Finite actions on the Klein four-orbifold and prism manifolds

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Abstract. We describe the finite group actions, up to equivalence, which can act on the orbifold $\Sigma(2,2,2)$, and their quotient types. This is then used to consider actions on prism manifolds $M(b,d)$ which preserve a longitudinal fibering, but do not leave any Heegaard Klein bottle invariant. If $\varphi\colon G \to \mathrm{Homeo}(M(b,d))$ is such an action, we show that $M(b,d) = M(b,2)$ and $M(b,2)/\varphi$ fibers over a certain collection of 2-orbifolds with positive Euler characteristic which are covered by $\Sigma(2,2,2)$. For the standard actions, we compute the fundamental group of $M(b,2)/\varphi$ and indicate when it is a Seifert fibered manifold.

Keywords: finite group action; prism 3-manifold; equivalence of actions; orbifold; Klein four-group

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1. Introduction

Let M be a manifold and let G be a finite group. A *G-action on M* is a monomorphism $\varphi\colon G \to \mathrm{Homeo}\,(M)$ where $\mathrm{Homeo}\,(M)$ is the group of self homeomorphisms of M. Two group actions $\varphi\colon G \to \mathrm{Homeo}\,(M)$ and $\varphi'\colon G' \to \mathrm{Homeo}\,(M')$ are equivalent if there is a homeomorphism $h\colon M \to M'$ such that $\varphi'(G') = h\circ\varphi(G)\circ h^{-1}$. If $\varphi\colon G \to \mathrm{Homeo}\,(M)$ is an action, we obtain an orbifold covering map $\nu_\varphi\colon M \to M/\varphi$, and the orbifold M/φ is referred to as the quotient type of the action.

Let $\varphi : G \to \mathbb{S}^2$ be a finite orientation preserving geometric action on the two-sphere $\mathbb{S}^2$. If G is isomorphic to the *Klein four-group* $\mathbb{Z}_2\times\mathbb{Z}_2$, then the quotient space $\mathbb{S}^2/\varphi$ is called the *Klein four-orbifold*, and is denoted by $\Sigma(2,2,2)$. There are numerous references discussing the Klein four-group, such as [1].

In this work we begin by classifying, up to equivalence, the finite groups which act on the Klein four-orbifold $\Sigma(2,2,2)$ and their quotient types. The orbifold fundamental group of $\Sigma(2,2,2)$ is the Klein four-group $\mathbb{Z}_2\times\mathbb{Z}_2$. We show that if $\varphi\colon G \to \mathrm{Homeo}\,(\Sigma(2,2,2))$ is a G-action on $\Sigma(2,2,2)$, then G is isomorphic to one of the following groups: $\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $Dih(\mathbb{Z}_3)$ or $Dih(\mathbb{Z}_6)$. Furthermore, there is only one equivalence class for each quotient type.

We then consider finite group actions on prism manifolds $M(b,d)$ which preserve the longitudinal fibering. With this fibering, $M(b,d)$ fibers over the orbifold

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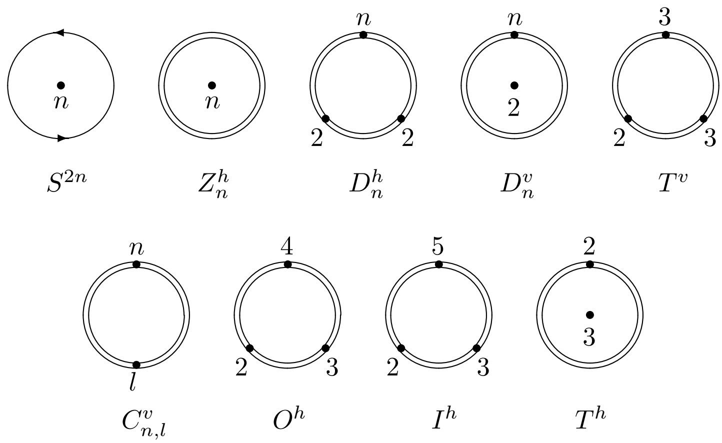
$\Sigma(2,2,d)$. In [2] we classified these actions, together with their quotient types, when the actions leave a Heegaard Klein bottle invariant. Actions can fail to preserve a Heegaard Klein bottle when $d = 2$ and the induced action on $\Sigma(2,2,2)$ is either $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ or $\mathrm{Dih}(\mathbb{Z}_6)$. We consider these actions, and show that the orbifold quotients fiber over the following 2-orbifolds: $\Sigma(2,3,3)$, T^h, $\Sigma(2,3,4)$, T^v, or O^h, all of which are covered by $\Sigma(2,2,2)$. For the standard actions, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ and $\mathrm{Dih}(\mathbb{Z}_6)$ on $M(b,2)$, we compute the fundamental groups of the quotient orbifolds. In certain cases, these orbifold quotients are Seifert fibered manifolds, and these are described as well.

We now define a prism manifold. Let $T = S^1 \times S^1$ be a torus where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is viewed as the set of complex numbers of norm 1 and $I = [0,1]$. The twisted I-bundle over a Klein bottle is the quotient space $W = T \times I/(u,v,t) \simeq (-u,\bar{v},1-t)$. Let D^2 be a unit disk with $\partial D^2 = S^1$ and let $V = S^1 \times D^2$ be a solid torus. Then the boundary of both V and W is a torus $S^1 \times S^1$. For relatively prime integers b and d, there exist integers a and b such that $ad - bc = -1$. The *prism manifold* $M(b,d)$ is obtained by identifying the boundary of V to the boundary of W by the homeomorphism $\psi\colon \partial V \to \partial W$ defined by $\psi(u,v) = (u^a v^b, u^c v^d)$ for $(u,v) \in \partial V = S^1 \times S^1$. The integers b and d determine $M(b,d)$, up to homeomorphism. An embedded Klein bottle K in $M(b,d)$ is called a *Heegaard Klein bottle* if for any regular neighborhood $N(K)$ of K, $N(K)$ is a twisted I-bundle over K and the closure of $M(b,d) - N(K)$ is a solid torus.

There are five orientable 2-orbifolds with a positive Euler number. All of them have underlying space a 2-sphere with the cone points indicated in the notation (see [7]). They are $\Sigma(2,2,n) = D_n$, $\Sigma(2,3,3) = T$, $\Sigma(2,3,4) = O$, $\Sigma(2,3,5) = I$, and $\Sigma(n,l) = C_{n,l}$.

The following is a list of all the nine non-orientable 2-orbifolds with a positive Euler characteristic.



2. Finite group actions on $\Sigma(2,2,2)$

In this section, we classify the finite groups (up to equivalence) which act on the orbifold $\Sigma(2,2,2)$ and their quotient types. It is convenient to view all topological spaces in the PL-category. Thus, all homeomorphisms map a vertex to a vertex, an edge to an edge, and a face to a face. In this sense, we initially view $\mathbb{S}^2$ as a tetrahedron which has four triangles or faces: $\triangle 124$, $\triangle 314$, $\triangle 234$ and $\triangle 321$. Then each triangle is further subdivided (barycentric subdivision) in order to describe several finite group actions on $\mathbb{S}^2$. See Figure 1.

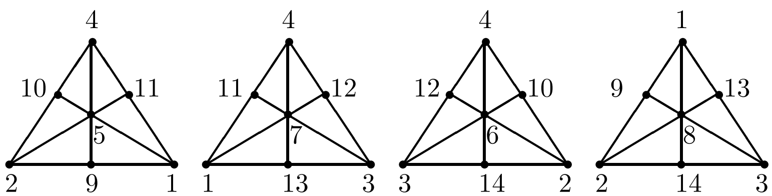


Figure 1

We view $\mathbb{S}_4$ as a group generated by $a = (1,2)(6,7)(10,11)(13,14)$ and $b = (2,4,3)(5,7,8)\ (9,11,13)(10,12,14)$ in $\mathbb{S}_{14}$. We can see that a is a reflection on the circle containing vertices 4, 5, 9, 8, 3 and 12 in $\mathbb{S}^2$. On the other hand, b is a $120°$ rotation about the axis passing through vertices 1 and 6. It is easy to check $ab = (1,2,4,3)(5,6,7,8)(9,10,12,13)(11,14)$. Although ab reverses an orientation, the map ab is called an *improper rotation*. The group generated by a and b is isomorphic to the symmetric group on 4 letters and we write $\mathbb{S}_4 = \langle a, b | a^2 = b^3 = (ab)^4 = 1\rangle$.

Consider an antipodal map $i = (1,6)(2,7)(3,5)(4,8)(9,12)(10,13)(11,14)$ on $\mathbb{S}^2$. It is easy to check that

$$ia = (1,7)(2,6)(3,5)(4,8)(9,12)(10,14)(11,13) = ai \text{ and}$$
$$ib = (1,6)(2,8,3,7,4,5)(9,14,13,12,11,10) = bi.$$

As a result, a, b and i generate a group $\mathbb{S}_4 \times \mathbb{Z}_2 = \langle a, b, i | a^2 = b^3 = (ab)^4 = i^2 = 1, [a,i] = [b,i] = 1\rangle$.

Lemma 1. *Let $O = \langle ai, b\rangle$. Then O is a normal subgroup of $\mathbb{S}_4 \times \mathbb{Z}_2$ isomorphic to $\mathbb{S}_4$, but not conjugate to $\mathbb{S}_4$.*

Proof: Since $[a,i] = [b,i] = 1$, it is easy to check $(ai)^2 = b^3 = [(ai)b]^4 = 1$, showing O is isomorphic to $\mathbb{S}_4$. To show normality, note that $aba^{-1} = (ai)b(ai)^{-1} \in O$. Notice that a and i are orientation reversing maps on $\mathbb{S}^2$, and hence the element ai preserves the orientation. Since b preserves orientation, O consists of all orientation preserving elements whereas $\mathbb{S}_4$ contains a which is orientation reversing. This shows that the groups are not conjugate. $\square$

Lemma 2. *The group $\langle b, (ab)^2\rangle$ is a normal subgroup in $\mathbb{S}_4 \times \mathbb{Z}_2$ isomorphic to the alternating group on four letters, and we write $\mathbb{A}_4 = \langle b, (ab)^2\rangle$.*

Proof: Notice that $(ab)^2 = (1,4)(2,3)(5,7)(6,8)(9,12)(10,13)$ and $b(ab)^2 = (1,3,4)\ (5,8,6)\ (9,14,10)(11,13,12)$. Thus, $[(ab)^2]^2 = b^3 = [b(ab)^2]^3 = 1$ showing $\langle b,(ab)^2\rangle$ isomorphic to the alternating group on four letters. To show normality, observe that $aba^{-1} = aba = ababb^{-1} = (ab)^2b^{-1}$ and $a(ab)^2a^{-1} = baba = bababb^{-1} = b(ab)^2b^{-1}$ are both elements of $\mathbb{A}_4$. Since $[a,i] = [b,i] = 1$, $\mathbb{A}_4$ is a normal subgroup of $\mathbb{S}_4 \times \mathbb{Z}_2$. $\square$

Lemma 3. *Let $T = \langle b, [(ai)b]^2\rangle$. Then $T = \mathbb{A}_4$ and T is a normal subgroup of O and $\mathbb{S}_4$.*

Proof: Since $[(ai)b]^2 = (ab)^2$, it follows that $T = \mathbb{A}_4$. As for normality, $T = \mathbb{A}_4$ is actually normal in $\mathbb{S}_4 \times \mathbb{Z}_2$ by Lemma 1, and since $T \leq O \leq \mathbb{S}_4 \times \mathbb{Z}_2$, the result follows. $\square$

Lemma 4. *The group $\langle x = (ab)^2,\ y = a(ab)^2a^{-1}\rangle$ is the unique normal Klein four-subgroup of $\mathbb{S}_4 \times \mathbb{Z}_2$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, $\langle x = (ab)^2,\ y = a(ab)^2a^{-1}\rangle$ is contained in $\langle b,(ab)^2\rangle = \mathbb{A}_4$.*

Proof: Note that $x = (ab)^2 = (1,4)(2,3)(5,7)(6,8)(9,12)(10,13)$ and $y = a(ab)^2a^{-1} = (1,3)(2,4)(5,6)(7,8)(9,12)(11,14)$. Letting

$$z = (1,2)(3,4)(5,8)(6,7)(10,13)(11,14),$$

we see that $z = xy = yx$ showing that x and y generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is also easy to check $axa^{-1} = y$, $aya^{-1} = x$, $bxb^{-1} = y$ and $byb^{-1} = z$. Since $[a,i] = [b,i] = 1$, the group generated by x and y is normal in $\mathbb{S}_4 \times \mathbb{Z}_2$. The group $\langle x,y\rangle$ is the unique $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $\mathbb{S}_4$. (See [1].) Since $y = a(ab)a^{-1} = baba = b(ab)^2b^{-1}$, it follows that $\langle x,y\rangle$ is contained in $\langle b,(ab)^2\rangle = \mathbb{A}_4$.

Suppose N is a normal subgroup of $\mathbb{S}_4 \times \mathbb{Z}_2$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and not contained in $\mathbb{S}_4$. We may write $N = \langle u, vi\rangle$ or $\langle ui, vi\rangle$ where u and v are in $\mathbb{S}_4$. Note that since i commutes with every element in $\mathbb{S}_4$, it follows that $\langle u,v\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, using normality of N, it can be shown that $\langle u,v\rangle$ is a normal subgroup of $\mathbb{S}_4$, and therefore $\langle u,v\rangle = \langle x,y\rangle$. Thus N is either $\langle xi,y\rangle$, $\langle x,yi\rangle$ or $\langle xi,yi\rangle$. Since $a(xi)a^{-1} = yi$, $a(yi)a^{-1} = xi$ and $b^{-1}(xy)b = y$, we see that none of these groups are normal in $\mathbb{S}_4 \times \mathbb{Z}_2$, thus giving a contradiction. $\square$

Corollary 5. *The group $\langle x,y,i\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and is a normal subgroup of $\mathbb{A}_4 \times \mathbb{Z}_2$ and of $\mathbb{S}_4 \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \langle i\rangle$.*

Lemma 6. *The orbifold quotient $\mathbb{S}^2/(\langle x\rangle \times \langle y\rangle) = \Sigma(2,2,2)$ where $\langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof: Notice that $\mathbb{S}^2$ is tiled by $\triangle 123$ under this group action. For instance, $x(\triangle 123) = \triangle 432$, $y(\triangle 432) = \triangle 214$ and $x(\triangle 214) = \triangle 341$. Thus, we may choose $\triangle 123$ for a fundamental region for $\Sigma(2,2,2)$. Now, the vertex 9 is fixed under $z = xy$, the vertex 13 is fixed under y and the vertex 14 is fixed under x creating three each cone points of order two. The edge $\overline{2,9}$ is identified to the edge $\overline{1,9}$ by using z. Likewise, the edges $\overline{3,13}$ and $\overline{1,13}$ are identified via y and $\overline{2,14}$ and $\overline{3,14}$ are identified via x. $\square$

Lemma 7. *The orbifold quotient* $\mathbb{S}^2/(\langle x\rangle \times \langle y\rangle \times \langle i\rangle) = D_2^h$ *where* $\langle x\rangle \times \langle y\rangle \times \langle i\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof: Since $\langle x\rangle \times \langle y\rangle \times \langle i\rangle$ contains the Klein four-group generated x and y, the antipodal map i on $\mathbb{S}^2$ induces a map $\overline{i}$ on $\Sigma(2,2,2)$. Since $\overline{i}$ is an orientation reversing involution on the underlying space of $\Sigma(2,2,2)$ which is a 2-sphere, it follows that $\overline{i}$ is either a reflection or the antipodal map. We claim that $\overline{i}$ is a reflection on $\Sigma(2,2,2)$ fixing all cone points of order two. Observe that $yi(9) = 9$ and $yi(14) = 14$. This implies $\overline{i}$ is a reflection on $\Sigma(2,2,2)$ fixing all cone points of order two. Thus $\Sigma(2,2,2)/\langle\overline{i}\rangle = D_2^h$. $\square$

Proposition 8. *Let* $\varphi\colon \mathbb{Z}_2 \to$ *Homeo* $(\Sigma(2,2,2))$ *be an action such that* $\Sigma(2,2,2)/\varphi$ *is homeomorphic to* D_2^h. *Then* $\varphi(\mathbb{Z}_2)$ *is conjugate to* $\langle\overline{i}\rangle$.

Proof: Let $\nu\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\langle x,y\rangle = D_2^h$ and $\eta\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\varphi$ be the orbifold covering maps to the quotient spaces. By assumption there exists a homeomorphism $h\colon D_2^h \to \Sigma(2,2,2)/\varphi$. Since $\nu_*(\pi_1(\Sigma(2,2,2)))$ and $\eta_*(\pi_1(\Sigma(2,2,2)))$ are the unique orientation preserving subgroups of $\pi(D_2^h)$ and $\pi_1(\Sigma(2,2,2)/\varphi)$ respectively, it follows that $h_*(\nu_*(\pi_1(\Sigma(2,2,2)))) = \eta_*(\pi_1(\Sigma(2,2,2)))$. Lifting h to $\Sigma(2,2,2)$ conjugates the two actions. $\square$

Lemma 9. *The orbifold quotient* $\mathbb{S}^2/[(\langle x\rangle \times \langle y\rangle) \circ \langle a\rangle] = D_2^v$ *where* $(\langle x\rangle \times \langle y\rangle) \circ \langle a\rangle \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ \mathbb{Z}_2$ *and* $axa^{-1} = y$ *and* $aya^{-1} = x$.

Proof: Recall $x = (ab)^2$ and $y = a(ab)^2a^{-1}$ where the two elements generate the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. As $axa^{-1} = y$ and $aya^{-1} = x$, we obtain $\langle x,y\rangle \circ \langle a\rangle = (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ \mathbb{Z}_2$ where $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponds to $\Sigma(2,2,2)$. Further, an orientation reversing map a on $\mathbb{S}^2$ induces a reflection map $\overline{a}$ on $\Sigma(2,2,2)$. In addition, $a(9) = 9$, $a(13) = 14$ and $a(14) = 13$ showing $\overline{a}$ fixes one cone point but it exchanges the remaining cone points on $\Sigma(2,2,2)$. A fundamental region for D_2^v is a triangle $\triangle 239$ lying on $\triangle 123$ containing points 8 and 14. Notice that $xya(2) = 2$ and a fixes vertices 3, 8, 9 on $\mathbb{S}^2$. Because the edges $\overline{2,14}$ and $\overline{3,14}$ are identified in $\Sigma(2,2,2)$, a loop $[\overline{2,9,8,3}]$ is a fixed set under $\overline{a}$. As a result, $D_2^v = \Sigma(2,2,2)/\langle\overline{a}\rangle$ where $\langle\overline{a}\rangle = \mathbb{Z}_2$. $\square$

Proposition 10. *Let* $\varphi\colon \mathbb{Z}_2 \to$ *Homeo* $(\Sigma(2,2,2))$ *be an action such that* $\Sigma(2,2,2)/\varphi$ *is homeomorphic to* D_2^v. *Then* $\varphi(\mathbb{Z}_2)$ *is conjugate to* $\langle\overline{a}\rangle$.

Proof: The proof is similar to Proposition 8. $\square$

Remark 11. Note that the $\mathbb{Z}_2$-actions on $\Sigma(2,2,2)$ in Proposition 8 and Proposition 10 are not equivalent because these two quotient spaces of $\Sigma(2,2,2)$ by $\mathbb{Z}_2$-actions are topologically distinct orbifolds.

Proposition 12. *The orbifold quotient* $\mathbb{S}^2/\mathbb{A}_4 = \Sigma(2,3,3)$ *where* $\mathbb{A}_4 = \langle b,x\rangle$.

Proof: Since $\langle x,y\rangle$ is a normal subgroup of $\mathbb{A}_4 = \langle b, x = (ab)^2\rangle$, we have $\langle b,x\rangle/\langle x,y\rangle = \langle \overline{b} |\ \overline{b}^3 = 1\rangle \cong \mathbb{Z}_3$. Thus b induces an orientation preserving homeomorphism $\overline{b}$ on $\Sigma(2,2,2)$ with $\overline{b}^3 = id$, such that $\Sigma(2,2,2)/\langle\overline{b}\rangle = \mathbb{S}^2/\mathbb{A}_4$. Observe that $xyb(9) = 14$, $xyb(14) = 13$ and $b(13) = 9$. Therefore, $\overline{b}$ is a $120°$ rotation operating on $\Sigma(2,2,2)$ permuting the cone points. Now, $xyb(8) = 8$ and $yb(2) = 2$, so that these two vertices become distinct cone points of order 3 in $\Sigma(2,2,2)/\langle\overline{b}\rangle$. This implies that $\Sigma(2,2,2)/\langle\overline{b}\rangle = \Sigma(2,3,3)$ proving the result. □

As above using the uniqueness of $\langle x,y\rangle$ in $\langle b,x\rangle$, we have the following corollary:

Corollary 13. *Let $\varphi\colon \mathbb{Z}_3 \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to $\Sigma(2,3,3)$. Then $\varphi(\mathbb{Z}_3)$ is conjugate to $\langle\overline{b}\rangle$.*

Remark 14. It follows that $\mathbb{A}_4 = \langle b,x\rangle$ is a normal subgroup of $O = \langle ai, b\rangle$.

Proposition 15. *The orbifold quotient $\mathbb{S}^2/O = \Sigma(2,3,4)$ where $O = \langle ai,b\rangle$.*

Proof: Since $\langle x,y\rangle$ is a normal subgroup of $O = \langle ai,b\rangle$, we consider the group $O/\langle x,y\rangle = \langle\overline{ai},\overline{b}\rangle$ acting on $\mathbb{S}^2/\langle x,y\rangle = \Sigma(2,2,2)$. Now $\overline{ai}$ is an orientation preserving involution, and $\overline{b}$ is also an orientation preserving homeomorphism of order three. Since $(ai)b(ai)^{-1}b = x$, we obtain a dihedral action $\langle\overline{b}\rangle \circ_{-1} \langle\overline{ai}\rangle \simeq Dih(\mathbb{Z}_3)$ on $\Sigma(2,2,2)$. By Proposition 12, $\mathbb{S}^2/\mathbb{A}_4 = \Sigma(2,3,3)$ where $\mathbb{A}_4 = \langle b,x\rangle$, and we obtain an orientation preserving involution $\overline{\overline{ai}}$ on $\Sigma(2,3,3)$, which must be a rotation. Note that $\Sigma(2,2,2)/[\langle\overline{b}\rangle\circ_{-1}\langle\overline{ai}\rangle] = \Sigma(2,3,3)/\langle\overline{\overline{ai}}\rangle = \mathbb{S}^2/O$. The vertex $14 \in \mathbb{S}^2$ projects to an order two cone point in $\mathbb{S}^2/\mathbb{A}_4 = \Sigma(2,3,3)$, and the vertices 2 and 8 in $\mathbb{S}^2$ project to the cone points of order three. Since $bxb(ai)(14) = 14$, the cone point of order two is fixed under the involution $\overline{\overline{ai}}$ on $\Sigma(2,3,3)$. On the other hand, $(bx)^{-1}(ai)(2) = 8$ showing that $\overline{\overline{ai}}$ exchanges the two cone points of order three in $\Sigma(2,3,3)$. This implies $\Sigma(2,3,3)/\langle\overline{\overline{ai}}\rangle = \Sigma(2,3,4)$. □

Corollary 16. *Let $\varphi\colon Dih(\mathbb{Z}_3) \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to $\Sigma(2,3,4)$. Then $\varphi(Dih(\mathbb{Z}_3))$ is conjugate to $\langle\overline{b}\rangle \circ_{-1} \langle\overline{ai}\rangle$.*

Proposition 17. *The orbifold quotients $\mathbb{S}^2/[\langle a,b\rangle \times \langle i\rangle] = \Sigma(2,3,4)/\langle\widehat{i}\rangle = \mathbb{S}^2/[\mathbb{S}_4 \times \mathbb{Z}_2] = \Sigma(2,2,2)/Dih(\mathbb{Z}_6) = O^h$.*

Proof: We know that the orientation reversing map i commutes with a and b on $\mathbb{S}^2$, hence it induces an orientation reversing involution $\widehat{i}$ on $\Sigma(2,3,4) = \mathbb{S}^2/O$ where $O = \langle ai,b\rangle$. Since each cone point must be left fixed, it follows that $\widehat{i}$ is a reflection. This implies that $\Sigma(2,3,4)/\langle\widehat{i}\rangle = O^h$, and since $[\langle a,b\rangle \times \langle i\rangle] = \langle ai,b\rangle \times \langle i\rangle$ it follows that $\mathbb{S}^2/[\langle a,b\rangle \times \langle i\rangle] = \Sigma(2,3,4)/\langle\widehat{i}\rangle = O^h$.

Inasmuch as $\langle x,y\rangle$ is a normal subgroup of $\langle ai,b\rangle \times \langle i\rangle$, we obtain a $\langle\overline{b},\overline{ai}\rangle \times \langle\overline{i}\rangle$ action on $\Sigma(2,2,2)$ such that $\Sigma(2,2,2)/[\langle\overline{b},\overline{ai}\rangle \times \langle\overline{i}\rangle] = O^h$. Now $\langle\overline{b},\overline{ai}\rangle \times \langle\overline{i}\rangle = \langle\overline{b},\overline{i}\rangle \circ_{-1} \langle\overline{ai}\rangle = (\mathbb{Z}_3 \times \mathbb{Z}_2) \circ_{-1} \mathbb{Z}_2 = Dih(\mathbb{Z}_6)$ proving the result. □

Corollary 18. *Let $\varphi\colon Dih(\mathbb{Z}_6) \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to O^h. Then $\varphi(Dih(\mathbb{Z}_6))$ is conjugate to $[\langle\overline{bi}\rangle \circ_{-1} \langle\overline{ai}\rangle]$.*

Proposition 19. *The orbifold quotients $\Sigma(2,3,3)/\langle\overline{\overline{i}}\rangle = \mathbb{S}^2/[\langle b,(ab)^2\rangle \times \langle i\rangle] = \mathbb{S}^2/[\mathbb{A}_4 \times \mathbb{Z}_2] = \Sigma(2,2,2)/\mathbb{Z}_6 = T^h$ where $\overline{\overline{i}}$ is an involution on $\Sigma(2,3,3)$ induced by i.*

Proof: The map i induces an orientation reversing involution $\overline{\overline{i}}$ on $\Sigma(2,3,3) = \mathbb{S}^2/\mathbb{A}_4$. The vertex 14 in $\mathbb{S}^2$ projects to the cone point of order two and the vertices 2 and 8 in $\mathbb{S}^2$ each project to a cone point of order 3. Recall $\langle x,y\rangle \trianglelefteq \mathbb{A}_4$. Now, $yi(14) = 14$ and $yi(8) = 2$, which indicates that the induced map $\overline{\overline{i}}$ fixes the order two cone point but the order three cone points are exchanged. This implies that $\Sigma(2,3,3)/\langle\overline{\overline{i}}\rangle = \mathbb{S}^2/[\langle b,(ab)^2\rangle \times \langle i\rangle] = \mathbb{S}^2/[\mathbb{A}_4 \times \mathbb{Z}_2] = T^h$. Observe also that b and i induce homeomorphisms $\overline{b}$ and $\overline{i}$ on $\Sigma(2,2,2) = \mathbb{S}^2/\langle x,y\rangle$ respectively, such that $\Sigma(2,2,2)/\langle\overline{b},\overline{i}\rangle = \Sigma(2,3,3)/\langle\overline{\overline{i}}\rangle = \mathbb{S}^2/[\langle b,(ab)^2\rangle \times \langle i\rangle] = T^h$. Now $\langle\overline{b},\overline{i}\rangle = \mathbb{Z}_3 \times \mathbb{Z}_2 = \mathbb{Z}_6 = \langle\overline{bi}\rangle$. $\square$

Corollary 20. *Let $\varphi\colon \mathbb{Z}_6 \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to T^h. Then $\varphi(\mathbb{Z}_6)$ is conjugate to $\langle\overline{bi}\rangle$.*

Proposition 21. *There exist an orientation reversing involution $\overline{\overline{a}}$ on $\Sigma(2,3,3)$ and a dihedral action $Dih(\mathbb{Z}_3)$ on $\Sigma(2,2,2)$ such that $\Sigma(2,3,3)/\langle\overline{\overline{a}}\rangle = \Sigma(2,2,2)/Dih(\mathbb{Z}_3) = T^v$.*

Proof: Since $\langle x,y\rangle \trianglelefteq \langle a,b\rangle$, the maps a and b on $\mathbb{S}^2$ induce actions $\overline{a}$ and $\overline{b}$ on $\Sigma(2,2,2)$ respectively. We therefore have $\langle a,b\rangle/\langle x,y\rangle = \langle\overline{a},\overline{b}|\overline{a}^2 = \overline{b}^3 = 1, \overline{a}\overline{b}\overline{a}^{-1} = \overline{b}^{-1}\rangle = [\langle\overline{b}\rangle \circ_{-1} \langle\overline{a}\rangle] = Dih(\mathbb{Z}_3)$ acting on $\Sigma(2,2,2)$. Furthermore, $\overline{a}$ induces an orientation reversing involution $\overline{\overline{a}}$ on $\Sigma(2,3,3) = \Sigma(2,2,2)/\langle\overline{b}\rangle$. Now $\overline{\overline{a}}$ must leave the cone point of order two fixed, and hence is a reflection. This implies $\Sigma(2,3,3)/\langle\overline{\overline{a}}\rangle = T^v$. Since $\Sigma(2,3,3)/\langle\overline{\overline{a}}\rangle = \Sigma(2,2,2)/\langle\overline{a},\overline{b}\rangle$, the result follows. $\square$

Corollary 22. *Let $\varphi\colon Dih(\mathbb{Z}_3) \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to T^v. Then $\varphi(Dih(\mathbb{Z}_3))$ is conjugate to $\langle\overline{a},\overline{b}\rangle$.*

Remark 23. This $Dih(\mathbb{Z}_3)$-action on $\Sigma(2,2,2)$ is not conjugate to the dihedral action in Corollary 16, since we have different quotient orbifolds. We remark that $\pi_1(T^v) = \mathbb{S}_4 \trianglelefteq \pi_1(O^h)$.

We will now consider the dihedral group $Dih(\mathbb{Z}_m) = \langle r,s|r^m = s^2 = 1, srs^{-1} = r^{-1}\rangle$. In [4], we illustrated a dihedral action $Dih(\mathbb{Z}_m)$ on $\mathbb{S}^2$ such that $\mathbb{S}^2/Dih(\mathbb{Z}_m) = \Sigma(2,2,m)$. It is known that when m is odd, the only distinct normal subgroups of $Dih(\mathbb{Z}_m)$ are of the form $\langle r^d\rangle$ where $d|m$. When m is even, we have the previous listed groups together with the groups $\langle r^2,s\rangle$ and $\langle r^2,rs\rangle$. Suppose N is a normal subgroup of $Dih(\mathbb{Z}_m)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. This implies that N is either $\langle r^2,s\rangle$ or $\langle r^2,rs\rangle$, and since $r^4 = 1$ we have $m = 4$.

For this case, we triangulate the 2-sphere $\mathbb{S}^2$ in a different manner from all of the cases above. We locate the vertices 1 through 8 on the equator line. The vertices 9 and 10 are placed onto its (north/south) poles. We define two kinds of rotations r and s by $r = (1,3,5,7)(2,4,6,8)$ and $s = (1,5)(2,4)(6,8)(9,10)$. An antipodal map is defined by $i = (1,5)(2,6)(3,7)(4,8)(9,10)$ and a reflection map l is defined by $l = (1,2)(3,8)(4,7)(5,6)$. It is easy to check that $srs^{-1} = r^{-1}$ and $[r,i] = [s,i] = [i,l] = 1$. In addition we also have $lrl^{-1} = r^{-1}$ and $lsl^{-1} = rs$.

To obtain $\Sigma(2,2,2)$, we consider the normal subgroup $\langle r^2, s\rangle = \langle r^2\rangle \times \langle s\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$. Observe that the union of the four triangles $\triangle 129 \cup \triangle 239 \cup \triangle 349 \cup \triangle 459$ is a fundamental region for the $\langle r^2, s\rangle$-action on $\mathbb{S}^2$. Note that vertex 9 is fixed by r^2, vertices 1 and 5 are fixed by $r^2 s$, and vertex 3 is fixed by s. The map s identifies $\overline{1,2}$ and $\overline{4,5}$, also $\overline{2,3}$ and $\overline{4,3}$. Furthermore, r^2 identifies $\overline{1,9}$ and $\overline{5,9}$. Therefore $\mathbb{S}^2/[\langle r^2\rangle \times \langle s\rangle] = \Sigma(2,2,2)$. Let $\overline{r}$ be the induced map on $\Sigma(2,2,2)$, and note that $\overline{r}$ is an orientation preserving involution. The vertices 9, 1 and 3 project to distinct cone points of order two in $\Sigma(2,2,2)$. Since r fixes the vertex 9 and sends vertex 1 to 3, $\overline{r}$ is a rotation fixing a cone point of order two and exchanging the other two cone points of order two. This implies $\Sigma(2,2,2)/\langle\overline{r}\rangle = \Sigma(2,2,4)$. We have the following proposition.

Proposition 24. *The orbifold quotients* $\mathbb{S}^2/\langle r,s\rangle = \Sigma(2,2,4)$, $\mathbb{S}^2/\langle r^2,s\rangle = \Sigma(2,2,2)$, *and for the induced map* $\overline{r}$ *on* $\Sigma(2,2,2)$ *we obtain* $\Sigma(2,2,2)/\overline{r} = \Sigma(2,2,4)$. *Moreover,* $\Sigma(2,2,2)$ *does not cover* $\Sigma(2,2,m)$ *for* $m \neq 4$.

Observe that $\pi_1(\Sigma(2,2,4)) = \langle r,s | r^4 = s^2 = 1, srs^{-1} = r^{-1}\rangle$ and if $\nu\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\langle\overline{r}\rangle = \Sigma(2,2,4)$ is the orbifold covering map, then this is the covering corresponding to the normal subgroup $\langle r^2, s\rangle$.

We now consider the subgroup $\langle r^2, rs\rangle$. Now $rs = (1,7)(2,6)(3,5)(9,10)$, which is a rotation about an axis passing through the vertices 4 and 8. As above, we obtain $\mathbb{S}^2/\langle r^2, rs\rangle = \Sigma(2,2,2)$, an induced orientation preserving involution $\overline{s}$ on $\Sigma(2,2,2)$, and an orbifold covering map $\nu'\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\langle\overline{s}\rangle = \Sigma(2,2,4)$ corresponding to the subgroup $\langle r^2, rs\rangle$. Recall that $lsl^{-1} = rs$, $[l,i] = 1$ and $lrl^{-1} = r^{-1}$. Thus l is in the normalizer of both the groups $[\langle r\rangle \circ_{-1} \langle s\rangle] = Dih(\mathbb{Z}_4)$ and $[\langle r\rangle \circ_{-1} \langle s\rangle] \times \langle i\rangle = Dih(\mathbb{Z}_4) \times \mathbb{Z}_2$.

Proposition 25. *Let* $\varphi\colon \mathbb{Z}_2 \to$ *Homeo*$(\Sigma(2,2,2))$ *be an action such that* $\Sigma(2,2,2)/\varphi$ *is homeomorphic to* $\Sigma(2,2,4)$. *Then* $\varphi(\mathbb{Z}_2)$ *is conjugate to* $\langle\overline{r}\rangle$.

PROOF: We will begin by showing $\overline{r}$ and $\overline{s}$ are conjugate. Recall $\pi_1(\Sigma(2,2,4)) = \langle r,s | r^4 = s^2 = 1, srs^{-1} = r^{-1}\rangle$. Now $\nu\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\langle\overline{r}\rangle = \Sigma(2,2,4)$ is the covering corresponding to $\langle r^2, s\rangle$ and $\nu'\colon \Sigma(2,2,2) \to \Sigma(2,2,2)/\langle\overline{s}\rangle = \Sigma(2,2,4)$ is the covering corresponding to $\langle r^2, rs\rangle$. Since l is in the normalizer of $\langle r,s\rangle$, the map l induces a homeomorphism $\widehat{l}\colon \Sigma(2,2,4) \to \Sigma(2,2,4)$. Note that $l^{-1}sl = rs$ and $lrl^{-1} = r^{-1}$ implies that $\widehat{l}$ lifts to a homeomorphism $\overline{l}\colon \mathbb{S}^2/\langle r^2, s\rangle \to \mathbb{S}^2/\langle r^2, rs\rangle$ which conjugates $\overline{r}$ to $\overline{s}$.

Now any covering $\Sigma(2,2,2) \to \Sigma(2,2,4)$ must either correspond to the subgroup $\langle r^2, s\rangle$ or $\langle r^2, rs\rangle$. The argument above can be used to show that any $\mathbb{Z}_2$-action on $\Sigma(2,2,2)$ must be conjugate to $\langle \overline{r}\rangle$. □

In [4], we showed how an action $Dih(\mathbb{Z}_m) \times \mathbb{Z}_2$ on $\mathbb{S}^2$ for m even gave the quotient type $\mathbb{S}^2/(Dih(\mathbb{Z}_m) \times \mathbb{Z}_2) = D^h_m$. For m odd, we obtained the action $Dih(\mathbb{Z}_m) \circ \mathbb{Z}_2$ on $\mathbb{S}^2$ such that $\mathbb{S}^2/(Dih(\mathbb{Z}_m) \circ \mathbb{Z}_2) = D^h_m$. The dihedral group consists of orientation preserving elements, and the $\mathbb{Z}_2$ subgroup is generated by an orientation reversing element. If $\eta\colon \Sigma(2,2,2) \to D^h_m$ is a regular covering, then $\eta_*(\pi_1(\Sigma(2,2,2))) \subset Dih(\mathbb{Z}_m)$ is a normal subgroup, and hence $m = 4$.

We now show how to obtain the quotient D^h_4. Since i commutes with both r and s, it induces an orientation reversing involution $\widehat{i}$ on $\mathbb{S}^2/\langle r, s\rangle = \Sigma(2,2,4)$. To obtain D^h_4, notice that $\widehat{i}$ is a reflection on $\Sigma(2,2,4)$ fixing all cone points since $si(1) = 1$, $r^2 i(3) = 3$ and $si(5) = 5$. Thus $D^h_4 = \Sigma(2,2,4)/\langle\widehat{i}\rangle$ and $\pi_1(D^h_4) = \langle r, s\rangle \times \langle i\rangle = Dih(\mathbb{Z}_4) \times \mathbb{Z}_2$. Furthermore i induces an involution $\overline{i}$ on $\Sigma(2,2,2)$ such that $\Sigma(2,2,2)/\langle \overline{r}, \overline{i}\rangle = \Sigma(2,2,2)/(\mathbb{Z}_2 \times \mathbb{Z}_2) = D^h_4$.

Proposition 26. *The orbifold quotients $\mathbb{S}^2/[\langle r, s\rangle \times \langle i\rangle] = D^h_4$, and for the induced maps $\overline{r}$ and $\overline{i}$ on $\Sigma(2,2,2) = \mathbb{S}^2/\langle r^2, s\rangle$ we obtain $\Sigma(2,2,2)/\langle \overline{r}, \overline{i}\rangle = D^h_4$. Furthermore, $\Sigma(2,2,2)$ is not a regular cover of D^h_m for $m \neq 4$.*

Proposition 27. *Let $\varphi\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to Homeo\,(\Sigma(2,2,2))$ be an action such that $\Sigma(2,2,2)/\varphi$ is homeomorphic to D^h_4. Then $\varphi(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is conjugate to $\langle \overline{r}, \overline{i}\rangle$.*

Proof: The proof is similar to that of Proposition 25. Since l is in the normalizer of $[\langle r\rangle \circ_{-1} \langle s\rangle] \times \langle i\rangle = Dih(\mathbb{Z}_4) \times \mathbb{Z}_2 = \pi_1(D^h_4)$, it induces a homeomorphism $\widehat{l}\colon D^h_4 \to D^h_4$. Moreover since $l\langle r^2, s\rangle l^{-1} = \langle r^2, rs\rangle$, the proof follows as in Proposition 25. □

Again in [4], we showed how to obtain a $Dih(\mathbb{Z}_m) \times \mathbb{Z}_2$-action on $\mathbb{S}^2$ which gave the quotient type $\mathbb{S}^2/[Dih(\mathbb{Z}_m) \times \mathbb{Z}_2] = D^v_m$ for m odd. For m even, we obtained a $Dih(\mathbb{Z}_m) \circ \mathbb{Z}_2$-action on $\mathbb{S}^2$ such that $\mathbb{S}^2/[Dih(\mathbb{Z}_m) \circ \mathbb{Z}_2] = D^v_m$. The $\mathbb{Z}_2$ groups are generated by orientation reversing involutions, and thus if $\Sigma(2,2,2) \to D^v_m$ is a covering, it follows that $m = 4$.

Proposition 28. *The orbifold quotient $\mathbb{S}^2/[Dih(\mathbb{Z}_4) \circ \mathbb{Z}_2] = \mathbb{S}^2/[\langle r, s\rangle \circ \langle l\rangle] = D^v_4$ where $l^{-1}sl = rs$ and $lrl^{-1} = r^{-1}$. Furthermore, $\Sigma(2,2,2)$ is not a regular cover of D^v_m.*

Proof: A fundamental region for the dihedral action $Dih(\mathbb{Z}_4) = \langle r\rangle \circ_{-1} \langle s\rangle$ on $\mathbb{S}^2$ consists of the union of the triangles $\triangle 129 \cup \triangle 239$. The edge $\overline{1,9}$ is identified to the edge $\overline{3,9}$ via r and the edge $\overline{1,2}$ is identified to the edge $\overline{3,2}$ by s. The vertex 9 projects to the cone point of order four since r has order four and fixes vertex 9. The vertices 1 and 2 project to distinct cone points, each of order two. This follows since vertex 1 is fixed by sr and vertex 2 is fixed by s, and each of these homeomorphisms have order two. The reflection l induces an orientation reversing involution $\widehat{l}$ on $\Sigma(2,2,4) = \mathbb{S}^2/\langle r, s\rangle$, which must also be a reflection. Now $l(9) = 9$

and $l(1) = 2$. This implies $\widehat{l}$ leaves the cone point of order four fixed and exchanges the cone points of order two. Hence $\mathbb{S}^2/[\langle r,s\rangle \circ \langle l\rangle] = \Sigma(4,2,2)/\langle\widehat{l}\rangle = D_4^v$.

We have already seen that if $\nu\colon \Sigma(2,2,2) \to D_m^v$ is a regular covering, then it follows that $\nu_*(\pi_1(\Sigma(2,2,2)))$ is contained in $Dih(\mathbb{Z}_m)$ and $m = 4$. Furthermore $\nu_*(\pi_1(\Sigma(2,2,2))) = \langle r^2, s\rangle$ or $\langle r^2, rs\rangle$. However $lrl^{-1} = rs \notin \langle r^2, s\rangle$, and $lrsl^{-1} = r^{-1}rs = s \notin \langle r^2, rs\rangle$. Thus $\nu_*(\pi_1(\Sigma(2,2,2)))$ cannot be a normal subgroup of $\pi_1(D_4^v) = Dih(\mathbb{Z}_4) \circ \mathbb{Z}_2$, and thus $\Sigma(2,2,2)$ is not a regular cover of D_m^v. □

Theorem 29. *Let $\varphi\colon G \to Homeo\,(\Sigma(2,2,2))$ be a G-action on $\Sigma(2,2,2)$. Then G is isomorphic to one of the following groups: $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $Dih(\mathbb{Z}_3)$ or $Dih(\mathbb{Z}_6)$. There is only one equivalence class for each quotient type.*

(1) *If $G = \mathbb{Z}_2$, then $(\Sigma(2,2,2))/\mathbb{Z}_2$ is either D_2^h, D_2^v or $\Sigma(2,2,4)$.*
(2) *If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $(\Sigma(2,2,2))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is D_4^h.*
(3) *If $G = \mathbb{Z}_3$, then $(\Sigma(2,2,2))/\mathbb{Z}_3$ is $\Sigma(2,3,3)$.*
(4) *If $G = \mathbb{Z}_6$, then $(\Sigma(2,2,2))/\mathbb{Z}_6$ is T^h.*
(5) *If $G = Dih(\mathbb{Z}_3)$, then $(\Sigma(2,2,2))/Dih(\mathbb{Z}_3)$ is either $\Sigma(2,3,4)$ or T^v.*
(6) *If $G = Dih(\mathbb{Z}_6)$, then $(\Sigma(2,2,2))/Dih(\mathbb{Z}_6)$ is O^h.*

Proof: There are 14 closed 2-orbifolds B, with positive Euler characteristic $\chi(B) > 0$. The proof will follow when we show that $\Sigma(2,2,2)$ cannot cover the remaining closed 2-orbifolds with positive Euler characteristic not covered above. These orbifolds are as follows: $\Sigma(2,3,5)$, I^h, S^{2m}, Z_h^m, $\Sigma(0,m,m)$ and $C_{m,m}^v$. Now $\pi_1(\Sigma(2,3,5)) = I$ is the icosahedral group, which is a simple group. Hence, $\pi_1(\Sigma(2,2,2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ cannot be a normal subgroup of I. The groups $\pi_1(I^h) = I \times \mathbb{Z}_2$ and $\pi_1(C_{m,m}^v) = \mathbb{Z}_m \circ_{-1} \mathbb{Z}_2$, where the $\mathbb{Z}_2$ subgroup in both cases is generated by an orientation reversing map. This implies $\pi_1(\Sigma(2,2,2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ must be a normal subgroup of I or $\mathbb{Z}_m$, which is impossible. For all $m \in \mathbb{N}$, $\pi_1(S^{2m}) = \mathbb{Z}_{2m}$, $\pi_1(\Sigma(0,2,2)) = \mathbb{Z}_m$ and $\pi_1(Z_h^m) = \mathbb{Z}_m \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by an orientation reversing map. Again $\pi_1(\Sigma(2,2,2))$ would have to be a subgroup of $\mathbb{Z}_{2m}$ or $\mathbb{Z}_m$, which is impossible. □

Let $\Psi\colon \mathrm{Homeo}\,(\Sigma(2,2,2)) \to Out(\pi_1(\Sigma(2,2,2)))$ be the homomorphism which sends any homeomorphism to the outer automorphism that it induces.

Corollary 30. *If $\varphi\colon G \to Homeo\,(\Sigma(2,2,2))$ is a finite group action such that $\Sigma(2,2,2)/\varphi = \Sigma(2,3,4)$, then $\varphi\Psi\colon G \to Out(\pi_1(\Sigma(2,2,2)))$ is an isomorphism. Thus every subgroup of $Out(\pi_1(\Sigma(2,2,2)))$ is realizable by a finite subgroup of $Homeo\,(\Sigma(2,2,2))$.*

Proof: If $\Sigma(2,2,2)/\varphi = \Sigma(2,3,4)$, then $G = Dih(\mathbb{Z}_3)$ and by Corollary 16 we may assume $\varphi(G) = \langle\overline{b}\rangle \circ_{-1} \langle\overline{ai}\rangle$. It follows that only the identity element in this group induces the identity outer automorphism, and thus $\varphi\Psi\colon G \to Out(\pi_1(\Sigma(2,2,2)))$ is one-to-one. Since $Out(\pi_1(\Sigma(2,2,2))) = Aut(\pi_1(\Sigma(2,2,2)))$ $= Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{S}_3 = Dih(\mathbb{Z}_3)$, the result follows. □

3. Prism manifolds fibering over $\Sigma(2,2,2)$

In this section, we consider the finite group actions on a prism manifold $M(b,d)$ which preserve a longitudinal fibering, but do not leave any Heegaard Klein bottle invariant. Actions which leave a Heegaard Klein bottle invariant are said to split, and these were classified in [3]. We consider the orbifold quotients for these actions, and show that they fiber over certain of the orbifolds mentioned in Section 1. Under certain conditions, these quotient spaces are Seifert fibered manifolds and these are identified as well. In addition, we compute the orbifold fundamental groups of certain quotient spaces.

Let $\mathbb{S}^3$ be the 3-sphere viewed as the set of quaternions $\{u+vj \mid u,v\in\mathbb{C}$ and $|u|^2+|v|^2=1\}$. Many of the computations in this paper use the formulae $uj=j\overline{u}$ and $(u+vj)^{-1}=\overline{u}-vj$.

Let $\sigma\colon \mathbb{S}^3\times\mathbb{S}^3\to Isom^+(\mathbb{S}^3)$ be the map to the orientation preserving isometries defined by $\sigma(q_1,q_2)(q)=q_1qq_2^{-1}$ for the elements $q,q_1,q_2\in\mathbb{S}^3$. The kernel of σ is $\langle(-1,-1)\rangle\simeq\mathbb{Z}_2$.

We begin by considering actions which preserve the longitudinal fibering $h_l\colon M(b,d)\to\Sigma(2,2,d)$, which is induced by the fibering $\mathbb{F}_l=\langle pS^1\rangle_{p\in\mathbb{S}^3}$ on $\mathbb{S}^3$ with $H_l\colon\mathbb{S}^3\to\mathbb{S}^2$ defined by $H_l(u+vj)=u/\bar{v}$. By Theorem 13 in [2], if $\varphi\colon G\to \mathrm{Diff}(M(b,d))$ is an action which preserves a longitudinal fibering and does not split, then $d=2$ and G/G_0 is either $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$, or $\mathrm{Dih}(\mathbb{Z}_6)$, where G_0 is the subgroup of G consisting of elements which leave every longitudinal fiber invariant. Hence we consider only the prism manifolds $M(b,2)$. Note that $M(b,2)=\mathbb{S}^3/\langle\sigma(i,1),\sigma(j,1),\sigma(1,e^{\frac{2\pi i}{b}})\rangle$. Now $\sigma(j,1)(pe^{i\theta})=j(ue^{i\theta}+ve^{-i\theta}j)=\bar{u}e^{-i\theta}j+\bar{v}e^{i\theta}j^2=-\bar{v}e^{i\theta}+\bar{u}e^{-i\theta}j$, and therefore $H_l\sigma(j,1)(pe^{i\theta})=H_l(-\bar{v}e^{i\theta}+\bar{u}e^{-i\theta}j)=-\bar{v}e^{i\theta}/\overline{\bar{u}e^{-i\theta}}=-\bar{v}/u$. For the induced action $\bar{\sigma}(j,1)$ on $\mathbb{S}^2$, we have $\bar{\sigma}(j,1)(u/\bar{v})=-\bar{v}/u$. Hence for any $z\in\mathbb{S}^2$, it follows that $\bar{\sigma}(j,1)(z)=-1/z$. A similar computation shows that $\bar{\sigma}(i,1)(z)=-z$ and $\bar{\sigma}(1,e^{\frac{2\pi i}{b}})=id_{\mathbb{S}^2}$. The fixed-point sets are as follows: $\mathrm{fix}(\bar{\sigma}(i,1))=\{0,\infty\}$, $\mathrm{fix}(\bar{\sigma}(j,1))=\{i,-i\}$, and $\mathrm{fix}(\bar{\sigma}(ij,1))=\{1,-1\}$. Note that the points in each fixed-point set are being identified under the $\langle\bar{\sigma}(i,1),\ \bar{\sigma}(j,1)\rangle$-action on $\mathbb{S}^2$. It follows that $\mathbb{S}^2/\langle\bar{\sigma}(i,1),\ \bar{\sigma}(j,1)\rangle=\Sigma(2,2,2)$ where each fixed-point set projects to a cone point of order 2. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "H_l"] \arrow[d, "\nu"] & \mathbb{S}^2 \arrow[d, "\bar{\nu}"] \\
M(b,2) \arrow[r, "h_l"] & \Sigma(2,2,2)
\end{tikzcd}$$

where the maps ν and $\bar{\nu}$ are covering maps.

It follows that a Klein bottle K in $M(b,2)$ is a fibered Heegaard Klein bottle if and only if $h_l(K)$ is an arc in $\Sigma(2,2,2)$ whose endpoints contain exactly two of the cone points.

Consider the element $y = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(1+j)$ in $\mathbb{S}^3$. A computation shows that $y^2 = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(i+j)$ and $y^3 = -1$. Furthermore $yiy^{-1} = j$ and $yjy^{-1} = ij$. Thus (y,j) is in the normalizer of the group $\langle (i,1),(j,1),(1,e^{\frac{2\pi i}{b}})\rangle$ in $\mathbb{S}^3 \times \mathbb{S}^3$, and thus $\sigma(y,j)$ induces an isometry $\widehat{\sigma}(y,j)$ on $M(b,2)$. It was shown in [3] that $\widehat{\sigma}(y,j)$ has order 6, and thus we obtain a fiber-preserving $\mathbb{Z}_6$-action on $M(b,2)$.

We consider $\sigma(y,j)^2 = \sigma(y^2,-1)$ which induces the same $\mathbb{Z}_3$-action on $M(b,2)$ as $\sigma(y^2,1)$. Now $\sigma(y^2,1)(u+vj) = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(i+j)(u+vj) = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}[(iu-\bar{v})+(iv+\bar{u})j] = [\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) + \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iv+\bar{u})j]$.

Lemma 31. *The isometry $\sigma(y^2,1)$ is fixed-point free on $\mathbb{S}^3$.*

Proof: Suppose $\sigma(y^2,1)(u+vj) = [\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) + \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iv+\bar{u})j] = u+vj$. Then $\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) = u$ and $\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iv+\bar{u}) = v$. Considering the first of these two equations, we have $\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) = \frac{1}{2}(1+i)(iu-\bar{v}) = u$, which simplifies to $\bar{v} = -(1-2i)u$. Similarly the second equation simplifies to $\bar{u} = (1-2i)v$. Consider now the quotient $\frac{\bar{u}}{\bar{v}} = \frac{(1-2i)v}{-(1-2i)u} = \frac{v}{-u}$, which implies $0 = u\bar{u} + v\bar{v}$ giving a contradiction. $\square$

We now want to investigate the induced map on $\mathbb{S}^2$. We see that

$$H_l(\sigma(y^2,1)(u+vj)) = \frac{\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v})}{\frac{1}{\sqrt{2}}e^{\frac{-\pi i}{4}}(-i\bar{v}+u)} = \frac{i(iu-\bar{v})}{-i\bar{v}+u} = \frac{-u-i\bar{v}}{-i\bar{v}+u} = \frac{\frac{-u}{\bar{v}}i+1}{\frac{u}{\bar{v}}i+1}.$$

This implies that if $\bar{\sigma}(y^2,1)$ is the induced map on $\mathbb{S}^2$, then $\bar{\sigma}(y^2,1)(z) = \frac{1-iz}{1+iz}$. Solving the equation $\frac{1-iz}{1+iz} = z$ to find the fixed points, we see that $z = (1-i)(\frac{-1\pm\sqrt{3}}{2})$. We have the following lemma.

Lemma 32. *The map $\sigma(y^2,1)$ leaves two fibers $F_\pm = \{u+vj \mid \frac{u}{\bar{v}} = (1-i)(\frac{-1\pm\sqrt{3}}{2})\}$ in $\mathbb{S}^3$ invariant.*

The map $\bar{\sigma}(y^2,1)$ induces a $\mathbb{Z}_3$-action on $\Sigma(2,2,2) = \mathbb{S}^2/\langle\bar{\sigma}(i,1),\bar{\sigma}(j,1)\rangle$, which we denote by $\widehat{\bar{\sigma}}(y^2,1)$. Since $\bar{\sigma}(y^2,1)(z) = \frac{1-iz}{1+iz}$, we see that $\bar{\sigma}(y^2,1)$ permutes the fixed-point sets of $\bar{\sigma}(i,1)$, $\bar{\sigma}(j,1)$ and $\bar{\sigma}(ij,1)$. Thus $\widehat{\bar{\sigma}}(y^2,1)$ permutes the three cone points of order two in $\Sigma(2,2,2)$, and $\Sigma(2,2,2)/\langle\widehat{\bar{\sigma}}(y^2,1)\rangle = \Sigma(2,3,3)$. Furthermore, since $\bar{\nu}\circ H_l = h_l \circ \nu$, it follows that $\nu(F_\pm)$ are regular fibers in $M(b,2)$.

Proposition 33. *The induced map $\widehat{\sigma}(y^2,1)\colon M(b,2) \to M(b,2)$ is fixed point free if and only if $b \not\equiv 0(mod\ 3)$. Furthermore $fix(\widehat{\sigma}(y^2,1)) = \nu(F_+)\cup\nu(F_-)$ if and only if $b \equiv 0(mod\ 3)$.*

Proof: Since the only fixed points of $\widehat{\bar{\sigma}}(y^2,1)$ are $\bar{\nu}((1-i)(\frac{-1\pm\sqrt{3}}{2}))$, it follows that $\widehat{\sigma}(y^2,1)$ will leave invariant only the two fibers $\nu(F_\pm)$. Now the only elements of the group of covering translations leaving $F_\pm$ invariant are of the form $\sigma(1,e^{\frac{2\pi i}{b}})^k = \sigma(1,e^{\frac{2\pi ik}{b}})$ where $0<k<b$. Suppose $\sigma(1,e^{\frac{2\pi ik}{b}})(u+vj) =$

$\sigma(y^2,1)(u+vj)$ for $u+vj \in F_+$. Thus $ue^{\frac{-2\pi ik}{b}} + ve^{\frac{2\pi ik}{b}}j = [\frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) + \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iv+\bar{u})j]$. Consider first $ue^{\frac{-2\pi ik}{b}} = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iu-\bar{v}) = \frac{1}{2}(1+i)(iu-\bar{v})$. We obtain the equation $2ue^{\frac{-2\pi ik}{b}} = (-u+iu)-(\bar{v}+i\bar{v})$, and dividing by $\bar{v}$ and rearranging yields $\frac{u}{\bar{v}}[2e^{\frac{-2\pi ik}{b}}+(1-i)] = -(1+i)$. Substituting $\frac{u}{\bar{v}} = (1-i)(\frac{-1+\sqrt{3}}{2})$ and simplifying, we obtain $e^{\frac{-2\pi ik}{b}} = \frac{-1}{2} - \frac{\sqrt{3}}{2}$. Setting $ve^{\frac{2\pi ik}{b}} = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(iv+\bar{u})$ and using a similar argument as above gives $e^{\frac{2\pi ik}{b}} = \frac{-1}{2} + \frac{\sqrt{3}}{2}$. This is true if and only if 3 divides b. The case when $u+vj \in F_-$ is similar. □

Recall that the manifold $N = \{m; (o_1,0) : (2,1),(3,\beta_2),(3,\beta_3)\}$ where $b = 6m+3+2(\beta_2+\beta_3)$, m is the obstruction class, and $g.c.d.\{b,12\} = 1$, is a tetrahedral manifold fibering over $\Sigma(2,3,3)$ with fundamental group $T^* \times \mathbb{Z}_b$. (See [6] and note that b and m have been exchanged.)

Theorem 34. *The quotient orbifold $M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle$ fibers over $\Sigma(2,3,3)$, and is a tetrahedral manifold N if and only if $g.c.d.\{b,6\} = 1$. When $g.c.d.\{b,6\} \neq 1$, $\nu(F_+)\cup\nu(F_-)$ projects to the only non-manifold points in $M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle$, and each projects to a cone point of order 3 in $\Sigma(2,3,3)$. The orbifold fundamental group $\pi_1(M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle) \simeq T^* \times \mathbb{Z}_b$ where T^* is the binary tetrahedral group.*

Proof: Since b is odd, $b \not\equiv 0(mod\ 3)$ is equivalent to $g.c.d.\{b,6\} = 1$. The first two statements follow from Proposition 33 and the above discussion. To compute the fundamental group, note that

$$M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle = \mathbb{S}^3/\langle\sigma(i,1),\sigma(j,1),\sigma(1,e^{\frac{2\pi i}{b}}),\sigma(y^2,1)\rangle.$$

A computation using $y^3 = -1$ shows that $y^2jy^4 = -y^2jy = i$. Hence the group $\langle i,j,y^2\rangle = \langle j,y\rangle \simeq T^*$. This implies the quotient space

$$\mathbb{S}^3/\langle\sigma(i,1),\sigma(j,1),\sigma(y^2,1),\sigma(1,e^{\frac{2\pi i}{b}})\rangle = \mathbb{S}^3/\langle\sigma(j,1),\sigma(y,1),\sigma(1,e^{\frac{2\pi i}{b}})\rangle,$$

and thus the result follows. □

Corollary 35. *Let $M(b,2)$ be a prism manifold with $g.c.d.\{b,6\} = 1$. Then we may write $b = 6m+7$ or $b = 6m+11$.*

(1) *If $b = 6m+7$, then $M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle = \{m; (o_1,0) : (2,1),(3,1),(3,1)\}$.*
(2) *If $b = 6m+11$, then $M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle = \{m; (o_1,0) : (2,1),(3,2),(3,2)\}$.*

Remark 36. Since $y^3 = -1$, it follows that $M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle = M(b,2)/\langle\widehat{\sigma}(y,1)\rangle$.

In both [5] and [6], M is a tetrahedral manifold if and only if M fibers over $\Sigma(2,3,3)$, $\pi_1(M) \simeq T^* \times \mathbb{Z}_b$ and $g.c.d.\{b,6\} = 1$. We have shown that when b is odd and $g.c.d.\{b,6\} \neq 1$, there exists an orbifold N which fibers over $\Sigma(2,3,3)$ with fundamental group $T^* \times \mathbb{Z}_b$.

We now consider the isometry $\sigma(y,j)$ which induces a $\mathbb{Z}_6$-action $\widehat{\sigma}(y,j)$ on $M(b,2)$. Now $\sigma(y,j)(u+vj) = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}(1+j)(u+vj)j^{-1} = \frac{1}{2}(1+i)[(v+\bar{u}) + (\bar{v}-u)j]$. Since $\sigma(y,j)^2$ is fixed-point free, $\sigma(y,j)$ does not have a fixed-point.

To compute the induced map $\bar{\sigma}(y,j)$ on $\mathbb{S}^2$, we see that $H_l(\sigma(y,j)(u+vj)) = \frac{\frac{1}{2}(1+i)(v+\bar{u})}{\frac{1}{2}(1-i)(v-\bar{u})} = \frac{i(v+\bar{u})}{(v-\bar{u})} = \frac{i(1+\frac{\bar{u}}{v})}{(1-\frac{\bar{u}}{v})}$. For $z \in \mathbb{S}^2$, this implies that $\bar{\sigma}(y,j)(z) = \frac{i(1+\bar{z})}{(1-\bar{z})}$. As $\sigma(y,j)^3 = \sigma(1,j)$, we obtain $\sigma(1,j)(u+vj) = (u+vj)j^{-1} = (v-uj)$. Thus $H_l(\sigma(y,j)^3(u+vj)) = \frac{v}{-\bar{u}}$ and $\bar{\sigma}(y,j)^3(z) = \frac{-1}{\bar{z}}$. A computation shows that $\bar{\sigma}(y,j)^4(z) = \frac{1-z}{i(1+z)}$ and $\bar{\sigma}(y,j)^5(z) = \frac{-1+i\bar{z}}{1+i\bar{z}}$.

Proposition 37. *The maps $\bar{\sigma}(y,j)$, $\bar{\sigma}(y,j)^3$, $\bar{\sigma}(y,j)^5$ are fixed-point free on $\mathbb{S}^2$. However $\bar{\sigma}(y,j)^2$ and $\bar{\sigma}(y,j)^4$ have the same fixed-point set consisting of the points $(1-i)(\frac{-1\pm\sqrt{3}}{2})$.*

Proof: Suppose $\bar{\sigma}(y,j)(z) = z$, and so we obtain the equation $i(1+\bar{z}) = z - z\bar{z}$. Substituting $z = a+bi$ into this equation and simplifying, yields $-1 = a^2+b^2$, giving a contradiction. Since $\frac{-1}{\bar{z}} = z$ implies $-1 = z\bar{z}$ which is impossible, it follows that $\bar{\sigma}(y,j)^3$ is fixed-point free. If $\bar{\sigma}(y,j)^5(z) = z$, then we obtain the equation $-1+i\bar{z} = z + iz\bar{z}$. Again substituting $z = a+bi$ gives a contradiction. Finally suppose that $\bar{\sigma}(y,j)^4(z) = z$, and so we obtain the equation $1-z = zi(1+z)$. By solving this equation, we obtain $z = (1-i)(\frac{-1\pm\sqrt{3}}{2})$. □

Since $\bar{\sigma}(y,j)(z) = \frac{i(1+\bar{z})}{(1-\bar{z})}$, we see that $\bar{\sigma}(y,j)$ permutes the fixed-point sets of $\bar{\sigma}(i,1)$, $\bar{\sigma}(j,1)$ and $\bar{\sigma}(ij,1)$. Thus if $\widehat{\bar{\sigma}}(y,j)$ is the induced map on $\Sigma(2,2,2)$, then $\widehat{\bar{\sigma}}(y,j)$ permutes the three cone points of order two in $\Sigma(2,2,2)$ and $\widehat{\bar{\sigma}}(y,j)^3 = \widehat{\bar{\sigma}}(1,j)$ fixes each cone point. Observe that $\bar{\sigma}(1,j)((1-i)(\frac{-1+\sqrt{3}}{2})) = \frac{-2}{(1+i)(-1+\sqrt{3})}$ $= \frac{-(1-i)}{-1+\sqrt{3}} = (1-i)(\frac{-1-\sqrt{3}}{2})$.

We recall the definition of an amalgamated direct product. A group G is a *direct product of the groups A and B with amalgamated subgroup C if $G = A \cup B$, $A \cap B = C$, and the centralize of A in G contains B*. We write $G = A \times_C B$.

For the tetrahedral group $T^* = \langle x_0, y_0 | x_0^2 = y_0^3 = (x_0y_0)^3 = -1\rangle$ and the group $\mathbb{Z}_4 = \langle z_0 | z_0^2 = -1\rangle$ form the amalgamated direct product $T^* \times_{\mathbb{Z}_2} \mathbb{Z}_4$. Now form the group $\mathbb{Z}_b \circ (T^* \times_{\mathbb{Z}_2} \mathbb{Z}_4)$ as follows: if w generates $\mathbb{Z}_b$, then $[w,x_0] = [w,y_0] = 1$ and $z_0wz_0^{-1} = w^{-1}$.

Theorem 38. *The quotient orbifold $M(b,2)/\langle\widehat{\sigma}(y,j)\rangle$ fibers over T^h. The orbifold fundamental group $\pi_1(M(b,2)/\langle\widehat{\sigma}(y,j)\rangle) \simeq \mathbb{Z}_b \circ (T^* \times_{\mathbb{Z}_2} \mathbb{Z}_4)$.*

Proof: Note that $\mathbb{Z}_6 = \langle\widehat{\sigma}(y,j)\rangle = \langle\widehat{\sigma}(y,j)^2, \widehat{\sigma}(y,j)^3\rangle = \langle\widehat{\sigma}(y,1), \widehat{\sigma}(1,j)\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_2$. We have seen in Theorem 34 that the orbifold quotient $O = M(b,2)/\langle\widehat{\sigma}(y^2,1)\rangle$ fibers over $\Sigma(2,3,3)$ where $\Sigma(2,3,3) = \Sigma(2,2,2)/\langle\widehat{\bar{\sigma}}(y^2,1)\rangle$. Now $\widehat{\sigma}(1,j)$ on $M(b,2)$ induces a map $s(1,j)$ on O, which projects to a map $\bar{s}(1,j)$ on $\Sigma(2,3,3)$. Note that $\widehat{\bar{\sigma}}(1,j)$ fixes each cone point of order two, and $\bar{\sigma}(1,j)$ exchanges the points $(1-i)(\frac{-1+\sqrt{3}}{2})$ and $(1-i)(\frac{-1-\sqrt{3}}{2})$ in $\mathbb{S}^2$, which are the points that project to the order three cone points in $\Sigma(2,3,3)$. This implies that $\bar{s}(1,j)$ is a reflection leaving the cone point of order two fixed and exchanging the two cone points of order 3. Thus $\Sigma(2,3,3)/\langle\bar{s}(1,j)\rangle = T^h$. We therefore have $M(b,2)/\langle\widehat{\sigma}(y,j)\rangle = O/\langle s(1,j)\rangle$ fibering over T^h.

As in Theorem 34, to compute the fundamental group notice that

$$\begin{aligned} M(b,2)/\langle\widehat{\sigma}(y,j)\rangle &= \mathbb{S}^3/\langle\sigma(j,1),\sigma(y,j),\sigma(1,e^{\frac{2\pi i}{b}})\rangle \\ &= \mathbb{S}^3/\langle\sigma(j,1),\sigma(y,1),\sigma(1,j),\sigma(1,e^{\frac{2\pi i}{b}})\rangle. \end{aligned}$$

Setting $\sigma(j,1)=x_0$, $\sigma(y,1)=y_0$, $\sigma(1,j)=z_0$ and $\sigma(1,e^{\frac{2\pi i}{b}})=w$, proves the result. □

We will now consider certain dihedral groups acting on $M(b,2)$. Let $x=\frac{1}{\sqrt{2}}(i+j)\in\mathbb{S}^3$. A computation shows that $xix^{-1}=j$, $xjx^{-1}=i$ and $(xy)^2=-i$. Therefore we see that both $\sigma(x,1)$ and $\sigma(x,j)$ induce involutions $\widehat{\sigma}(x,1)$ and $\widehat{\sigma}(x,j)$ on $M(b,2)$ respectively. Denote by $\bar{\sigma}(x,1)$ and $\bar{\sigma}(x,j)$ the induced involutions on $\mathbb{S}^2$. A further computation shows that $xyx^{-1}y=i$, and so we obtain the following two dihedral actions on $M(b,2)$: $Dih_1(\mathbb{Z}_3)=\langle\widehat{\sigma}(y,1)\rangle\circ_{-1}\langle\widehat{\sigma}(x,1)\rangle$ and $Dih_2(\mathbb{Z}_3)=\langle\widehat{\sigma}(y,1)\rangle\circ_{-1}\langle\widehat{\sigma}(x,j)\rangle$.

We consider first $Dih_1(\mathbb{Z}_3)=\langle\widehat{\sigma}(y,1)\rangle\circ_{-1}\langle\widehat{\sigma}(x,1)\rangle$. Now $\widehat{\sigma}(x,1)(u+vj)=\frac{1}{\sqrt{2}}(i+j)(u+vj)=\frac{1}{\sqrt{2}}[(iu-\bar{v})+(iv+\bar{u})j]$.

Lemma 39. *The isometry $\sigma(x,1)$ is fixed-point free on $\mathbb{S}^3$.*

Proof: Suppose $\widehat{\sigma}(x,1)(u+vj)=u+vj$. We obtain the following equations: $\frac{1}{\sqrt{2}}(iu-\bar{v})=u$ and $\frac{1}{\sqrt{2}}(iv+\bar{u})=v$. Solving the first of these equations for u we obtain $u=\frac{-\bar{v}}{i-\sqrt{2}}$, and substituting this into the second equation yields $v=0$. This implies $u=0$ giving a contradiction. □

We now investigate the induced map $\bar{\sigma}(x,1)$ on $\mathbb{S}^2$. The projection

$$H_l(\sigma(x,1)(u+vj))=H_l(\frac{1}{\sqrt{2}}[(iu-\bar{v})+(iv+\bar{u})j])=\frac{iu-\bar{v}}{-i\bar{v}+u}=\frac{i\frac{u}{\bar{v}}-1}{\frac{u}{\bar{v}}-i}.$$

Thus for $z\in\mathbb{S}^2$ we have $\bar{\sigma}(x,1)(z)=\frac{iz-1}{z-i}$. Solving the equation $\frac{iz-1}{z-i}=z$ to find the fixed points, we see that $z=(1\pm\sqrt{2})i$. This gives the following lemma.

Lemma 40. *The map $\sigma(x,1)$ leaves two fibers $E_\pm=\{u+vj|\ \frac{u}{\bar{v}}=(1\pm\sqrt{2})i\}$ in $\mathbb{S}^3$ invariant.*

Recall that the induced maps $\bar{\sigma}(i,1)$ and $\bar{\sigma}(j,1)$ on $\mathbb{S}^2$ are defined by $\bar{\sigma}(i,1)(z)=-z$ and $\bar{\sigma}(j,1)(z)=\frac{-1}{z}$, and $\mathbb{S}^2/\langle\bar{\sigma}(i,1),\bar{\sigma}(j,1)\rangle=\Sigma(2,2,2)$. We next consider the dihedral action $\langle\bar{\sigma}(y,1)\rangle\circ_{-1}\langle\bar{\sigma}(x,1)\rangle$ on $\mathbb{S}^2$. Now $\bar{\sigma}(ij,1)((1+\sqrt{2})i)=\frac{1}{(1+\sqrt{2})i}=(1-\sqrt{2})i$, and thus $\bar{\nu}((1+\sqrt{2})i)=\bar{\nu}((1-\sqrt{2})i)=w_0$ in $\Sigma(2,2,2)$. Let $\bar{\nu}(\{0,\infty\})=[0]$, $\bar{\nu}(\{i,-i\})=[i]$ and $\bar{\nu}(\{1,-1\})=[1]$. In addition for the fixed points $(1-i)(\frac{-1\pm\sqrt{3}}{2})$ of $\bar{\sigma}(y,1)$, let $\bar{\nu}((1-i)(\frac{-1+\sqrt{3}}{2}))=w_+$ and $\bar{\nu}((1-i)(\frac{-1-\sqrt{3}}{2}))=w_-$. Observe that $\bar{\sigma}(x,1)(\{0,\infty\})=\{i,-i\}$ and $\bar{\sigma}(x,1)(\{1,-1\})=\{1,-1\}$. This implies that if $\widehat{\bar{\sigma}}(y,1)$ and $\widehat{\bar{\sigma}}(x,1)$ are the induced maps on $\Sigma(2,2,2)$, then $\widehat{\bar{\sigma}}(x,1)([1]) = [1]$, $\widehat{\bar{\sigma}}(x,1)([0]) = [i]$ and $\widehat{\bar{\sigma}}(x,1)(w_0) = w_0$. A computation shows that $\bar{\sigma}(x,1)((1-i)(\frac{-1+\sqrt{3}}{2})) = (1+i)(\frac{1-\sqrt{3}}{2})$, and since $\bar{\sigma}(ij,1)((1+i)(\frac{1-\sqrt{3}}{2})) = (1-i)(\frac{-1-\sqrt{3}}{2})$, it follows that $\widehat{\bar{\sigma}}(x,1)(w_+) = w_-$.

Consider the orbifold $M(b,2)/\langle\widehat{\sigma}(y,1)\rangle = O$ which we see fibers over the space $\Sigma(2,3,3) = \Sigma(2,2,2)/\langle\widehat{\bar{\sigma}}(y,1)\rangle$. Denote by $\mu\colon M(b,2) \to M(b,2)/\langle\widehat{\sigma}(y,1)\rangle$ and $\bar{\mu}\colon \Sigma(2,2,2) \to \Sigma(2,3,3)$ the orbifold covering maps.

The maps $\widehat{\sigma}(x,1)$ and $\widehat{\bar{\sigma}}(x,1)$ induce maps $s(x,1)$ and $\bar{s}(x,1)$ on O and $\Sigma(2,3,3)$ respectively. It follows by the above that $\bar{s}(x,1)$ exchanges the two cone points of order three, $\bar{\mu}(w_+)$ and $\bar{\mu}(w_-)$. Furthermore the fixed-point set of $\bar{s}(x,1)$ consists of $\bar{\mu}([1])$, which is a cone point of order two, and the manifold point $\bar{\mu}(w_0)$. This implies that $\Sigma(2,3,3)/\langle\bar{s}(x,1)\rangle = \Sigma(2,3,4)$.

Recall that the manifold $N = \{m; (o_1, 0) : (2,1), (3,\beta_2), (4,\beta_3)\}$ where $b = 12m + 6 + 4\beta_2 + 3\beta_3$, m is the obstruction class, and $g.c.d.\{b, 24\} = 1$, is a octahedral manifold fibering over $\Sigma(2,3,4)$ with fundamental group $O^* \times \mathbb{Z}_b$. (See [6] and note that b and m have been exchanged.)

We therefore have the following theorem.

Theorem 41. *The quotient orbifold $M(b,2)/\langle\widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle$ fibers over $\Sigma(2,3,4)$ and is an octahedral manifold N if and only if $g.c.d.\{b,6\} = 1$. The orbifold fundamental group $\pi_1(M(b,2)/\langle\widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle) \simeq O^* \times \mathbb{Z}_b$ where O^* is the binary octahedral group.*

Proof: By the above we have seen that $M(b,2)/\langle\widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle = O/\langle s(x,1)\rangle$, and that $O/\langle s(x,1)\rangle$ fibers over $\Sigma(2,3,3)/\langle\bar{s}(x,1)\rangle = \Sigma(2,3,4)$, proving the beginning of the first statement. As for the fundamental group, notice that $\pi_1(M(b,2)/\langle\widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle)$ is isomorphic to the group

$$\langle\sigma(i,1), \sigma(j,1), \sigma(1, e^{\frac{2\pi i}{b}}), \sigma(x,1), \sigma(y,1)\rangle.$$

Since $j = yiy^{-1}$ and $(xy)^2 = -i$, we see that

$$\langle\sigma(i,1), \sigma(j,1), \sigma(1, e^{\frac{2\pi i}{b}}), \sigma(x,1), \sigma(y,1)\rangle = \langle\sigma(x,1), \sigma(y,1), \sigma(1, e^{\frac{2\pi i}{b}})\rangle,$$

which is isomorphic to $O^* \times \mathbb{Z}_b$.

By Theorem 34, we see that $M(b,2)/\langle\widehat{\sigma}(y,1)\rangle = O$ is a manifold if and only if $g.c.d.\{b,6\} = 1$. We need only show that the map $s(x,1)$ on O is fixed-point free to finish the proof. Let F_1 be the fiber in $\mathbb{S}^3$ such that $H_l(F_1) = 1 \in \mathbb{S}^2$. Now $\mu\nu(E_+)$ projects to $\bar{\mu}(w_0)$ and $\mu\nu(F_1)$ projects to $\bar{\mu}\bar{\nu}(1)$, which are the fixed points of $\bar{s}(x,1)$. Therefore the only fibers left invariant by $s(x,1)$ in O are $\mu\nu(E_\pm) = E$ and $\mu\nu(F_1) = F$. For $u+vj \in \mathbb{S}^3$, we have $\sigma(x,1)(u+vj) = \frac{1}{\sqrt{2}}[(ui-\bar{v})+(vi+\bar{u})j]$.

We first show $s(x,1)|_E$ is fixed point free. Let $u+vj \in E_+$, and thus $\frac{u}{v} = (1+\sqrt{2})i$. If $s(x,1)(u+vj) = u+vj$, we have the following equations: $\frac{1}{\sqrt{2}}(ui-\bar{v}) = u$ and $\frac{1}{\sqrt{2}}(vi+\bar{u}) = v$. Dividing first equation by $\bar{v}$ and multiplying by $\sqrt{2}$, we obtain $\frac{u}{\bar{v}}i - 1 = \sqrt{2}\frac{u}{\bar{v}}$. Using $\frac{u}{\bar{v}} = (1+\sqrt{2})i$, we see that the left side of this equation is a real number but the right side is a complex number, giving a contradiction. Since

$\sigma(i,1)$, $\sigma(j,1)$, $\sigma(ij,1)$ and $\sigma(y,1)$ do not leave E_+ invariant, we need only check that $\sigma(x,1)(u+vj) \neq \sigma(1, e^{\frac{2k\pi i}{b}})(u+vj)$ for some $0 < k < b$ where $u+vj \in E_+$. Now $\sigma(1, e^{\frac{2k\pi i}{b}})(u+vj) = ue^{-\frac{2k\pi i}{b}} + ve^{\frac{2k\pi i}{b}}j$. If $\sigma(x,1)(u+vj) = \sigma(1, e^{\frac{2k\pi i}{b}})(u+vj)$, then we obtain the equations $ue^{-\frac{2k\pi i}{b}} = \frac{1}{\sqrt{2}}(ui - \bar{v})$ and $ve^{\frac{2k\pi i}{b}} = \frac{1}{\sqrt{2}}(vi + \bar{u})$. Simplifying the first equation and substituting $\frac{u}{\bar{v}} = (1+\sqrt{2})i$, we get the equation $e^{-\frac{2k\pi i}{b}} = i$. Since b is odd, this is impossible. Thus $s(x,1)|_E$ is fixed point free.

We now show $s(x,1)|_F$ is fixed point free. Note that $\sigma(x,1)$ does not leave F_1 invariant, and so we consider $\sigma(i,1) \circ \sigma(x,1)$ which does leave F_1 invariant. For $u+vj \in \mathbb{S}^3$, $\sigma(i,1) \circ \sigma(x,1)(u+vj) = \frac{1}{\sqrt{2}}[(-u - i\bar{v}) + (-v + i\bar{u})j]$. For $u+vj \in F_1$, we have $\frac{u}{\bar{v}} = 1$. If $\sigma(i,1) \circ \sigma(x,1)(u+vj) = u+vj$, we obtain the equations $\frac{1}{\sqrt{2}}(-u - i\bar{v}) = u$ and $\frac{1}{\sqrt{2}}(-v + i\bar{u}) = v$. Dividing by $\bar{v}$ and substituting $\frac{u}{\bar{v}} = 1$ gives a contradiction. As above, we need only see if $\sigma(i,1) \circ \sigma(x,1)(u+vj) = \sigma(1, e^{\frac{2k\pi i}{b}})(u+vj)$ for some $0 < k < b$. In this case, we obtain the equations $ue^{-\frac{2k\pi i}{b}} = \frac{1}{\sqrt{2}}(-u - i\bar{v})$ and $ve^{\frac{2k\pi i}{b}} = \frac{1}{\sqrt{2}}(-v + i\bar{u})$. Dividing the first equation by $\bar{v}$ and using $\frac{u}{\bar{v}} = 1$, we obtain $e^{-\frac{2k\pi i}{b}} = \frac{1}{\sqrt{2}}(-1-i)$, which is impossible. Thus $s(x,1)|_F$ is fixed point free, completing the proof. $\square$

Corollary 42. *Let $M(b,2)$ be a prism manifold with g.c.d.$\{b,6\} = 1$. Then we may write $b = 12m + r$ where $r = 13$, 17, 19, or 23.*

(1) *If $r = 13$, then $M(b,2)/\langle \widehat{\sigma}(x,1), \widehat{\sigma}(y^2,1) \rangle = \{m; (o_1, 0) : (2,1), (3,1), (4,1)\}$.*
(2) *If $r = 17$, then $M(b,2)/\langle \widehat{\sigma}(x,1), \widehat{\sigma}(y^2,1) \rangle = \{m; (o_1, 0) : (2,1), (3,2), (4,1)\}$.*
(3) *If $r = 19$, then $M(b,2)/\langle \widehat{\sigma}(x,1), \widehat{\sigma}(y^2,1) \rangle = \{m; (o_1, 0) : (2,1), (3,1), (4,3)\}$.*
(4) *If $r = 23$, then $M(b,2)/\langle \widehat{\sigma}(x,1), \widehat{\sigma}(y^2,1) \rangle = \{m; (o_1, 0) : (2,1), (3,2), (4,3)\}$.*

We now consider the dihedral group $Dih_2(\mathbb{Z}_3) = \langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,j) \rangle$ acting on $M(b,2)$. By Theorem 34, $M(b,2)/\langle \widehat{\sigma}(y,1) \rangle$ fibers over

$$\Sigma(2,3,3) = \Sigma(2,2,2)/\langle \bar{\sigma}(y,1) \rangle.$$

Since the subgroup generated by $\widehat{\sigma}(y,1)$ in $Dih_2(\mathbb{Z}_3)$ is a normal subgroup of $Dih_2(\mathbb{Z}_3)$, the map $\widehat{\sigma}(x,j)$ induces $s(x,j)$ on $M(b,2)/\langle \widehat{\sigma}(y,1) \rangle$. Likewise, $\bar{\sigma}(x,j)$ on $\mathbb{S}^2$ induces $\widehat{\bar{\sigma}}(x,j)$ and $\bar{s}(x,j)$ on $\Sigma(2,2,2)$ and $\Sigma(2,3,3)$ respectively.

We will show that the induced map $\bar{s}(x,j)$ on $\Sigma(2,3,3)$ fixes all cone points on this orbifold space. Observe first that $\bar{s}(x,j)$ is orientation reversing and must fix the cone point of order two. To show that $\bar{s}(x,j)$ fixes the other cones points, we find $\bar{\sigma}(x,j)$ acting on $\mathbb{S}^2$. Notice that for $u+vj \in \mathbb{S}^3$, $\sigma(x,j)(u+vj) = \frac{1}{\sqrt{2}}(i+j)(u+vj)j^{-1} = \frac{1}{\sqrt{2}}[(\bar{u} + vi) + (-ui + \bar{v})j]$. Hence,

$$H_l(\sigma(x,j)(u+vj)) = \frac{\frac{1}{\sqrt{2}}(\bar{u} + vi)}{\frac{1}{\sqrt{2}}(\bar{u}i + v)} = \frac{(\frac{\bar{u}}{v} + i)}{(\frac{\bar{u}}{v}i + 1)}.$$

As a result, we obtain $\bar{\sigma}(x,j)(z) = \frac{\bar{z}+i}{\bar{z}i+1}$. A computation shows that $\bar{\sigma}(x,j)((1-i)(\frac{-1+\sqrt{3}}{2})) = \frac{(-1+\sqrt{3})(1+i)}{2(2-\sqrt{3})}$ and $\bar{\sigma}(ij,1)(\frac{(-1+\sqrt{3})(1+i)}{2(2-\sqrt{3})}) = (1-i)(\frac{-1+\sqrt{3}}{2})$. Thus $\bar{s}(x,j)$ fixes the cone point $\bar{\nu}\bar{\mu}((1-i)(\frac{-1+\sqrt{3}}{2}))$ of order three, and hence must fix the other cone point of order three. Therefore $\bar{s}(x,j)$ is a reflection fixing all the cone points. This implies $T^v = \Sigma(2,3,3)/\langle \bar{s}(x,j)\rangle = \Sigma(2,2,2)/\langle \bar{\sigma}(y,1), \bar{\sigma}(x,j)\rangle$ and $M(b,2)/\langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,j)\rangle$ is fibered over the base space T^v.

Let $H = \langle x_0, y_0 | x_0^2 = 1, (x_0y_0)^4 = y_0^3 = -1\rangle$. Form the group $\mathbb{Z}_b \circ H$ as follows: if w generates $\mathbb{Z}_b$, then $x_0wx_0^{-1} = w^{-1}$ and $[y_0, w] = 1$.

We have the following theorem:

Theorem 43. *The quotient orbifold $M(b,2)/\langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,j)\rangle$ is fibered over the base space T^v. The orbifold fundamental group $\pi_1(M(b,2)/\langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,j)\rangle) \simeq \mathbb{Z}_b \circ H$.*

PROOF: Identifying x_0, y_0 and w with $\sigma(x,j)$, $\sigma(y,1)$ and $\sigma(1, e^{\frac{2\pi i}{b}})$ respectively proves the result. $\square$

Consider now the dihedral group

$$Dih(\mathbb{Z}_6) = \langle \widehat{\sigma}(y,j), \widehat{\sigma}(x,1)\rangle = \langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,1), \widehat{\sigma}(1,j)\rangle$$

acting on $M(b,2)$. By an earlier argument, $M(b,2)/\langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle$ fibers over $\Sigma(2,3,4)$, where $\Sigma(2,3,4) = \Sigma(2,2,2)/\langle \widehat{\bar{\sigma}}(y,1), \widehat{\bar{\sigma}}(x,1)\rangle$. The map $\sigma(1,j)$ commutes with $\sigma(y,1)$ and $\sigma(x,1)$, hence the map $\sigma(1,j)$ on $\mathbb{S}^3$ induces a map $\widehat{\sigma}(1,j)$ on $M(b,2)$ and $s(1,j)$ on $M(b,2)/\langle \widehat{\sigma}(y,1), \widehat{\sigma}(x,1)\rangle$. Moreover, $\bar{\sigma}(1,j)$ on $\mathbb{S}^2$ induces a map $\widehat{\bar{\sigma}}(1,j)$ on $\Sigma(2,2,2)$ and $\bar{s}(1,j)$ on $\Sigma(2,3,4)$. Note that $s(1,j)$ also induces $\bar{s}(1,j)$. Observe that $\sigma(1,j)(u+vj) = v-uj$, hence $H_l(\sigma(1,j)(u+vj)) = H_l(v-uj) = \frac{v}{-\bar{u}}$. Thus $\bar{\sigma}(1,j)(z) = -\frac{1}{\bar{z}}$. Since $\bar{\sigma}(1,j)$ is an orientation reversing map, so is $\bar{s}(1,j)$ on $\Sigma(2,3,4)$. Since the cone points in $\Sigma(2,3,4)$ have different orders, they are left invariant by $\bar{s}(1,j)$. We conclude that $\bar{s}(1,j)$ is a reflection through a circle containing all the cone points. Thus $\Sigma(2,3,4)/\langle \bar{s}(1,j)\rangle = O^h$ and we have the following theorem.

For the octahedral group $O^* = \langle x_0, y_0, z_0 | x_0^2 = y_0^3 = (x_0y_0)^4 = -1\rangle$ and the group $\mathbb{Z}_4 = \langle z_0 | z_0^2 = -1\rangle$ form the amalgamated direct product $O^* \times_{\mathbb{Z}_2} \mathbb{Z}_4$. Now form the group $\mathbb{Z}_b \circ (O^* \times_{\mathbb{Z}_2} \mathbb{Z}_4)$ as follows: if w generates $\mathbb{Z}_b$, then $[w, x_0] = [w, y_0] = 1$ and $z_0wz_0^{-1} = w^{-1}$.

Theorem 44. *The quotient orbifold $M(b,2)/\langle \widehat{\sigma}(y,j), \widehat{\sigma}(x,1)\rangle$ is fibered over the base space O^h. The orbifold fundamental group $\pi_1(M(b,2)/\langle \widehat{\sigma}(y,j), \widehat{\sigma}(x,1)\rangle) \simeq \mathbb{Z}_b \circ (O^* \times_{\mathbb{Z}_2} \mathbb{Z}_4)$.*

PROOF: To obtain the fundamental group, identify $\sigma(x,1)$, $\sigma(y,1)$, $\sigma(1,j)$ and $\sigma(1, e^{\frac{2\pi i}{b}})$ with x_0, y_0, z_0 and w respectively. $\square$

Next we will state a result from [3], which will be used in the proof of the theorem below.

Theorem 45 ([3]). *Let G be a finite group of isometries operating on $M(b,d)$ preserving the longitudinal fibering, and suppose G does not preserve any Heegaard Klein bottle. Denote $\overline{G}$ to be the induced action on $\Sigma(2,2,d)$. Then $M(b,d) = M(b,2)$, $\overline{G} \simeq \mathbb{Z}_3$, $\mathbb{Z}_6$, $Dih(\mathbb{Z}_3)$ or $Dih(\mathbb{Z}_6)$. Moreover, G is one of the following groups.*

(1) $G = \langle \widehat{\sigma}(y, e^{\frac{p\pi i}{q}}) \rangle$ *or* $G = \langle \widehat{\sigma}(y, e^{\frac{p\pi i}{q}}), \widehat{\sigma}(1, e^{\frac{\pi i}{n}}) \rangle$ *when* $\overline{G} \simeq \mathbb{Z}_3$.
(2) $G = \langle \widehat{\sigma}(y, e^{i\theta} j) \rangle$ *or* $G = \langle \widehat{\sigma}(y, 1) \rangle, \widehat{\sigma}(1, e^{\frac{\pi i}{q}}), \widehat{\sigma}(1, e^{i\theta} j) \rangle$ *when* $\overline{G} \simeq \mathbb{Z}_6$.
(3) $G = \langle \widehat{\sigma}(x, e^{\frac{\pi i}{q}}), \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}) \rangle$, $G = \langle \widehat{\sigma}(x, e^{\frac{\pi i}{q}}),\ \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}), \widehat{\sigma}(1, e^{\frac{\pi i}{m}}) \rangle$, $G = \langle \widehat{\sigma}(x, e^{i\theta} j),\ \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}) \rangle$ *or* $G = \langle \widehat{\sigma}(x, e^{i\theta} j),\ \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}), \widehat{\sigma}(1, e^{\frac{\pi i}{m}}) \rangle$ *when* $\overline{G} \simeq Dih(\mathbb{Z}_3)$.
(4) $G = \langle \widehat{\sigma}(x, e^{\frac{\pi i}{q}}), \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}), \widehat{\sigma}(1, e^{i\theta} j) \rangle$ *or*
$G = \langle \widehat{\sigma}(x, e^{\frac{\pi i}{q}}), \widehat{\sigma}(y, e^{\frac{p\pi i}{n}}),\ \widehat{\sigma}(1, e^{i\theta} j), \widehat{\sigma}(1, e^{\frac{\pi i}{m}}) \rangle$ *when* $\overline{G} \simeq Dih(\mathbb{Z}_6)$.

Theorem 46. *Let G be a finite group of isometries acting on $M(b,d)$ which preserves the longitudinal fibering and does not leave any Heegaard Klein bottle invariant. Denote by G_0 the normal subgroup of G consisting of those elements which leave each fiber invariant. Then $M(b,d) = M(b,2)$, and G/G_0 is either $\mathbb{Z}_3$, $\mathbb{Z}_6$, $Dih(\mathbb{Z}_3)$ or $Dih(\mathbb{Z}_6)$.*

(1) *If G/G_0 is $\mathbb{Z}_3$, then the quotient orbifold $M(b,2)/G$ fibers over $\Sigma(2,3,3)$.*
(2) *If G/G_0 is $\mathbb{Z}_6$, then the quotient orbifold $M(b,2)/G$ fibers over T^h.*
(3) *If G/G_0 is $Dih(\mathbb{Z}_3)$, then the quotient orbifold $M(b,2)/G$ fibers over $\Sigma(2,3,4)$ or T^v.*
(4) *If G/G_0 is $Dih(\mathbb{Z}_6)$, then the quotient orbifold $M(b,2)/G$ fibers over O^h.*

Proof: Noting that $\overline{G} = G/G_0$, the first part follows from the theorem above. Suppose G/G_0 is $\mathbb{Z}_3$. Then $G = \langle \widehat{\sigma}(y, e^{\frac{p\pi i}{q}}) \rangle$ or $G = \langle \widehat{\sigma}(y, e^{\frac{p\pi i}{q}}), \widehat{\sigma}(1, e^{\frac{\pi i}{n}}) \rangle$. We obtain a commutative diagram

$$\begin{array}{ccc} M(b,2) & \xrightarrow{h_l} & \Sigma(2,2,2) \\ \Big\downarrow \nu' & & \Big\downarrow \bar{\nu}' \\ M(b,2)/G & \xrightarrow{\bar{h}_l} & \Sigma(2,2,2)/\overline{G} \end{array}$$

where ν, $\bar{\nu}'$ are orbifold covering maps, and h_l, $\bar{h}_l$ are maps identifying fibers to points.

The induced action $\overline{G}$ on $\Sigma(2,2,2)$ is $\langle \widehat{\bar{\sigma}}(y,1) \rangle$, and we have seen that $\Sigma(2,2,2)/\langle \widehat{\bar{\sigma}}(y,1) \rangle = \Sigma(2,3,3)$. Thus $M(b,2)/G$ fibers over $\Sigma(2,3,3)$. The proof for all other cases is similar. □

References

[1] Dummit D., Foote R., *Abstract Algebra*, Wiley, Hoboken, NJ, 2004.
[2] Kalliongis J., Ohashi R., *Finite group actions on prism manifolds which preserve a Heegaard Klein bottle*, Kobe J. Math. **28** (2011), no. 1, 69–89.

[3] Kalliongis J., Ohashi R., *Classifying non-splitting fiber preserving actions on prism manifolds*, Topology Appl. **178** (2014), 200–218.
[4] Kalliongis J., Ohashi R., *Finite actions on the* 2*-sphere and the projective plane*, preprint.
[5] McCullough D., *Isometries of elliptic* 3*-manifolds*, J. London Math. Soc. **65** (2002), no. 1, 167–182.
[6] Orlik P., *Seifert Manifolds*, Lecture Notes in Mathematics, 291, Springer, Berlin-New York, 1972.
[7] Scott P., *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **5** (1983), 401–48.

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