The weak Gelfand-Phillips property in spaces of compact operators

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Abstract. For Banach spaces X and Y, let $K_{w^*}(X^*, Y)$ denote the space of all $w^* - w$ continuous compact operators from X^* to Y endowed with the operator norm. A Banach space X has the wGP property if every Grothendieck subset of X is relatively weakly compact. In this paper we study Banach spaces with property wGP. We investigate whether the spaces $K_{w^*}(X^*, Y)$ and $X \otimes_\epsilon Y$ have the wGP property, when X and Y have the wGP property.

Keywords: Grothendieck sets; property wGP

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1. Introduction

A bounded subset A of a Banach space X is called a *limited* subset of X if for each w^*-null sequence (x_n^*) in X^*,

$$\lim_n (\sup\{|x_n^*(x)| : x \in A\}) = 0.$$

If A is a limited subset of X, then $T(A)$ is relatively compact for any operator $T : X \to c_0$ ([2, p. 56], [31, p. 23]).

The space X has the *Gelfand-Phillips* (GP) property if every limited subset of X is relatively compact. The following spaces have the Gelfand-Phillips property: Schur spaces; spaces with w^*-sequential compact dual unit balls (for example subspaces of weakly compactly generated spaces, separable spaces, Asplund spaces (or spaces whose duals have the Radon-Nikodým property), reflexive spaces, and spaces whose duals do not contain ℓ_1); dual spaces X^* with X not containing ℓ_1; Banach spaces with the separable complementation property, i.e., every separable subspace is contained in a complemented separable subspace (for example $L_1(\mu)$ spaces, where μ is a positive measure) [31, p. 31], [2, Proposition], [11, Theorem 3.1 and p. 384], [10, Proposition 5.2], [14, Corollary 5].

The space X has the *BD property* if every limited subset of X is relatively weakly compact [13]. Gelfand-Phillips spaces and spaces not containing ℓ_1 have the BD property ([2, Proposition], [31, p. 47, 67]).

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A subset A of X is called a *Grothendieck set* if every operator $T : X \to c_0$ maps A onto a relatively weakly compact set [23]. A Banach space X has the *weak Gelfand-Phillips* (wGP) property if every Grothendieck subset of X is relatively weakly compact [23].

Every limited set is a Grothendieck set. If X has the wGP property, then X has the BD property. Properties BD and wGP are inherited by closed subspaces.

In [20, Corollary 4.11] it was shown that if $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ and both X and Y have the BD property, then $K_{w^*}(X^*, Y)$ has the BD property. In [23, Theorem 2] it was shown that if X has the wGP property and the Gelfand-Phillips property and Y has the wGP property, then $K_{w^*}(X^*, Y)$ has the wGP property.

In this note we study Banach spaces with the wGP property. We prove that X has the wGP property if and only if every Grothendieck operator $T : Y \to X$ is weakly compact, for every Banach space Y. We show that if X does not contain a copy of ℓ_1, then X has the wGP property.

We prove that if $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then $K_{w^*}(X^*, Y)$ has property wGP if and only if X and Y have property wGP. We also show that if $L_{w^*}(X^*, Y)$ has property wGP, then at least one of the spaces X and Y does not contain ℓ_2.

2. Definitions and notation

Throughout this paper, X, Y, E and F will denote Banach spaces. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X. The canonical basis of ℓ_1 will be denoted by (e_n^*). An operator $T : X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$. The $w^* - w$ continuous (resp. compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$ (resp. $K_{w^*}(X^*, Y)$). The injective (resp. projective) tensor product of two Banach spaces X and Y will be denoted by $X \otimes_\epsilon Y$ (resp. $X \otimes_\pi Y$). The space $X \otimes_\epsilon Y$ can be embedded into the space $K_{w^*}(X^*, Y)$, by identifying $x \otimes y$ with the rank one operator $x^* \to \langle x^*, x\rangle\, y$.

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every limited set is weakly precompact, e.g., see [2, Proposition]. The operator $T : X \to Y$ is *weakly precompact* (*or almost weakly compact*) if $T(B_X)$ is weakly precompact.

A topological space S is called *dispersed* (*or scattered*) if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [28, Main theorem].

The operator $T : X \to Y$ is called *completely continuous* (or *Dunford -Pettis*) if T maps weakly convergent sequences to norm convergent sequences. A Banach space X has the *Dunford-Pettis property* (DPP) if every weakly compact operator $T : X \to Y$ is completely continuous, for every Banach space Y. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can check [5], [6], and [7] for a guide to the extensive classical literature dealing with the DPP.

A Banach space X has the *Grothendieck property* if every w^*-convergent sequence in X^* is weakly convergent [7, p. 179].

A series $\sum x_n$ of elements of X is *weakly unconditionally convergent* (*wuc*) if $\sum |x^*(x_n)| < \infty$, for each $x^* \in X^*$. An operator $T : X \to Y$ is called *unconditionally converging* if T maps weakly unconditionally convergent (wuc) series in X into unconditionally convergent series in Y. A Banach space X has *property* (V) if every unconditionally converging operator T from X to any Banach space Y is weakly compact [27]. $C(K)$ spaces and reflexive spaces have property (V) ([27, Theorem 1, Proposition 7]).

The Banach-Mazur distance $d(E, F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infimum is taken over all isomorphisms T from E onto F. A Banach space E is called an $\mathcal{L}_\infty$-*space* (resp. $\mathcal{L}_1$-*space*) [1, p. 7] if there is a $\lambda \geq 1$ so that every finite dimensional subspace of E is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$ spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) ([1, Proposition 1.26]). The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$- space) ([1, Proposition 1.27]). The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the *DPP* ([1, Corollary 1.30]).

3. Banach spaces with property *wGP*

For better clarity, we start with two diagrams that show the implications among several properties of sets and properties of Banach spaces. The first one shows the implications among the displayed properties of sets which hold in any Banach space. In the second one classes of Banach spaces which are defined or characterized by the validity of an implication (not valid in general) are included.

$$
\begin{tikzcd}
& \text{rel. weakly compact} \arrow[dr, Rightarrow] & & \\
\text{rel. compact} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{weak precompact} \arrow[r, Rightarrow] & \text{Grothendieck} \\
& \text{limited} \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \text{rel. weakly compact} \arrow[dl, Rightarrow, "\text{[Schur]}"'] & \text{Grothendieck} \arrow[l, Rightarrow, "\text{[wGP]}"'] \\
\text{rel. compact} & & \text{weak precompact} \arrow[ul, Rightarrow, "\text{[wsc]}"] \\
& \text{limited} \arrow[ul, Rightarrow, "\text{[GP]}"] \arrow[uu, Rightarrow, "\text{[BD]}"] &
\end{tikzcd}
$$

An operator $T : X \to Y$ is called a *Grothendieck* operator if T^* takes w^*-null sequences in Y^* to weakly null sequences in X^* [9].

The following result [9, Lemma 1.3] appeared with no proof. We include its proof for the convenience of the reader.

Proposition 1. *Let $T : Y \to X$ be an operator. The following are equivalent.*

(i) *$T(B_Y)$ is a Grothendieck set.*

(ii) *If $S : X \to c_0$ is an operator, then $ST : Y \to c_0$ is weakly compact.*

(iii) *T is a Grothendieck operator.*

Proof: (i) $\Leftrightarrow$ (ii) follows from definitions.

(ii) $\Rightarrow$ (iii) Suppose (x_n^*) is w^*-null in X^*. Let $S : X \to c_0$ be defined by $S(x) = (x_n^*(x))$. Note that $S^*(e_n^*) = x_n^*$. Since T^*S^* is weakly compact, $(T^*S^*(e_n^*)) = (T^*(x_n^*))$ is relatively weakly compact. Thus $(T^*(x_n^*))$ is weakly null, and T is a Grothendieck operator.

(iii) $\Rightarrow$ (ii) Let $S : X \to c_0$ be an operator. If $(x_n^*) = (S^*(e_n^*))$, then (x_n^*) is w^*-null and $(T^*(x_n^*))$ is weakly null, since T is a Grothendieck operator. Note that $T^*S^*(B_{\ell_1})$ is contained in the closed and absolutely convex hull of $\{T^*(x_n^*) : n \in \mathbb{N}\}$, which is relatively weakly compact ([7, p. 51]). Thus T^*S^* is weakly compact. Hence ST is weakly compact. $\square$

Corollary 2. (i) ([23, p. 177]) *If X has the Grothendieck property, then X has property wGP if and only if X is reflexive.*

(ii) *If X^* has property (V) and X is not reflexive, then X^* does not have property wGP.*

Proof: (i) Suppose X has the Grothendieck property and the wGP property. Let $i : X \to X$ be the identity map. Then $i^* : X^* \to X^*$ maps w^*-null sequences to weakly null ones, and B_X is a Grothendieck set (by Proposition 1). Hence B_X is relatively weakly compact, and X is reflexive.

(ii) If X^* has property (V), then X^* has the Grothendieck property ([6, p. 40], [18, Corollary 32(ii)]). Apply (i). $\square$

Proposition 3. *Suppose X has the DPP and the Grothendieck property. Then X has property BD if and only if X is weakly sequentially complete.*

Proof: Since X has the DPP and the Grothendieck property, a subset of X is weakly precompact if and only if it is limited ([2, p. 57], [31, Proposition 1.1.7, p. 26], [18, Corollary 5(i)]). Suppose X has property BD. Then X is weakly sequentially complete by the previous results and the second diagram.

Conversely, let A be a limited set in X. Since A is weakly precompact [2, Proposition 4.], A is relatively weakly compact. $\square$

Corollary 4. (i) *If X has property BD, in particular if it has property wGP, then X does not contain ℓ_∞.*

(ii) *If X^* has property BD, in particular if it has property wGP, then ℓ_1 is not complemented in X and c_0 does not embed in X^*.*

(iii) *If X is an infinite dimensional $\mathcal{L}_1$-space, then X^* does not have property BD.*

(iv) *Let (Ω, Σ, μ) be a finite measure space. If Y is an infinite dimensional complemented subspace of $L_1(\mu)$, then Y^* does not have property BD.*

(v) *If X is an infinite dimensional $\mathcal{L}_\infty$-space, then X^{**} does not have property BD.*

(vi) *If X is complemented in a $C(K)$ space, then X^{**} does not have property BD.*

Proof: (i) The space ℓ_∞ has the DPP and the Grothendieck property, and it is not weakly sequentially complete (since it contains c_0). By Proposition 3, ℓ_∞ does not have property BD (see also [31, Example 1.1.8]). Thus if X has property BD, then X does not contain ℓ_∞.

(ii) Suppose X^* has property BD. Then X^* does not contain ℓ_∞ (by (i)). Hence $\ell_1 \overset{c}{\not\hookrightarrow} X$ and $c_0 \not\hookrightarrow X^*$ ([3, Theorem 4], [5, Theorem 10, p. 48]).

(iii) Since X is an infinite dimensional $\mathcal{L}_1$-space, $\ell_1 \overset{c}{\hookrightarrow} X$ ([1, Proposition 1.24]). Apply (ii).

(iv) Since Y is a complemented subspace of $L_1(\mu)$, Y is an $\mathcal{L}_1$-space ([1, Proposition 1.26]). Apply (iii).

(v) If X is an $\mathcal{L}_\infty$-space, then X^* is an $\mathcal{L}_1$-space ([1, Proposition 1.27]). Apply (iii).

(vi) If X is complemented in a $C(K)$ space, then X is an $\mathcal{L}_\infty$-space ([1, Proposition 1.26]). Apply (v). $\square$

The next result gives elementary operator theoretic characterizations of weak precompactness, relative weak compactness, and relative norm compactness for Grothendieck sets.

Theorem 5. *Let X be a Banach space. The following statements are equivalent.*

(i) *For every Banach space Y, every Grothendieck operator $T : Y \to X$ is weakly precompact* (*weakly compact, resp. compact*).

(ii) *Same as* (i) *with $Y = \ell_1$.*

(iii) *Every Grothendieck subset of X is weakly precompact* (*relatively weakly compact, resp. relatively compact*).

Proof: We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) in the relatively weakly compact case. The arguments for the remaining implications of the theorem follow the same pattern.

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Let K be a Grothendieck subset of X and let (x_n) be a sequence in K. Define $T : \ell_1 \to X$ by $T(b) = \sum b_i\, x_i$. Note that $T^*(x^*) = (x^*(x_i))$. Suppose (x_n^*) is w^*-null in X^*. Let $S : X \to c_0$ be defined by $S(x) = (x_n^*(x))$. Note that $S^*(e_n^*) = x_n^*$. Since K is a Grothendieck set, $\{ST(e_n^*) : n \in \mathbb{N}\} = \{S(x_n) : n \in \mathbb{N}\}$ is relatively weakly compact. Note that $ST(B_{\ell_1})$ is contained in the closed and absolutely convex hull of $\{S(x_n) : n \in \mathbb{N}\}$, which is relatively weakly compact ([7, p. 51]). Then ST is weakly compact. Hence $(T^*(x_n^*)) = (T^*S^*(e_n^*))$ is weakly null, and T is a Grothendieck operator. Therefore T is weakly compact. Hence $(T(e_n^*)) = (x_n)$ has a weakly convergent subsequence.

(iii) $\Rightarrow$ (i) Let $T : Y \to X$ be a Grothendieck operator. By Proposition 1, $T(B_Y)$ is a Grothendieck subset of X. Hence $T(B_Y)$ is relatively weakly compact and T is weakly compact. $\square$

Corollary 6. (i) ([23, p. 177]) *If B_{X^*} is w^*-sequentially compact, then X has the wGP property.*

(ii) ([23, p. 177]) *If X is separable, then X has the wGP property.*

(iii) *If X is any subspace of a weakly compactly generated Banach space, then X has the wGP property.*

(iv) *If X^* does not contain copies of ℓ_1, then X has the wGP property.*

(v) *If X^* has the Radon-Nikodým property, then X has the wGP property.*

Proof: (i) Suppose B_{X^*} is w^*-sequentially compact and let $T : Y \to X$ be a Grothendieck operator. Let us show that T is weakly compact. By Gantmacher theorem it is enough to prove that T^* is weakly compact. Let (x_n^*) be a sequence in B_{X^*}. By passing to a subsequence, we can suppose that (x_n^*) is w^*-convergent. Then $(T^*(x_n^*))$ is weakly convergent. Hence T^*, thus T, is weakly compact. Apply Theorem 5.

(ii) If X is separable, then B_{X^*} is w^*-sequentially compact ([5, p. 226]). Apply (i).

(iii) If X is a subspace of a weakly compactly generated Banach space, then (B_{X^*}, w^*) is Eberlein compact (by [15, Theorem 13.20, p. 583], [16, Theorem 1.2.3, p. 12]) and hence it is sequentially compact by Eberlein-Šmulian theorem. Apply (i).

(iv) If X^* does not contain a copy of ℓ_1, then B_{X^*} is weakly precompact, by Rosenthal's ℓ_1 theorem ([5, p. 201]). Thus, by Alaoglu's theorem, B_{X^*} is w^*-sequentially compact ([5, p. 226]). Apply (i).

(v) If X^* has the Radon-Nikodým property, then X is Asplund (by [15, Theorem 11.8, p. 486, Theorem 11.15, p. 496], [16, Theorem 1.1.1, p. 6, p. 33]) and hence B_{X^*} is w^*-sequentially compact as it is fragmented by the norm. □

Corollary 7. *If X has the Grothendieck property and Y has property wGP (resp. every Grothendieck subset of Y is weakly precompact), then every operator $T : X \to Y$ is weakly compact (resp. weakly precompact).*

Proof: We only prove the result when Y has property *wGP*. The proof of the other case is similar. Let $T : X \to Y$ be an operator. Since X has the Grothendieck property, B_X is a Grothendieck set. Then $T(B_X)$ is a Grothendieck set and T is a Grothendieck operator. Since Y has property *wGP*, T is weakly compact, by Theorem 5. □

Hagler and Odell provided an example of a space without a copy of ℓ_1 whose dual ball is not *weak** sequentially compact ([21], [5, p. 239]). Such a space has property *wGP* (by Theorem 9). Thus Corollary 7 generalizes (ii) of [7, Theorem, p. 179].

A Banach space X *has property* (*CBH*) if X^* contains a bounded sequence which has no w^*-convergent convex block [31, p. 40].

Lemma 8 ([32, Lemma 1]). *Let A be a bounded subset of a Banach space Y. Then A is relatively weakly compact if and only if given any sequence (x_n) in A, there exists a sequence (y_n) with $y_n \in co\{x_i : i \geq n\}$ that converges weakly.*

Theorem 9. *If X does not have property (CBH), in particular if X does not contain a copy of ℓ_1, then X has the wGP property.*

Proof: Suppose $T : Y \to X$ is a Grothendieck operator. Let (x_n^*) be a sequence in B_{X^*}. Since X does not have property (CBH), every bounded sequence in X^* has a w^*-convergent convex block. Let (y_n^*) be a w^*-convergent convex block of (x_n^*). Let (k_n) be a strictly increasing sequence of natural numbers and (a_n) a sequence of positive real numbers with $\sum_{i=k_n}^{k_{n+1}-1} a_i = 1$, so that

$$y_n^* = \sum_{i=k_n}^{k_{n+1}-1} a_i x_i^*.$$

Since T is a Grothendieck operator, $(T^*(y_n^*))$ is weakly convergent. Note that $y_n^* \in co\{x_i^* : i \geq n\}$ and $T^*(y_n^*) \in co\{T^*(x_i^*) : i \geq n\}$ for each n. Then $T^*(B_{X^*})$ is relatively weakly compact, by Lemma 8. Then T^*, and thus T, is weakly compact. By Theorem 5, X has the wGP property.

If X does not contain a copy of ℓ_1, then every bounded sequence in X^* has a w^*-convergent convex block ([22, Lemma 3A, p. 4], [31, Lemma 2.2.1, p. 47]). □

The fact that X has the wGP property if X does not contain a copy of ℓ_1 was remarked in [23, p. 178].

Corollary 10. *If X is generated by a weakly precompact set, then X has property wGP.*

Proof: Suppose X is generated by a weakly precompact set. Then X does not have property (CBH), by [31, Corollary 2.3.1]. Apply Theorem 9. □

Schlumprecht constructed a Banach space Y such that Y contains no copies of ℓ_1 and Y does not have the Gelfand-Phillips property ([31, Theorem 5.2.4, p. 149]). The space Y has property wGP by Theorem 9. There is a $C(K)$ space which has the BD property, but does not have the Gelfand-Phillips property ([31, Proposition 5.1.7, p. 144]). This space is generated by a weakly precompact set, and thus it has property wGP by Corollary 10.

Corollary 11. (i) *Suppose K is a dispersed compact Hausdorff space and $\ell_1 \not\hookrightarrow X$. Then $C(K,X)$ has property wGP.*
(ii) *Suppose $\ell_1 \not\hookrightarrow X$, $\ell_1 \not\hookrightarrow Y$, and $L(X,Y^*) = K(X,Y^*)$. Then $X \otimes_\pi Y$ has property wGP.*
(iii) *Suppose $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon-Nykodým property. Then $K_{w^*}(X^*,Y)$ and $X \otimes_\epsilon Y$ have property wGP.*

Proof: (i) Suppose $\ell_1 \not\hookrightarrow C(K)$ and $\ell_1 \not\hookrightarrow X$. Then $\ell_1 \not\hookrightarrow C(K,X)$ [8, Theorem 3.1.2]. Apply Theorem 9.

(ii) Suppose $\ell_1 \not\hookrightarrow X$, $\ell_1 \not\hookrightarrow Y$, and $L(X,Y^*) = K(X,Y^*)$. Then $\ell_1 \not\hookrightarrow X \otimes_\pi Y$, by [12, Theorem 3]. Apply Theorem 9.

(iii) If $\ell_1 \not\hookrightarrow X$ and Y^* has the Radon-Nykodým property, then $\ell_1 \not\hookrightarrow K_{w^*}(X^*, Y)$ ([4, Theorem 1.14], [17, Corollary 3]). By Theorem 9, $K_{w^*}(X^*, Y)$ has property wGP. Hence $X \otimes_\epsilon Y$ has property wGP. $\square$

Proposition 12. *If for every separable subspace L of the Banach space X there exists a complemented subspace M of X containing L such that M has property wGP (resp. BD), then X has property wGP (resp. BD).*

Proof: We only prove the result for property wGP. The proof for property BD is similar. It suffices to show that every countable Grothendieck subset of X is relatively weakly compact. Let A be such a set, and let L be the separable closed linear span of A. Suppose M is a complemented subspace M of X containing L such that M has property wGP. Let P be a projection from X onto M. Note that $A = P(A)$. Then A is a Grothendieck subset of M. Hence A is relatively weakly compact, since M has property wGP. $\square$

Corollary 13 ([23, p. 179])**.** *Let (Ω, Σ, μ) be a positive measure space. Then the space $L_p(\mu)$, $1 \leq p < \infty$, has property wGP.*

Proof: The space $L_p(\mu)$ has the separable complementation property ([25, Theorem 1.b.8], [24, Proposition 1]), i.e., every separable subspace L of $L_p(\mu)$ is contained in a complemented separable subspace M of $L_p(\mu)$. Since M has property wGP (by Corollary 6), $L_p(\mu)$ has property wGP by Proposition 12. $\square$

4. Property wGP in spaces of operators

In this section we give sufficient conditions for the relative weak compactness of Grothendieck subsets of spaces of compact operators. If $K_{w^*}(X^*, Y)$ has property wGP, then X and Y have property wGP, since this property is inherited by subspaces.

We recall the following well-known isometries ([29, p. 60]):
1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$, $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \to T^*)$,
2) $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \to T^{**})$.

If H is a subset of $L_{w^*}(X^*, Y)$, $x^* \in X^*$ and $y^* \in Y^*$, let $H(x^*) = \{T(x^*) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$.

In the proofs of the next two theorems we will need the following results, which give criterions for weak precompactness and relative weak compactness in the space $K_{w^*}(X^*, Y)$.

Theorem 14 ([17, Theorem 2])**.** *Let H be a subset of $K_{w^*}(X^*, Y)$ such that either*

(i) *$H(x^*)$ is relatively weakly compact for each $x^* \in X^*$, and*
(ii) *$H^*(y^*)$ is weakly precompact for each $y^* \in Y^*$, or*
(i)' *$H(x^*)$ is weakly precompact for each $x^* \in X^*$, and*
(ii)' *$H^*(y^*)$ is relatively weakly compact for each $y^* \in Y^*$.*

Then H is weakly precompact.

Theorem 15 ([20, Theorem 4.8])**.** *Suppose that $L_{w^*}(X^*,Y)=K_{w^*}(X^*,Y)$. Let H be a subset of $K_{w^*}(X^*,Y)$ such that*

(i) *$H(x^*)$ is relatively weakly compact for each $x^*\in X^*$, and*
(ii) *$H^*(y^*)$ is relatively weakly compact for each $y^*\in Y^*$.*

Then H is relatively weakly compact.

The following result extends slightly [23, Lemma 1].

Theorem 16. *If X has the wGP property and every Grothendieck set in Y is weakly precompact, or if Y has the wGP property and every Grothendieck set in X is weakly precompact, then every Grothendieck set in $K_{w^*}(X^*,Y)$ is weakly precompact.*

Proof: Suppose X has the *wGP* property and every Grothendieck set in Y is weakly precompact. Let H be a Grothendieck subset of $K_{w^*}(X^*,Y)$. For fixed $x^*\in X^*$, the map $T\to T(x^*)$ is a bounded operator from $K_{w^*}(X^*,Y)$ into Y. It is easily verified that continuous linear images of Grothendieck sets are Grothendieck sets. Then $H(x^*)$ is a Grothendieck subset of Y, hence weakly precompact. For fixed $y^*\in Y^*$, the map $T\to T^*(y^*)$ is a bounded operator from $K_{w^*}(X^*,Y)$ into X. Then $H^*(y^*)$ is a Grothendieck subset of X, hence relatively weakly compact. By Theorem 14, H is weakly precompact. □

Theorem 17. *Suppose $L_{w^*}(X^*,Y)=K_{w^*}(X^*,Y)$. The following statements are equivalent:*

(i) *X and Y have property wGP;*
(ii) *$K_{w^*}(X^*,Y)$ has property wGP;*
(iii) *$X\otimes_\epsilon Y$ has property wGP.*

Proof: Suppose X and Y have property *wGP*. Let H be a Grothendieck subset of $K_{w^*}(X^*,Y)$. By the proof of Theorem 16 and Theorem 15, H is relatively weakly compact. Then $K_{w^*}(X^*,Y)$, and thus $X\otimes_\epsilon Y$, has property *wGP*. That (iii) implies (i) is clear. □

Corollary 18. *Suppose X and Y have property wGP. If Y (or X) has the Schur property, then $L_{w^*}(X^*,Y)=K_{w^*}(X^*,Y)$ has property wGP. Further, $X\otimes_\epsilon Y$ has property wGP.*

Proof: Let $T\in L_{w^*}(X^*,Y)$. Then T is weakly compact (since T^* is w^*-w continuous). If Y (or X) has the Schur property, then T is compact. Apply Theorem 17. □

Corollary 19. *If X has property wGP, then $\ell_1[X]$ has property wGP, where $\ell_1[X]$ is the Banach space of all unconditionally convergent series in X with the norm $\|(x_n)\|=\sup\{\sum|x^*(x_n)|:x^*\in B_{X^*}\}$.*

Proof: Since ℓ_1 has the Schur property and property *wGP* and X has property *wGP*, $\ell_1\otimes_\epsilon X$ has property *wGP* by Corollary 18. Recall that $\ell_1\otimes_\epsilon X\simeq\ell_1[X]$ [30, p. 48]. □

Corollary 20. *Suppose that $W(X,Y) = K(X,Y)$. The following statements are equivalent:*

(i) *X^* and Y have property wGP;*
(ii) *$K(X,Y)$ has property wGP;*
(iii) *$X^* \otimes_\epsilon Y$ has property wGP.*

Proof: Apply Theorem 17 (with X^* instead of X) and the isometries 2) at the beginning of this section. □

Observation 1. If X^* has property wGP and Y^* has the Grothendieck property, then every operator $T : X \to Y$ is weakly compact. To see this, let $T : X \to Y$ be an operator. By Corollary 7, $T^* : Y^* \to X^*$ is weakly compact. Thus T is weakly compact.

We will need the following version of Corollary 18, replacing Y by Y^* and X by X^*.

Corollary 21. *Suppose X^* has property wGP, Y^* has property wGP and the Schur property, and Y^{**} has the Grothendieck property. Then $L(X,Y^*) = K(X,Y^*)$ has property wGP. Consequently, ℓ_1 is not complemented in $X \otimes_\pi Y$.*

Proof: By Observation 1 applied to Y^* instead of Y, $L(X,Y^*) = W(X,Y^*)$. Taking into account the isometries 2) at the beginning of this section, the first statement follows from Corollary 18 applied to X^* instead of X and Y^* instead of Y. Since $L(X,Y^*) \simeq (X \otimes_\pi Y)^*$ ([7, p. 230]) has property wGP, ℓ_1 is not complemented in $X \otimes_\pi Y$, by Corollary 4(ii). □

Corollary 22. *Suppose X^* has property wGP, and Y is the second Bourgain-Delbaen space or $Y = c_0$. Then $L(X,Y^*) = K(X,Y^*)$ has property wGP and ℓ_1 is not complemented in $X \otimes_\pi Y$.*

Proof: The second Bourgain-Delbaen space Y is a separable $\mathcal{L}_\infty$-space, somewhat reflexive, so that $c_0 \not\hookrightarrow Y$ and $Y^* \simeq \ell_1$ [1]. Note that Y and c_0 satisfy the hypotheses of Corollary 21. Apply Corollary 21. □

Corollary 23. *Suppose that $W(X^*,Y^*) = K(X^*,Y^*)$, both X^{**}, Y^* have property wGP, and Y^{**} has the Grothendieck property. Then $L(X^*,Y^*) = K(X^*,Y^*)$ and this space has property wGP. The dual of the space of all nuclear operators $N_1(X,Y)$ also has property wGP. Consequently, ℓ_1 is not complemented in $N_1(X,Y)$.*

Proof: By Corollary 20 applied to X^* instead of X and Y^* instead of Y, $K(X^*,Y^*)$ has property wGP. By Observation 1 applied to X^* instead of X and Y^* instead of Y, $L(X^*,Y^*) = W(X^*,Y^*)$; hence by assumption $L(X^*,Y^*) = K(X^*,Y^*)$. Note that $L(X^*,Y^*) \simeq (X^* \otimes_\pi Y)^*$. It is known that $N_1(X,Y)$ is a quotient of $X^* \otimes_\pi Y$ ([30, p. 41]). Hence the dual of $N_1(X,Y)$ is a closed subspace of $(X^* \otimes_\pi Y)^*$, so it inherits property wGP of $(X^* \otimes_\pi Y)^* \simeq L(X^*,Y^*)$. Apply Corollary 4(ii). □

Theorem 24. *Suppose that $L_{w^*}(X^*, Y)$ has property wGP. Then X and Y have property wGP and either $\ell_2 \not\hookrightarrow X$ or $\ell_2 \not\hookrightarrow Y$. If moreover Y is a dual space Z^*, the condition $\ell_2 \not\hookrightarrow Y$ implies $\ell_1 \not\hookrightarrow Z$.*

Proof: Suppose that $L_{w^*}(X^*, Y)$ has property wGP. Then X and Y have property wGP, since property wGP is inherited by closed subspaces. Suppose $\ell_2 \hookrightarrow X$ and $\ell_2 \hookrightarrow Y$. Then $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ by [19, Theorem 20]. Since $c_0 \hookrightarrow L_{w^*}(X^*, Y)$ and X and Y do not have the Schur property, $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$ by [19, Corollary 2]. This contradiction proves the first assertion.

Now suppose $Y = Z^*$ and $\ell_1 \hookrightarrow Z$. Then $L_1 \hookrightarrow Z^*$ ([26, Theorem 3.4], [5, p. 212]). Also, the Rademacher functions span ℓ_2 inside of L_1, hence $\ell_2 \hookrightarrow Z^*$. $\square$

A similar argument shows that if $L_{w^*}(X^*, Y)$ has property wGP, then X and Y have property wGP and either $\ell_p \not\hookrightarrow X$ or $\ell_q \not\hookrightarrow Y$, for $1 < p' \leq q < \infty$, where p and p' are conjugate ($c_0 \hookrightarrow K_{w^*}(X^*, Y)$ by [19, Theorem 20] applied to $G = \ell_q$).

Corollary 25. *Suppose that $W(X, Y)$ has property wGP. Then X^* and Y have property wGP and either $\ell_1 \not\hookrightarrow X$ or $\ell_2 \not\hookrightarrow Y$. If moreover Y is a dual space Z^*, the condition $\ell_2 \not\hookrightarrow Y$ implies $\ell_1 \not\hookrightarrow Z$.*

Proof: Apply Theorem 24 and the isometries 2) at the beginning of this section. $\square$

Corollary 26. *Suppose that X and Y have the DPP and Y^{**} has the Grothendieck property. The following statements are equivalent:*

(i) *X^* and Y^* have property wGP and either $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$;*
(ii) *$L(X, Y^*) = K(X, Y^*)$ has property wGP.*

Proof: (i) $\Rightarrow$ (ii) Suppose $\ell_1 \not\hookrightarrow Y$. Since Y also has the DPP, Y^* has the Schur property ([6, Theorem 3]). Hence $L(X, Y^*) = K(X, Y^*)$ has property wGP by Corollary 21. Suppose $\ell_1 \not\hookrightarrow X$. Then X^* has the Schur property [6]. By looking at adjoints, $W(X, Y^*) = K(X, Y^*)$. By Corollary 20 applied to Y^* instead of Y, $K(X, Y^*)$ has property wGP. By Observation 1 applied to Y^* instead of Y, $L(X, Y^*) = W(X, Y^*)$. Thus $L(X, Y^*) = K(X, Y^*)$ has property wGP.

(ii) $\Rightarrow$ (i) Apply Corollary 25. $\square$

Observation 2. If $T : Y \to X^*$ is an operator such that $T^*|_X$ is compact, then T is compact. To see this, let $T : Y \to X^*$ be an operator such that $T^*|_X$ is compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*-convergent to x^{**}. Then $(T^*(x_\alpha)) \xrightarrow{w^*} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$. Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively compact. Therefore $T^*(B_{X^{**}})$ is relatively compact, and thus T is compact.

It follows that if $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$.

Corollary 27. *Suppose that $X = C(K_1)$, $Y = C(K_2)$, where K_1 and K_2 are infinite compact Hausdorff spaces. If K_1 or K_2 is dispersed, then $L(X, Y^*) = K(X, Y^*)$ has property wGP. Further, ℓ_1 is not complemented in $X \otimes_\pi Y$.*

Proof: Since Y^* is an $L_1(\mu)$ space, it has property wGP by Corollary 13. Similarly, X^* has property wGP. Suppose that K_2 is dispersed. Then $\ell_1 \not\hookrightarrow C(K_2)$, $Y^{**} = C(K_2)^{**}$ is isomorphic to $\ell_\infty(I)$ for some set I (by [28, Main theorem]), and Y^{**} has the Grothendieck property ([7, p. 156]). By Corollary 26, $L(X, Y^*) = K(X, Y^*)$ has property wGP. Now suppose that K_1 is dispersed. A similar argument shows that X^{**} has the Grothendieck property and $L(Y, X^*) = K(Y, X^*)$ has property wGP. Then $L(X, Y^*) = K(X, Y^*)$ has property wGP (by Observation 2 and Corollary 20). Apply Corollary 4(ii). $\square$

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