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A graph associated to proper non-small ideals of a commutative ring

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Abstract. In this paper, a new kind of graph on a commutative ring is introduced and investigated. Small intersection graph of a ring R, denoted by $G(R)$, is a graph with all non-small proper ideals of R as vertices and two distinct vertices I and J are adjacent if and only if $I \cap J$ is not small in R. In this article, some interrelation between the graph theoretic properties of this graph and some algebraic properties of rings are studied. We investigated the basic properties of the small intersection graph as diameter, girth, clique number, cut vertex, planar property and independence number. Further, it is shown that the independence number of a small graph of a ring R is equal to the number of its maximal ideals and the domination number of small graph is at most 2.

Keywords: small ideal; small intersection graph; clique number; independence number; domination number; planar property

Classification: 05C40, 05C25, 13A15

1. Introduction

In 1988, Beck [3] introduced the concept of the zero-divisor graph. Since then, others have introduced and studied many researches in this area. One of the most important graphs which have been studied is the intersection graph. Bosak [5] in 1964 defined the intersection graph of semigroups. In 1969, Csákány and Pollák [8] studied the graph of subgroups of a finite group. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [6]. The intersection graph of ideals of rings and submodules of modules has been investigated by several authors ([1], [10], [12], [13] and [14]).

In this paper, we introduce small intersection graph of ideals of a commutative ring, a new kind of intersection graph of rings. If the Jacobson radical of a ring R is zero, then the small intersection graph coincides with the intersection graph which is introduced by Chakrabarty, et al. [6]. The small intersection graph helps us to consider the algebraic properties of rings using graph theoretical tools. In our investigation of $G(R)$, maximal ideals play an important role to find some connections between the graph theoretic properties of this graph and some algebraic properties of rings. For instance, see Theorem 2.6 and 3.6.

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In Section 2, we show that the small intersection graph of a ring R is connected if and only if $|\max(R)| \neq 2$. Also if $G(R)$ is a connected graph, then $diam(G(R)) \leq 2$ and $gr(G(R)) = 3$ provided $G(R)$ contains a cycle. For a ring R, it is proved that $G(R)$ cannot be a complete r-partite graph and $G(R)$ has no cut vertex. Moreover, if R is a ring with finitely many maximal ideals, then $G(R)$ cannot be a complete graph and we give an example of a ring R with infinite maximal ideals such that its small intersection graph is complete. At the end of this section, it is shown that if $G(R)$ contains an end vertex then $|\max(R)| = 2$.

In Section 3, it is shown that if $\omega(G(R))$ is finite, then the number of maximal ideals of R is finite, R is semiperfect and R has finitely many maximal ideals. This enables us to show that, if the set of proper non-small ideals is nonempty and finite, then the set of ideals of R is finite. Also, it is proved that $G(R)$ is a planar graph if and only if either $|\max(R)| = 2$ and $R = R_1 \times R_2$, where R_i ($i = 1, 2$) is a local principle ideal ring with maximal ideal M_i such that $M_i^n = 0$, for some $n \leq 4$ or $|\max(R)| = 3$ and R is semisimple. Among other results, it is shown that the independence number of a small graph of a ring R is equal to the number of its maximal ideals and the domination number of small graph is at most 2.

Throughout this paper R is a commutative ring with unity. Jacobson radical of R, denoted by $J(R)$, is the intersection of all maximal ideals of R and $\max(R)$ denotes the set of all maximal ideals of R. An ideal I of R ($I \leq R$) is small (denoted by $I \ll R$) if $I + K = R$, for some ideal K of R, implies $K = R$. A module M is said to be hollow module if every proper submodule of M is a small submodule.

Let I be an ideal of a ring R. It is said that idempotents of R/I can be lifted, if for every idempotent $a + I \in R/I$, there exists idempotent $e \in R$ such that $a + I = e + I$. A ring R is called *semiperfect* in case $R/J(R)$ is semisimple and every idempotent of $R/J(R)$ can be lifted (see [15]).

A graph G is called *connected*, if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The *diameter* of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. The *girth* of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $w(G)$, is called the *clique number* of G. For r a nonnegative integer, an *r-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a *star* graph. Note that a graph whose vertices-set is empty is a *null* graph and a graph whose edge-set is empty is an *empty* graph.

Let $G = (V, E)$ be a graph. The (open) *neighborhood* $N(v)$ of a vertex v of V is the set of vertices which are adjacent to v. For each $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A set of vertices S in G is a *dominating set*, if $N[S] = V$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G ([9]).

In a graph $G = (V, E)$, a set $S \subseteq V$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G.

2. Some basic properties of $G(R)$

We begin this section with the following remark which will be used in the next theorems and lemmas.

Remark 2.1. (i) Let R be a ring and I, J be two ideals of R. If M is a maximal ideal of R, then $I \cap J \subseteq M$ implies $I \subseteq M$ or $J \subseteq M$.

(ii) Let R be a ring with $\max(R) = \{M_i\}_{i \in I}$ and ν be a proper finite subset of I. Then $\bigcap_\nu M_i$ is not a small ideal of R. Otherwise, if $\bigcap_\nu M_i \ll R$, then $\bigcap_\nu M_i \subseteq M_j$ for each $j \in I \setminus \nu$. Hence $M_i \subseteq M_j$ for some $i \in \nu$, which is a contradiction.

We begin with the key definition of this paper.

Definition 2.2. Let R be a ring. The *small intersection graph* $G(R)$ is the graph with all non small proper ideals of R as vertices and two distinct vertices I and J are adjacent if and only if $I \cap J \not\ll R$.

Proposition 2.3. *Let R be a ring. Then $G(R)$ is a null graph if and only if R is a local ring.*

Proof: The proof is clear. □

Since all definitions of graph theory are for non-null graph, in this paper all graphs are considered non-null ([4]).

Theorem 2.4. *Let R be a ring. Then $G(R)$ is an empty graph if and only if $\max(R) = \{M_1, M_2\}$, where M_1 and M_2 ($M_1 \neq M_2$) are finitely generated hollow R-modules.*

Proof: Let $G(R)$ be an empty graph. If $|\max(R)| = 1$, then $G(R)$ is a null graph by Proposition 2.3, a contradiction. Suppose, $|\max(R)| \geq 3$ and M_1, M_2 and $M_3 \in \max(R)$. By Remark 2.1, M_1 and M_2 are adjacent, a contradiction. So $|\max(R)| = 2$. Let $\max(R) = \{M_1, M_2\}$ with $M_1 \neq M_2$. We show that M_1 and M_2 are hollow R-modules. Since $\frac{R}{M_2} = \frac{M_1 + M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}$, $M_1 \cap M_2$ is a maximal submodule of M_1. We show that this is the only maximal submodule of M_1. Let I be a maximal submodule of M_1. If $I \not\ll R$, then $I \cap M_1 = I$ implies I and M_1 are adjacent in $G(R)$, a contradiction. So $I \ll R$. Hence $I \subseteq J(R) = M_1 \cap M_2$, which implies that $I = M_1 \cap M_2$ by maximality of I. So M_1 is a local R-module with maximal submodule $M_1 \cap M_2$. Now, we show that M_1 is a finitely generated R-module. Let $x \in M_1 \setminus M_2$, so $Rx \not\ll R$ because $Rx \nsubseteq M_1 \cap M_2 = J(R)$. If $Rx \neq M_1$, then $Rx \cap M_1 = Rx$ which implies Rx and M_1 are adjacent in $G(R)$, a contradiction. So $Rx = M_1$. Hence M_1 is a finitely generated local R-module. So M_1 is a finitely generated hollow R-module by [15]. By the similar way M_2 is a finitely generated hollow R-module.

Conversely, let $\max(R) = \{M_1, M_2\}$, where M_1, M_2 are finitely generated hollow R-modules. By a similar argument as above, $M_1 \cap M_2$ is a maximal submodule of M_1 and M_2. Since M_1 and M_2 are local, $M_1 \cap M_2$ is the only maximal submodule of M_1 and M_2. Let $I \neq M_1, M_2$ be a non-small ideal of R. Then $I \subseteq M_1$ or $I \subseteq M_2$. Suppose, without loss of generality, $I \subseteq M_1$. Since M_1 is a finitely generated local R-module, $I \subseteq M_1 \cap M_2 = J(R)$. So $I \ll R$, a contradiction. So the only non-small ideals of R are M_1 and M_2 which are not adjacent. So $G(R)$ is an empty graph. □

In the following we give an example of a ring R with empty $G(R)$.

Example 2.5. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is clear that $\max(R) = \{0 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus 0\}$ and $J(R) = 0$. By drawing the $G(R)$, we see that $G(R)$ is an empty graph with two vertices and M_1, M_2 are hollow.

We are now in a position to show a finer relationship between the number of maximal ideals of R and the connectivity of $G(R)$.

Theorem 2.6. *Let R be a ring. The following statements are equivalent:*

(i) *$G(R)$ is not connected;*
(ii) *$|\max(R)| = 2$;*
(iii) *$G(R) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraphs of $G(R)$.*

Proof: (i) $\Rightarrow$ (ii) Assume that $G(R)$ is not connected. Let G_1 and G_2 be two components of $G(R)$ and I, J be two ideals of R such that $I \in G_1$ and $J \in G_2$. Let M_1, M_2 be maximal ideals of R such that $I \subseteq M_1$ and $J \subseteq M_2$. If $M_1 = M_2$, then $I - M_1 - J$ is a path in $G(R)$ which is a contradiction. So $M_1 \neq M_2$. If $M_1 \cap M_2 \not\ll R$, then $I - M_1 - M_2 - J$ is a path between G_1 and G_2, which is a contradiction. Hence $M_1 \cap M_2 \ll R$, which gives $|\max(R)| = 2$.

(ii) $\Rightarrow$ (iii) Let $|\max(R)| = 2$ and $J(R) = M_1 \cap M_2$, where M_1, M_2 are two maximal ideals of R. Let $G_i = \{I_t \leq R : I_t \subseteq M_i \text{ and } I_t \not\ll R\}$ for $i = 1, 2$. Let I, J be elements of G_1. If I and J are not adjacent then $I \cap J \ll R$, which implies $I \cap J \subseteq M_1 \cap M_2$. Hence $I \cap J \subseteq M_2$, which gives $I \subseteq M_2$ or $J \subseteq M_2$ by Remark 2.1. So $I \ll R$ or $J \ll R$, a contradiction. So G_1 is a complete subgraph of $G(R)$. By the similar way G_2 is a complete subgraph of $G(R)$. Now, we show that there is no path between G_1 and G_2. Suppose, on the contrary, I and J are adjacent for some ideals $I \in G_1$ and $J \in G_2$ (note that each vertex in $G(R)$ is contained in G_1 or G_2). Since $I \cap J \subseteq M_1 \cap M_2 = J(R)$, so $I \cap J \ll R$, a contradiction with adjacency of I and J. So none of elements of G_1 and G_2

are adjacent. Hence $G(R) = G_1 \cup G_2$, where G_i's are disjoint complete subgraph of $G(R)$.

(iii) $\Rightarrow$ (i) It is clear. $\square$

In the following we provide an example of a ring R with two maximal ideals such that $G(R)$ is not connected.

Example 2.7. Let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. It is clear that $\max(R) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4\}$ and $V(G(R)) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4, 0 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$. An inspection shows that $G(R)$ is not connected and $G(R) = G_1 \cup G_2$, where $G_1 = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, 0 \oplus \mathbb{Z}_4\}$ and $G_2 = \{\mathbb{Z}_4 \oplus 2\mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$.

Theorem 2.8. *Let R be a ring and $G(R)$ be a connected graph, then $diam(G(R)) \leq 2$.*

Proof: Let I and J be two non-adjacent vertices of $G(R)$. So $I \cap J \ll R$. Let $I \subseteq M_1$ and $J \subseteq M_2$ for some maximal ideals M_1, M_2 of R. If $I \cap M_2 \not\ll R$, then $I - M_2 - J$ is a path in $G(R)$, hence $d(I, J) = 2$. By the similar way if $J \cap M_1 \not\ll R$, then $d(I, J) = 2$. Suppose $I \cap M_2 \ll R$ and $J \cap M_1 \ll R$. Since $G(R)$ is connected, $|\max(R)| \geq 3$ by Theorem 2.6. Let $M_3 \in \max(R)$. Since $I \cap J \ll R$, so $I \cap J \subseteq J(R) \subseteq M_3$ which implies $I \subseteq M_3$ or $J \subseteq M_3$. Suppose, without loss of generality, $I \subseteq M_3$. Now, we show that $J \cap M_3 \not\ll R$. If $J \cap M_3 \ll R$, then $J \cap M_3 \subseteq J(R) \subseteq M_1$, which implies $J \subseteq M_1$. Hence $J = J \cap M_1 \ll R$, a contradiction. So $J \cap M_3 \not\ll R$. Thus $I - M_3 - J$ is a path in $G(R)$, so $d(I, J) = 2$. $\square$

Theorem 2.9. *Let R be a ring. If $G(R)$ contains a cycle, then $gr(G(R)) = 3$.*

Proof: If $|\max(R)| = 2$, then $G(R)$ is a union of two disjoint complete subgraph by Theorem 2.6. Thus if $G(R)$ contains a cycle, then $gr(G(R)) = 3$. If $|\max(R)| \geq 3$, then by Remark 2.1, $M_1 - M_2 - M_3 - M_1$ is a cycle in $G(R)$, where $M_i \in \max(R)$. So $gr(G(R)) = 3$. $\square$

A vertex x of a connected graph G is a cut vertex of G if there are vertices y and z of G such that x is in every path from y to z (and $x \neq y$, $x \neq z$). Equivalently, for a connected graph G, x is a cut vertex of G if $G - \{x\}$ is not connected.

Theorem 2.10. *Let R be a ring with $G(R)$ connected. Then $G(R)$ has no cut vertex.*

Proof: Let I be a cut vertex of $G(R)$, so $G(R) \setminus \{I\}$ is not connected. Thus there exist vertices J, K such that I lies on every path from K to J. By Theorem 2.8, the shortest path from I to J is of length 2. So $J - I - K$ is a path between J, K. Hence $J \cap K \ll R$, $J \cap I \not\ll R$ and $K \cap I \not\ll R$. At first we show that I is a maximal ideal of R. If not, there exists an ideal L of R such that $I \subset L$ (as I is non-small ideal, L is non-small). Since $J \cap I \subseteq J \cap L$ and $J \cap I \not\ll R$, $J \cap L \not\ll R$. By a similar way $K \cap L \not\ll R$. So $J - L - K$ is a path in $G(R) \setminus \{I\}$, a contradiction. So I is a maximal ideal of R. We claim that there exists a maximal ideal $M_i \neq I$ of R such that $J \not\subseteq M_i$. Otherwise, if $J \subseteq M_i$ for each

$I \neq M_i \in \max(R)$, then $J \subseteq (\bigcap_{M_i \neq I} M_i)$, so $J \cap I \subseteq \bigcap_{M_i \in \max(R)} M_i = J(R)$. Hence $J \cap I \ll R$, a contradiction. By the similar way there exists a maximal ideal $M_j \neq I$ of R such that $K \nsubseteq M_j$. Now, we show that for each $M_t \in \max(R)$, $K \subseteq M_t$ or $J \subseteq M_t$. Because $J \cap K \ll R$, so $J \cap K \subseteq J(R) \subseteq M_t$ for each $M_t \in \max(R)$. So $J \subseteq M_t$ or $K \subseteq M_t$ for each $M_t \in \max(R)$. Since $G(R)$ is connected, $|\max(R)| \geq 3$ by Theorem 2.6. Now, let $I \neq M_i$, $M_j \in \max(R)$ such that $K \nsubseteq M_i$ and $J \nsubseteq M_j$. So $K \subseteq M_j$ and $J \subseteq M_i$. Hence $J - M_i - M_j - K$ is a path in $G(R) \setminus \{I\}$, a contradiction. So $G(R)$ has no cut vertex. $\square$

Theorem 2.11. *Let R be a ring. Then $G(R)$ cannot be a complete r-partite graph ($r \in \mathbb{N}$).*

Proof: Let $G(R)$ be a complete r-partite graph with r parts $V_1, V_2, \ldots, V_r$. By Remark 2.1, M_i and M_j are adjacent, for each $M_i, M_j \in \max(R)$. Hence each V_i contains at most one maximal ideal of R. So by Pigeon hole principle $|\max(R)| \leq r$. Now, we show that $|\max(R)| = r$. Suppose, on the contrary, $\max(R) = \{M_1, M_2, \ldots, M_t\}$, where $t < r$. Let $M_i \in V_i$ for $1 \leq i \leq t$. So V_{t+1} contains no maximal ideal. Since $|\max(R)|$ is finite, $\bigcap_{j \neq i} M_j \not\ll R$, by Remark 2.1. Since $\bigcap_{j \neq i} M_j \cap M_i = J(R) \ll R$, so $\bigcap_{j \neq i} M_j$ and M_i are not adjacent. Hence $\bigcap_{j \neq i} M_j \in V_i$, because $M_i \in V_i$. Let I be a vertex in V_{t+1} and $I \subseteq M_k$ for some $M_k \in \max(R)$. So I is adjacent to M_k. Since $G(R)$ is a complete r-partite graph and $M_k \in V_k$, so I is adjacent to all elements of V_k. Thus I is adjacent to $\bigcap_{j \neq k} M_j$, which is a contradiction, because $I \cap (\bigcap_{j \neq k} M_j) \subseteq M_k \cap (\bigcap_{j \neq k} M_j) = J(R) \ll R$. Hence $|\max(R)| = r$. Now, consider the ideal $J = \bigcap_{i=3}^{r} M_i$. By Remark 2.1, $J \not\ll R$. Since $J \cap M_1 = \bigcap_{i \neq 2} M_i \not\ll R$, J is adjacent to M_1. By the similar way J is adjacent to M_2. So $J \notin V_1, V_2$. Moreover, $J \cap M_i = J \not\ll R$, for each $3 \leq i \leq r$. So J is adjacent to all maximal ideals M_i of R. So $J \notin V_i$ for each $1 \leq i \leq r$, which is a contradiction. $\square$

Theorem 2.12. *Let R be a ring with finitely many maximal ideals. Then*

(i) *there is no vertex in $G(R)$ which is adjacent to every other vertex;*
(ii) *$G(R)$ cannot be a complete graph.*

Proof: (i) Let $\max(R) = \{M_1, M_2, \ldots, M_t\}$. Suppose, on the contrary, there exists a vertex I in $G(R)$ such that I is adjacent to every other vertex. Let $I \subseteq M_i$. By Remark 2.1, $K = \bigcap_{j \neq i} M_j$ is not a small ideal of R. Since I is adjacent to every vertex, I and K are adjacent. Thus $I \cap K \not\ll R$. But $I \cap K \subseteq M_i \cap (\bigcap_{j \neq i} M_j) = J(R)$. So $I \cap K \ll R$, a contradiction. Thus there is no vertex in $G(R)$ which is adjacent to every other vertex.

(ii) By the similar argument as in (i), $G(R)$ cannot be a complete graph. $\square$

The following example shows that the condition "$\max(R)$ is finite" in Theorem 2.12 is not superficial.

Example 2.13. Let $R = \mathbb{Z}$. It is clear that $\max(R)$ is infinite and the only small ideal of R is $\{0\}$. Since for every non-zero ideals I and J of R, $I \cap J \neq \{0\}$, thus

I and J are adjacent in $G(R)$. So $G(R)$ is a complete graph and each vertex is adjacent to every other vertex.

Theorem 2.14. *Let R be a ring. Then the following statements hold:*

(i) *$G(R)$ contains an end vertex if and only if $|\max(R)| = 2$ and $G(R) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraph of $G(R)$ and $|V(G_i)| = 2$ for some $i = 1, 2$;*
(ii) *$G(R)$ cannot be a star graph.*

PROOF: (i) Let I be an end vertex of $G(R)$. Suppose, $|\max(R)| \geq 3$. By Remark 2.1, for each $M_i \in \max(R)$, M_i is adjacent to every other maximal ideals of R, so $deg(M_i) \geq 2$. Hence I is not a maximal ideal of R. Without loss of generality, suppose $I \subseteq M_1$, hence I and M_1 are adjacent. Since $deg(I) = 1$, so the only vertex of $G(R)$ which is adjacent to I is M_1 and there is no maximal ideal $M_i \neq M_1$ of R such that $I \subseteq M_i$. Also $I \cap M_2 \ll R$. So $I \cap M_2 \subseteq M_j$ for each $M_j \neq M_1, M_2$. Thus $I \subseteq M_j$, which is a contradiction. So $|\max(R)| = 2$. By Theorem 2.6, $G(R) = G_1 \cup G_2$, where G_1, G_2 are complete subgraph of $G(R)$. Let $I \in G_i$. Since G_i is a complete subgraph of $G(R)$ and $deg(I) = 1$, $|V(G_i)| = 2$. The converse is clear.

(ii) Let $G(R)$ be a star graph. So $G(R)$ contains an end vertex. So $|\max(R)| = 2$ by (i). By Theorem 2.6, $G(R)$ is not connected, which is a contradiction. So $G(R)$ cannot be a star graph. □

As we see in Example 2.7, $G(R)$ contains an end vertex.

For every nonnegative integer r, the graph G is called r-regular if the degree of each vertex of G is equal to r.

Theorem 2.15. *Let R be a ring. Then the following holds:*

(i) *if I and J are two vertices of $G(R)$ such that $I \subseteq J$, then $deg(I) \leq deg(J)$;*
(ii) *if $G(R)$ is an r-regular graph, then $|\max(R)| = 2$ and $|V(G(R))| = 2r+2$.*

PROOF: (i) Let I and J be two vertices of $G(R)$ such that $I \subseteq J$. Let K be a vertex adjacent to I. So $I \cap K \not\ll R$, which implies $J \cap K \not\ll R$. Thus K is adjacent to J. Hence $deg(I) \leq deg(J)$.

(ii) Let $G(R)$ be an r-regular graph. So for each $M_i \in \max(R)$, $deg(M_i) = r$. By Remark 2.1, M_i is adjacent to all maximal ideals of R, hence $\max(R)$ is finite. Suppose $|\max(R)| \geq 3$. Then $deg(M_1 \cap M_2) \leq deg(M_1)$ by (i) and $deg(M_1 \cap M_2) \neq deg(M_1)$, because $I = \bigcap_{j \neq 2} M_j$ is adjacent to M_1 but I is not adjacent to $M_1 \cap M_2$ (note that $I \not\ll R$ by Remark 2.1). Thus $deg(M_1 \cap M_2) < r$, a contradiction. So $|\max(R)| \leq 2$. If $|\max(R)| = 1$, then R is a local ring hence $G(R)$ is a null graph, which is a contradiction. So $|\max(R)| = 2$ and $G(R)$ is a union of two disjoint complete subgraph G_1, G_2 by Theorem 2.6. Let $\max(R) = \{M_1, M_2\}$ and $M_i \in G_i$. Since $deg(M_1) = r$, so $|G_1| = r + 1$. By the similar way $|G_2| = r + 1$. Hence $|V(G(R))| = 2r + 2$. □

3. Clique number, independence number, domination number and planar property

In this section, we will investigate clique number, independence number, domination number and planar property of the small graph. Now we start with the following proposition.

Proposition 3.1. *Let R be a ring. The following statements hold.*

(i) $\omega(G(R)) \geq |\max(R)|$.
(ii) *If $\omega(G(R)) < \infty$, then the number of maximal ideals of R is finite.*
(iii) $\omega(G(R)) = 1$ *if and only if* $\max(R) = \{M_1, M_2\}$, *where M_1 and M_2 are finitely generated hollow R-modules.*
(iv) *If the number of maximal ideals of R is finite, then* $\omega(G(R)) \geq 2^{|\max(R)|-1} - 1$.

Proof: (i) By Remark 2.1, the subgraph of $G(R)$ with vertex set $\{M_i\}_{M_i \in \max(R)}$ is a complete subgraph of $G(R)$. Hence $\omega(G(R)) \geq |\max(R)|$.

(ii) It is clear by (i).

(iii) It is clear by Theorem 2.4.

(iv) Let $\max(R) = \{M_1, M_2, \dots, M_t\}$ and for each $1 \leq i \leq t$, set $A_i = \{M_1, M_2, \dots, M_{i-1}, M_{i+1}, \dots, M_t\}$. Let $P(A_i)$ be the power set of A_i. For each $X \in P(A_i)$, set $T_X = \bigcap_{T \in X} T$. Then by Remark 2.1, the subgraph of $G(R)$ with vertex set $\{T_X\}_{X \in P(A_i)}$ is a complete subgraph of $G(R)$. Since $|P(A_i) \setminus \{\}| = 2^{|\max(R)|-1} - 1$, so $|\{T_X\}_{X \in P(A_i)}| = 2^{|\max(R)|-1} - 1$. Hence $\omega(G(R)) \geq 2^{|\max(R)|-1} - 1$. □

For any ring R, we use $\mathbb{I}(R)$ and $\mathbb{NSI}(R)$ to denote the set of ideals of R and the set of proper non-small ideals of R, respectively.

We now state our next theorem, which gives us some information on the structure of the rings for which their small intersection graphs have finite clique number.

Theorem 3.2. *Let R be a ring. If $\omega(G(R)) < \infty$, then the following holds.*

(i) *R is semiperfect.*
(ii) *$R = R_1 \times R_2 \times \cdots \times R_t$ where $t \geq 2$, (R_i, M_i) is a local ring and $G(R)$ is finite.*
(iii) $\omega(G(R)) \geq \max\{(\prod_{j=1, j\neq i}^{t} |\mathbb{I}(R_i)|) - 1 : 1 \leq i \leq t\}$.
(iv) *R is artinian.*

Proof: (i) Let R be a ring such that $\omega(G(R)) < \infty$. Then by Proposition 3.1, $\max(R)$ is finite. Hence $R/J(R)$ is semisimple. Now, we show that idempotent of $R/J(R)$ can be lifted. Let $a + J(R)$ be a nonzero idempotent of $R/J(R)$. As $a \notin J(R)$, $a^n \notin J(R)$ for each $n \in \mathbb{N}$. Hence $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \dots$ is a descending chain of non-small proper ideals of R (if $Ra^n = R$, then $a + J(R) = 1 + J(R)$). Since $\omega(G(R)) < \infty$, there exists $n \in \mathbb{N}$ such that $Ra^n = Ra^{n+1}$. Hence $a^n = a^{n+1}r$ for some $r \in R$. Let $e = a^n r^n$. Then $e = (a^{n+1}r)r^n = a^{n+1}r^{n+1}$. This implies that $e = e^2$ and $a + J(R) = a^n + J(R) = a^{n+1}r + J(R) = (a^{n+1} + J(R))(r + J(R)) =$

$(a+J(R))(r+J(R)) = ar+J(R)$. Hence $a+J(R) = (a+J(R))^2 = (a+J(R))^n = (ar+J(R))^n = e+J(R)$. Therefore R is semiperfect.

(ii) By [11, Theorem 23.11], $R = R_1 \times R_2 \times \cdots \times R_t$, where (R_i, M_i) is a local ring for each $1 \leq i \leq t$. As $G(R)$ is non-null, $t \geq 2$, by Proposition 2.3. We show that $G(R)$ is finite. It suffices to show that $\mathbb{I}(R_i)$ is finite for each $1 \leq i \leq t$. Let $\mathbb{I}(R_i)$ be infinite for some $1 \leq i \leq t$. Set

$$\mathcal{C} = \{R_1 \times R_2 \times \cdots \times R_{i-1} \times I \times R_{i+1} \times \cdots \times R_t : \ I \in \mathbb{I}(R_i)\}.$$

Then $\mathcal{C}$ is an infinite clique in $G(R)$, which is a contradiction. Hence $\mathbb{I}(R_i)$ is finite for each $1 \leq i \leq t$. Therefore $\mathbb{I}(R)$ is finite and so $G(R)$ is finite.

(iii) Set

$$\mathcal{C}_j = \{I < R : \ I = I_1 \times I_2 \times \cdots \times I_{j-1} \times R_j \times I_{j+1} \times \cdots \times I_t, \ I_k \in \mathbb{I}(R_k), \\ \forall\ 1 \leq k \neq j \leq t\}$$

for each $1 \leq j \leq t$. As $0 \times 0 \times \cdots \times R_j \times \cdots \times 0 \subseteq I$ for each $I \in \mathcal{C}_j$, $\mathcal{C}_j$ is a clique in R. Since $|\mathcal{C}_j| = (\prod_{i=1, j\neq i}^{t} |\mathbb{I}(R_i)|) - 1$, $\omega(G(R)) \geq \max\{(\prod_{j=1, j\neq i}^{t} |\mathbb{I}(R_i)|) - 1 : 1 \leq i \leq t\}$.

(iv) By the proof of (2), $\mathbb{I}(R)$ is finite, hence R is artinian. □

Now, we are in a position to write one of the most important properties of the ring R, which has been concluded from the graph property of its small intersection graph.

Corollary 3.3. *Let R be a ring such that $\mathbb{NSI}(R) \neq \emptyset$. Then $\mathbb{NSI}(R)$ is finite if and only if $\mathbb{I}(R)$ is finite.*

Proof: Let $\mathbb{NSI}(R) \neq \emptyset$. Then $G(R)$ is non-null. If $\mathbb{NSI}(R)$ is finite, then $\omega(G(R))$ is finite and so $G(R)$ is finite, by Theorem 3.2. Hence $\mathbb{I}(R)$ is finite. The converse is clear. □

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [4]. In the following theorem, rings for which their small intersection graph is planar are characterized.

Theorem 3.4. *Let R be a ring. Then $G(R)$ is a planar graph if and only if one of the following cases occurs.*

(i) $|\max(R)| = 2$ *and* $R = R_1 \times R_2$, *where* R_i $(i = 1, 2)$ *is a local principle ideal ring with maximal ideal* M_i *such that* $M_i^n = 0$ *for some* $n \leq 4$.
(ii) $|\max(R)| = 3$ *and* R *is semisimple.*

Proof: Let $G(R)$ be a planar graph. Then $G(R)$ contains no K_5 as subgraph and so $\omega(G(R)) \leq 4$. By Remark 2.1, $|\max(R)| \leq 3$. Since $G(R)$ is not a null graph, $|\max(R)| \neq 1$. So $|\max(R)| = 2$ *or* 3.

By Theorem 3.2, R is a direct product of local ring. If $|\max(R)| = 2$, then $R = R_1 \times R_2$ where R_i is a local ring with maximal ideal M_i $(i = 1, 2)$. Let $\{x_1, x_2, \ldots, x_n\}$ be a minimal generating set for M_1. If $n \geq 3$, then

$$\{0 \times R_2, x_1R_1 \times R_2, x_2R_1 \times R_2, x_3R_1 \times R_2, M_1 \times R_2\}$$

is a clique with five elements in $G(R)$, a contradiction. Hence $n \leq 2$. Let $M_1 = xR_1 + yR_1$, where $\{x, y\}$ is a minimal generating set for M_1. Then xR_1, yR_1 and $(x+y)R_1$ are distinct ideals of R_1 and

$$\{0 \times R_2, xR_1 \times R_2, yR_1 \times R_2, (x+y)R_1 \times R_2, M_1 \times R_2\}$$

is a clique with five elements in $G(R)$, a contradiction. Hence M_1 is principle and so R_1 is a principle ideal ring. This implies that

$$\mathbb{I}(R_1) = \{M_1^i : \ 1 \leq i \leq n\},$$

where n is the smallest number such that $M_i^n = 0$ and $n \leq 4$.

Similarly, R_2 is a principle ideal ring and

$$\mathbb{I}(R_2) = \{M_2^i : \ 1 \leq i \leq n\},$$

where n is the smallest number such that $M_2^n = 0$ and $n \leq 4$. Hence (i) holds.

If $|\max(R)| = 3$, then $R = R_1 \times R_2 \times R_3$ where R_i is a local ring with maximal ideal M_i, for each $1 \leq i \leq 3$. If R_1 is not a field, then $M_1 \neq 0$ and

$$\{0 \times 0 \times R_3, 0 \times R_2 \times R_3, M_1 \times 0 \times R_3, M_1 \times R_2 \times R_3, R_1 \times 0 \times R_3\}$$

is a clique with five elements in $G(R)$, a contradiction. Hence R_1 is a field. Similarly, R_2 and R_3 are fields. Hence R is semisimple and (ii) holds.

Conversely, assume that (i) holds. Then $G(R) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraphs of $G(R)$ by Theorem 2.6. By (i), $G_i \cong K_n$ (a complete graph with n vertices) where $n \leq 4$ for each $i = 1, 2$. Hence $G(R)$ is planar. If (ii) holds, then by drawing $G(R)$, it is clear that $G(R)$ is planar. □

In the following theorem, for a ring R, the domination number of $G(R)$ is determined.

Theorem 3.5. *Let R be a ring. Then the following hold:*

(i) $\gamma(G(R)) \leq 2$;
(ii) $\max(R)$ *is infinite if and only if* $\gamma(G(R)) = 1$;
(iii) $\max(R)$ *is finite if and only if* $\gamma(G(R)) = 2$.

Proof: (i) As $G(R)$ is non-null, $|\max(R)| \geq 2$. Set $S = \{M_1, M_2\}$ where $M_1, M_2 \in \max(R)$. Let I be a vertex of $G(R)$. If $I \subseteq M_1$ or $I \subseteq M_2$, then $I \cap M_1 \not\ll R$ or $I \cap M_2 \not\ll R$. Hence I is adjacent to M_1 or M_2. Assume that $I \nsubseteq M_1$ and $I \nsubseteq M_2$. If I is not adjacent to M_1, then $I \cap M_1 \ll R$. So $I \cap M_1 \leq M_2$. This gives $I \subseteq M_2$, a contradiction. Similarly, I is adjacent to M_2. Hence $\gamma(G(R)) \leq 2$.

(ii) If $\max(R)$ is infinite, then $R/J(R)$ is not semisimple. Hence $R/J(R)$ has an essential ideal $I/J(R)$, where I is an ideal of R. So I is not small and for each ideal K of R with $J(R) \subset K$ we have $K \cap I \not\ll R$. Let P be a proper non-small ideal of R. As $I \cap (P + J(R)) = J(R) + I \cap P \not\ll R$, $I \cap P \not\ll R$. Hence I is adjacent to every other vertex of $G(R)$, and so $\gamma(G(R)) = 1$.

Conversely, assume that $\gamma(G(R)) = 1$. Hence there is an ideal which is adjacent to every other vertex of $G(R)$. So $\max(R)$ is infinite by Theorem 2.12.

(iii) It is clear from Theorem 2.12 and (ii). $\square$

In the following theorem, it is shown that the independence number of $G(R)$ is equal to $|\max(R)|$, for a ring R with a finite number of maximal ideals.

Theorem 3.6. *Let R be a ring with a finite number of maximal ideals. Then $\alpha(G(R)) = |\max(R)|$.*

Proof: Suppose that $\max(R)$ is finite and $\max(R) = \{M_1, M_2, \ldots, M_n\}$. As $\{\bigcap_{j=1, i\neq j}^{n} M_j\}_{i=1}^{n}$ is an independent set in $G(R)$, $n \leq \alpha(G(R))$. Let $\alpha(G(R)) = m$ and $S = \{I_1, I_2, \ldots, I_m\}$ be a maximal independent set in $G(R)$. For each $I \in S$, $I \not\ll R$. Hence $I \nsubseteq M$ for some $M \in \max(R)$. If $m > n$, then by Pigeon hole principle, there exist $1 \leq i, j \leq n$ and $M \in \max(R)$ such that $I_i \nsubseteq M$ and $I_j \nsubseteq M$. Hence $I_i \cap I_j \nsubseteq M$. As S is an independent set in $G(R)$, I_i and I_j are not adjacent and $I_i \cap I_j \ll R$. Hence $I_i \cap I_j \subseteq M$, a contradiction. This proves that $\alpha(G(R)) = |\max(R)|$. If $\alpha(G(R)) = \infty$, then by a similar argument as above (Pigeon hole principle), we have a contradiction. Hence $\alpha(G(R)) = |\max(R)|$. $\square$

In the ring $R = \mathbb{Z}$, it can be easily seen that $|\max(R)| = \infty$ and $\alpha(G(R)) = 0$. So the condition "$\max(R)$ is finite" is not superficial in Theorem 3.6.

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