *are respectively equivalent to left or two-sided quasigroups.*
(b) *Left or two-sided quantum loops in* $(\mathbf{Set}, \times, \top)$ *are respectively equivalent to left or two-sided loops.*

Theorem 4.9. *Consider the symmetric, monoidal category* $(\mathbf{FinSet}, \times, \top)$ *of finite sets under the cartesian product.*

(a) *Left quantum quasigroups in* $(\mathbf{FinSet}, \times, \top)$ *are equivalent to triples* (A, L, R) *that consist of a left quasigroup* A *with an automorphism* L *and endomorphism* R [16].

(b) *Quantum quasigroups in* $(\mathbf{FinSet}, \times, \top)$ *are equivalent to triples* (A, L, R) *consisting of a quasigroup* A *equipped with automorphisms* L *and* R [17].

Corollary 4.10 ([16])**.** *Given a left quasigroup* $(A, \cdot, \backslash)$ *equipped with an automorphism* L *and endomorphism* R*, define* $\nabla \colon A \otimes A \to A; a \otimes b \mapsto ab$ *as a multiplication and* $\Delta \colon A \to A \otimes A; a \mapsto a^L \otimes a^R$ *as a comultiplication. Then* (A, ∇, Δ) *is a left quantum quasigroup in* $(\mathbf{Set}, \times, \top)$.

Corollary 4.11 ([17])**.** *Suppose that* $(A, \cdot, /, \backslash)$ *is a quasigroup equipped with two automorphisms* L *and* R*. Define* $\nabla \colon A \otimes A \to A; a \otimes b \mapsto ab$ *as a multiplication and* $\Delta \colon A \to A \otimes A; a \mapsto a^L \otimes a^R$ *as a comultiplication. Then* (A, ∇, Δ) *is a quantum quasigroup in* $(\mathbf{Set}, \times, \top)$.

5. Quantum idempotence

Definition 5.1. Let (A, ∇, Δ) be a bimagma in a symmetric, monoidal category $\mathbf{V}$. If the diagram

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\nabla"] & \\ A \arrow[ur, "\Delta"] \arrow[rr, "1_A"'] & & A \end{tikzcd}$$

commutes in $\mathbf{V}$, then the bimagma is said to satisfy the condition of *quantum idempotence*.

5.1 Classical idempotence. The first result justifies the terminology of Definition 5.1.

Proposition 5.2. *Let* (A, ∇) *be a magma in the category of sets with the cartesian product. Define* $\Delta \colon A \to A \otimes A; a \mapsto a \otimes a$.

(a) *The structure* (A, ∇, Δ) *is a counital, cocommutative, coassociative bimagma.*

(b) *The bimagma* (A, ∇, Δ) *is quantum idempotent if and only if the magma* (A, ∇) *is idempotent in the classical sense.*

Proof: (a) Use Remark 3.2(d) and Lemma 4.6.

(b) For each element a of A, one has $a\Delta\nabla = (a \otimes a)\nabla = aa$. □

Proposition 5.3. *Suppose that* (A, ∇, Δ) *is a nontrivial quantum-idempotent left quantum quasigroup within the category* $(\mathbf{Set}, \times, \top)$.

(a) *If* (A, ∇, Δ) *is unital, it is not counital.*

(b) *If* (A, ∇, Δ) *is counital, it is not unital.*

Proof: Suppose that (A, ∇, Δ) is both unital and counital. According to Theorem 4.8(a), (A, ∇) is an idempotent, unital left quasigroup. Let a be an element of A. Note that $a\Delta = a \otimes a$ by Lemma 4.6. Then $aa = a\Delta\nabla = a = a1$ implies $a = a\backslash(aa) = a\backslash(a1) = 1$, so that A is trivial. $\square$

5.2 Non-classical quantum idempotence. The next two paragraphs furnish natural non-classical examples of quantum idempotence. The results are readily extended to categories of modules over a commutative ring (or indeed more general entropic varieties), under the tensor product, using the free algebra functor as discussed in Proposition 4.4 and its corollary.

Theorem 5.4. *Let $(A, \cdot, \backslash)$ be a left quasigroup in which the identity*

$$(x\backslash y) \cdot (x\backslash y) = (x \cdot x)\backslash(y \cdot y) \tag{5.1}$$

is satisfied. Define $\Delta\colon A \to A \otimes A; a \mapsto a \otimes a \cdot a$ and $\nabla\colon a \otimes b \mapsto a\backslash b$.

- (a) *The structure (A, ∇, Δ) forms a left quantum quasigroup within the category $(\mathbf{Set}, \times, \top)$.*
- (b) *The bimagma (A, ∇, Δ) is quantum idempotent.*

Proof: (a) Since $(A, \cdot, \backslash)$ is a left quasigroup, then so is $(A, \backslash, \cdot)$. The identity (5.1) guarantees the bimagma condition (3.7). By Corollary 4.10, it follows that (A, ∇, Δ) forms a left quantum quasigroup in $(\mathbf{Set}, \times, \top)$.

(b) For each a in A, one has $a\Delta\nabla = (a \otimes a \cdot a)\nabla = a\backslash(a \cdot a) = a$ by (IL). $\square$

Corollary 5.5. *Under the conditions of Theorem* 5.4*, the bimagma (A, ∇, Δ) is cocommutative if and only if (A, ∇) is classically idempotent.*

Proof: For an element a of A, one has $a\Delta = a \otimes a \cdot a$ and $a\Delta\tau = a \cdot a \otimes a$. Thus $\Delta\tau = \Delta$ if and only if $a \cdot a = a$. $\square$

Remark 5.6. Commutative, diassociative loops (such as abelian groups) satisfy the conditions of Theorem 5.4, along with entropic left quasigroups, including sets equipped with right projections.

5.3 Commutative Moufang loops.

Theorem 5.7. *Suppose that $(A, \cdot, /, \backslash, 1)$ is a commutative Moufang loop. Define $\Delta\colon A \to A \otimes A; a \mapsto a^{-1} \otimes a^{-1}$ and $\nabla\colon a \otimes b \mapsto a \cdot b$.*

- (a) *In the category $(\mathbf{Set}, \times, \top)$, the structure (A, ∇, Δ) forms a unital, commutative and cocommutative quantum quasigroup.*
- (b) *If $(A, \cdot, /, \backslash, 1)$ has exponent 3, the bimagma (A, ∇, Δ) is quantum idempotent.*

Proof: (a) By the commutativity and diassociativity of $(A, \cdot, /, \backslash, 1)$, the inversion mapping $A \to A; a \mapsto a^{-1}$ is an automorphism of the multiplication ∇. By Corollary 4.11, it follows that (A, ∇, Δ) is a quantum quasigroup in $(\mathbf{Set}, \times, \top)$. The remaining statements are immediate.

(b) Consider an element a of A. Then

$$a\Delta\nabla = (a^{-1} \otimes a^{-1})\nabla = a^{-2} = a$$

since $(A, \cdot, /, \backslash, 1)$ has exponent 3. □

Remark 5.8. Consider the context of Theorem 5.7, with $(A, \cdot, /, \backslash, 1)$ as a commutative Moufang loop of exponent 3.

(a) If A is nontrivial, say with non-identity element a, then

$$a\Delta(\Delta \otimes 1_A) = (a^{-1} \otimes a^{-1})(\Delta \otimes 1_A) = a \otimes a \otimes a^{-1}$$

while

$$a\Delta(1_A \otimes \Delta) = (a^{-1} \otimes a^{-1})(1_A \otimes \Delta) = a^{-1} \otimes a \otimes a,$$

so that (A, ∇, Δ) is not coassociative.

(b) The quantum quasigroup (A, ∇, Δ) is associative if and only if the loop $(A, \cdot, /, \backslash, 1)$ is associative.

(c) The quantum quasigroup (A, ∇, Δ) is always unital.

(d) By Proposition 5.3, the quantum quasigroup (A, ∇, Δ) is counital only when A is trivial.

6. Quantum distributivity and the QYBE

Definition 6.1. Suppose that (A, ∇, Δ) is a bimagma in a symmetric, monoidal category.

(a) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum left distributivity* if the left composite G of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (1.1).

(b) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum right distributivity* if the right composite Ə of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (1.1).

(c) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum distributivity* if it has both the left and right quantum distributivity properties.

6.1 Classical distributivity. An analogue of Proposition 5.2 relates the concepts of classical and quantum distributivity. In particular, the bimagma structure of the present result is obtained from Proposition 5.2(a).

Proposition 6.2. *Let (A, ∇) be a magma in the category of sets with the cartesian product. Define $\Delta\colon A \to A \otimes A; a \mapsto a \otimes a$. Then the bimagma (A, ∇, Δ) is quantum left distributive if and only if the magma (A, ∇) is left distributive, in the classical sense that the identity*

$$x(yz) = (xy)(xz) \tag{6.1}$$

is satisfied.

Proof: The left composite in the bimagma (A, ∇, Δ) is $\mathsf{G}\colon a \otimes b \mapsto a \otimes ab$. At the elementary level, the two sides of the quantum Yang-Baxter equation (1.1) appear as the top and bottom halves of

$$\begin{tikzcd}
& x \otimes xy \otimes z \arrow[r, mapsto, "\mathsf{G}^{13}"] & x \otimes xy \otimes xz \arrow[dr, mapsto, "\mathsf{G}^{23}"] & \\
& & & x \otimes xy \otimes (xy)(xz) \\
x \otimes y \otimes z \arrow[uur, mapsto, "\mathsf{G}^{12}"] \arrow[ddr, mapsto, "\mathsf{G}^{23}"'] & & & \\
& & & x \otimes xy \otimes x(yz) \\
& x \otimes y \otimes yz \arrow[r, mapsto, "\mathsf{G}^{13}"'] & x \otimes y \otimes x(yz) \arrow[ur, mapsto, "\mathsf{G}^{12}"'] &
\end{tikzcd}$$

when applied to an element $x \otimes y \otimes z$ of $A \otimes A \otimes A$. It is then apparent that the diagram commutes if and only if the magma is left distributive in the classical sense. □

Corollary 6.3. *Let (A, ∇, Δ) be a nontrivial left quantum distributive left quantum quasigroup within the category $(\mathbf{Set}, \times, \top)$.*

(a) *If (A, ∇, Δ) is unital, it is not counital.*
(b) *If (A, ∇, Δ) is counital, it is not unital.*

Proof: Suppose that (A, ∇, Δ) is both unital and counital. According to Theorem 4.8(a), (A, ∇) is a unital left quasigroup. Let a be an element of A. Note that $a\Delta = a \otimes a$ by Lemma 4.6. Then according to Proposition 6.2, (A, ∇) is a classically left distributive, unital left quasigroup. Now

$$aa = (a1)(a1) = a(11) = a1$$

— with the notation of Remark 3.6(b) — implies

$$a = a\backslash(aa) = a\backslash(a1) = 1,$$

so that A is trivial. □

6.2 Non-classical quantum distributivity.

Proposition 6.4. *Let (A, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, equipped with comultiplication $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$. Then (A, ∇, Δ) is quantum left* *distributive if $LR = RL$ and the identity*

$$x^R(y^Rz) = (x^{RR}y^R)(x^{RL}z) \tag{6.2}$$

is satisfied.

Proof: Consider an element $x \otimes y \otimes z$ of $A \otimes A \otimes A$. Then

$$\begin{aligned}(x \otimes y \otimes z)\mathsf{G}^{23}\mathsf{G}^{13}\mathsf{G}^{12} &= (x \otimes y^L \otimes y^Rz)\mathsf{G}^{13}\mathsf{G}^{12}\\ &= \big(x^L \otimes y^L \otimes x^R(y^Rz)\big)\mathsf{G}^{12}\\ &= x^{LL} \otimes x^{LR}y^L \otimes x^R(y^Rz)\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}(x \otimes y \otimes z)\mathsf{G}^{12}\mathsf{G}^{13}\mathsf{G}^{23} &= (x^L \otimes x^Ry \otimes z)\mathsf{G}^{13}\mathsf{G}^{23}\\ &= (x^{LL} \otimes x^Ry \otimes x^{LR}z)\mathsf{G}^{23}\\ &= x^{LL} \otimes (x^Ry)^L \otimes (x^Ry)^R(x^{LR}z).\end{aligned} \tag{6.4}$$

By Remark 3.2(d), the maps R and L are endomorphisms of the magma (A, ∇). Thus the respective middle factors of (6.3) and (6.4) agree if R and L commute. For commuting magma endomorphisms L and R, the final factors of (6.3) and (6.4) agree if the identity (6.2) is satisfied. □

Corollary 6.5. *Let (A, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, equipped with comultiplication $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$. Then the bimagma (A, ∇, Δ) is quantum left distributive if $LR = RL$ and the identity*

$$x(yz) = (x^Ry)(x^Lz) \tag{6.5}$$

is satisfied.

6.3 Commutative Moufang loops of exponent 3. The following result is a continuation of Theorem 5.7, which showed that commutative Moufang loops of exponent three yield quantum idempotent quantum quasigroups in the category of sets under cartesian products.

Theorem 6.6. *Suppose that $(A, \cdot, /, \backslash, 1)$ is a commutative Moufang loop of exponent 3. Define $\Delta\colon A \to A \otimes A; a \mapsto a^{-1} \otimes a^{-1}$ and $\nabla\colon a \otimes b \mapsto a \cdot b$, as in Theorem 5.7. Then the quantum quasigroup (A, ∇, Δ) is quantum left distributive.*

Proof: With the given comultiplication, the identity (6.5) of Corollary 6.5 takes the form

$$x(yz) = (x^{-1}y)(x^{-1}z). \tag{6.6}$$

Then for elements x, y, z of A, one has

$$\begin{aligned}(x^{-1}y)(x^{-1}z) &= (x^{-1}y)(zx^{-1}) = (x^{-1} \cdot yz)x^{-1} \\ &= x^{-1}(x^{-1} \cdot yz) = x^{-1}x^{-1}(yz) = x(yz)\end{aligned}$$

by sequential application of the commutative, Moufang, commutative, diassociative, and exponent 3 properties of $(A, \cdot, /, \backslash, 1)$. $\square$

Remark 6.7. Manin actually takes the identity $x^2(yz) = (xy)(xz)$ as a defining axiom for commutative Moufang loops, within the class of loops [10, I.1.4(4)] (cf. [2, Theorem II.7B]). Substituting x^{-1} for x and using the exponent 3 condition $(x^{-1})^2 = x$ then produces (6.6) directly.

The commutativity and cocommutativity of (A, ∇, Δ) yield the following extension of Theorem 6.6.

Corollary 6.8. *The quantum quasigroup (A, ∇, Δ) is quantum distributive.*

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