About G-rings

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Abstract. In this paper, we are concerned with G-rings. We generalize the Kaplansky's theorem to rings with zero-divisors. Also, we assert that if $R \subseteq T$ is a ring extension such that $mT \subseteq R$ for some regular element m of T, then T is a G-ring if and only if so is R. Also, we examine the transfer of the G-ring property to trivial ring extensions. Finally, we conclude the paper with illustrative examples discussing the utility and limits of our results.

Keywords: G-ring; pullback; trivial extension

Classification: 13D05, 13D02

1. Introduction

All rings considered below are commutative with unit and all modules are unital. Let R be a commutative ring and let $Q(R)$ denote the total quotient ring of R. We call R a G-ring if $Q(R) = R[u^{-1}]$ for some regular element $u \in R$ (equivalently, if $Q(R)$ is finitely generated as a ring over R) [1]. This generalizes Kaplansky's definition of G-domain [12]. Also, he shows that if $R \subseteq T$ are domains and if T is algebraic over R and finitely generated as a ring over R, then R is a G-domain if and only if so is T [12, Theorem 22].

In this paper, we are concerned with G-rings. Our main result of Section 2 is to generalize the above Kaplansky's theorem to rings with zero-divisors. Also, we assert that if $R \subseteq T$ is a ring extension such that $mT \subseteq R$ for some regular element m of T, then T is a G-ring if and only if so is R. As an immediate consequence, we get a corollary on the transfer of the G-ring property to pullbacks issued from domains. Our main result of Section 3 examines the transfer of the G-ring property to trivial ring extensions; precisely, it states that if A is a ring, E is an A-module such that $Z(E) \subseteq Z(A)$ (where $Z(E) := \{a \in A; ae = 0$ for some $e \in E - \{0\}\}$ is the set of zero-divisors on E), then the trivial extension of A by E is a G-ring if and only if A is a G-ring. In Section 4, we conclude the paper with illustrative examples discussing the utility and limits of our results.

2. The G-ring property in a pullback

Let R be a ring and $R_u := R[1/u]$, where u is regular in R. We first give a zero-divisor extension of Kaplansky's theorem [12, Theorem 22].

DOI 10.14712/1213-7243.2015.196

Theorem 2.1. *Let R be a subring of T such that each regular element of R is regular in T (consequently, $K := Q(R) \subseteq L := Q(T)$). Assume that L is integral over K. Then:*

(1) *if R is a G-ring, then T is a G-ring;*
(2) *if T is a finitely generated R-algebra, then T is a G-ring if and only if R is a G-ring.*

Proof: (1) Assume that R is a G-ring. Hence, $K := Q(R) = R_u$ for some regular element $u \in R$. But, $K := R_u \subseteq T_u \subseteq L := Q(T)$. Hence, L is integral over T_u since L is integral over K. Therefore, $L = T_u$ since L is a fraction ring of T_u and so T is a G-ring.

(2) If R is a G-ring, then T is a G-ring by (1). Conversely, assume that T is a G-ring. Hence, $L = T_v$ for some regular element $v \in T$ and $T = R[w_1, \dots, w_k]$ for some $w_i \in T$ and for a positive integer k (since T is a finitely generated R-algebra). Then, the elements v^{-1}, $w_1, \dots, w_k$ are integral over K. So, we get Kaplansky's equations (see proof of [12, Theorem 22]) with a, b_i being regular elements of R. Let $R_1 := R[a^{-1}, b_1^{-1}, \dots, b_k^{-1}]$. As argued by [12, Theorem 22], $L = R_1[w_1, \dots, w_k, v^{-1}]$ and L is integral over R_1. Then, K is integral over R_1 and so $K = R_1$ since K is a fraction ring of R_1. Hence, R is a G-ring and this completes the proof of Theorem 2.1. □

Now, we provide a somewhat analogue of a zero-divisor extension of Kaplansky's result mentioned above. Precisely, we have:

Theorem 2.2. *Let $R \subseteq T$ be a ring extension such that $mT \subseteq R$, for some regular element $m \in T$. Then T is a G-ring if and only if R is a G-ring.*

The proof of this theorem requires the following lemma.

Lemma 2.3. *Let R be a ring and $R_f = R[1/f]$, where f is regular in R. Then R is a G-ring if and only if R_f is a G-ring.*

Proof: It is clear that $R_f = \{af^{-n}; a \in R \text{ and } n \in \mathbb{N}\}$. Hence, $Q(R_f) = Q(R)$ since af^{-n} is regular in R_f if and only if a is regular in R (because f is invertible in R_f).

Assume that R is a G-ring. Hence, $Q(R) = R_u$ for some regular element $u \in R$. But, $Q(R_f) = Q(R) = R_u \subseteq (R_f)_u \subseteq Q(R_f)$. Therefore, $Q(R_f) = (R_f)_u$ and so R_f is a G-ring.

Conversely, assume that R_f is a G-ring, that is, $Q(R_f) = (R_f)_u$ for some regular element $u \in R_f$. We may assume that $u \in R$ since $u = af^{-n}$ for some regular element $a \in R$ and $n \in \mathbb{N}$, and since f^{-n} is invertible in R_f. It is well-known and easy to see that $(R_f)_u = R_{fu}$. Therefore, $Q(R) = Q(R_f) = (R_f)_u \subseteq R_{fu} \subseteq Q(R)$ and so $Q(R) = R_{fu}$ and this completes the proof of Lemma 2.3. □

Proof of Theorem 2.2: Let $R \subseteq T$ be a ring extension such that $mT \subseteq R$, for some regular element m of T. Clearly, $m \in R$ and m is regular element of R. But $R_m = T_m$ since $R_m \subseteq T_m = \{am^{-n}; a \in T \text{ and } n \in \mathbb{N}\} = \{(am)m^{-(n+1)}; (am) \in R \text{ and } n \in \mathbb{N}\} \subseteq R_m$. Therefore, R is a G-ring if and only if T is a G-ring by Lemma 2.3 since $T_m = R_m$. □

The above result generates new families of examples of G-domains not covered by Kaplansky's result [12, Theorem 22] mentioned above. It also denies any similitude with this result as shown by the following corollary.

Corollary 2.4. *Let D be a domain which is not a G-domain, $K = Q(D)$ and T a domain such that $T/M = K$ for some nozero maximal ideal M of T. Let $f : T \to K$ be the canonical surjection and $R = f^{-1}(D)$. Then:*

(1) *T is a G-domain if and only if R is a G-domain;*
(2) *T is not finitely generated as a ring over R.*

Proof: (1) Results by Theorem 2.2 because $mT \subseteq M \subseteq kerf \subseteq R$ and $R_m = T_m$ for each nonzero m in M.

(2) Assume that T is finitely generated as a ring over R. Then $T = R[x_1, \ldots, x_n]$, for some $x_i \in T$, where n is a positive integer. Hence, $K = T/M = (R/M)[\bar{x_1}, \ldots, \bar{x_n}] = D[\bar{x_1}, \ldots, \bar{x_n}]$, a contradiction since D is not a G-domain. Therefore, T is not finitely generated as a ring over R. □

Remark 2.5. Part (1) of Corollary 2.4 generalizes [9, Theorem 2.7 (a), p. 341].

A pair of rings $A \subseteq B$ is called a G-ring pair if D is a G-ring for each ring D such that $A \subseteq D \subseteq B$. In [6, Theorem 2.1], Dobbs gives necessary and sufficient conditions to have a G-domain pair. In the context of Theorem 2.2, we obtain:

Corollary 2.6. *Let T, R, and m be as in Theorem 2.2. Then (R, T) is a G-ring pair if and only if T (resp., R) is a G-ring.*

Proof: Let S be a ring such that $R \subseteq S \subseteq T$. Hence, $mS \subseteq mT \subseteq R$ and m is regular in S. Therefore, Theorem 2.2 completes the proof of Corollary 2.6. □

Remark 2.7. In Theorem 2.2, the hypothesis "m is a regular element of T" is necessary (see Example 4.4).

3. G-ring property in trivial extension

Let A be a ring, E be an A-module and $R = A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Recall that a maximal ideal of R has always the form $M \propto E$, where M is a maximal ideal of A [11, Theorem 25.1(3)]. The author of [11] also confirms by a private communication that [11, Theorem 25.1] is not true, that is, an ideal J of R has not always the form: $J = I \propto E^{'}$, where $I = \{a \in A \,|\, (a, e) \in J \text{ for some } e \in E\}$ and $E^{'} = \{e \in E \,|\, (a, e) \in J \text{ for some } a \in A\}$. We only have that $J \subseteq I \propto E^{'}$

(see [14]). Nevertheless, it is easily seen that $J = I \propto E'$ if and only if $0 \propto E' \subseteq J$ if and only if $I \propto 0 \subseteq J$.

In this section, we study the possible transfer of the G-ring property for various trivial extension contexts.

Theorem 3.1. *Let A be a ring, E be an A-module such that $Z(E) \subsetneq Z(A)$ (where $Z(E)$ denotes the set of zero-divisors on E), and $R := A \propto E$ be the trivial ring extension of A by E. Then R is a G-ring if and only if A is a G-ring.*

Proof: Set $S = A - Z(A)$. Then $Z(R) = Z(A) \propto E$ and $Q(R) = Q(A) \propto E_S$ by [11, p. 164–165]. Assume that A is a G-ring. Hence, $Q(A) = A_a$ for some $a \in S$. Then, $(a, 0) \notin Z(R)$ and $E_a := E \otimes_A A_a = E \otimes_A Q(A) = E_S$. So, $Q(R) = Q(A) \propto E_S = A_a \propto E_a = \{(xa^{-n}, ea^{-m}); (x, e) \in R \text{ and } n, m \in \mathbb{N}\} = \{(xa^{p-n}, ea^{p-m})(a, 0)^{-p}; (x, e) \in R,\ n, m \in \mathbb{N} \text{ and } p = \sup(n, m)\} \subseteq R_{(a,0)} \subseteq Q(R)$. Therefore, $Q(R) = R_{(a,0)}$ and then R is a G-ring.

Conversely, assume that R is a G-ring. Hence, $Q(R) = R_{(a,e)}$ for some $(a, e) \notin Z(R)$. If $Q(R) := Q(A) \propto E_S$ and $p : Q(R) \to Q(A)$ is the map $p(x, y) = x$, we claim that $Q(A)(= p(R_{(a,e)})) = A_a$. Indeed, let $(x, y)(a, e)^{-n} \in R_{(a,e)}$, where $(x, y) \in R$ and $n \in \mathbb{N}$. Hence, $a^n p((x, y)(a, e)^{-n}) = p((a, 0)^n (x, y)(a, e)^{-n}) = p((x, y)((a^{-n}, 0)(a, e)^n)^{-1}) = p((x, y)((a^{-n}, 0)(a^n, e_n))^{-1})$ where $e_n \in E$. This is equal to $p((x, y)(1, a^{-n}e_n)^{-1}) = p((x, y)(1, -a^{-n}e_n)) = p(x, y - xa^{-n}e_n) = x \in A$, so $p((x, y)(a, e)^{-n}) = xa^{-n} \in A_a$. Therefore, $Q(A) = A_a$ and then A is a G-ring. □

If A is a domain and E is a torsion-free A-module, we obtain by Theorem 3.1:

Corollary 3.2. *Let A be a domain, E be a torsion-free A-module, and $R := A \propto E$ be the trivial ring extension of A by E. Then R is a G-ring if and only if A is a G-domain.*

If $R := A \propto E$ is a trivial extension of a ring A by an A-module E, we do not have in general that R is a G-ring if and only if A is a G-ring, as shown by the following result.

Proposition 3.3. *Let (A, M) be a local ring and E an A-module such that $ME = 0$. Then the trivial ring extension of A by E is a G-ring.*

Proof: The result holds since the trivial ring extension of A by E is a total ring (since $(M \propto E)(0, 1) = (0, 0)$ and $M \propto E$ is a maximal ideal of a local ring $A \propto E$). □

4. Examples

In this section, we exhibit a non-Noetherian coherent G-domain (Example 4.1). Then, we give non-coherent G-rings (Examples 4.2 and 4.3). We also show that if $f : R \to S$ is a faithfully flat ring extension such that S is a G-ring, then R is not a G-ring, in general (Examples 4.1(4) and 4.2(3)). Finally, we give a counter-example showing that the hypothesis "m is a regular element of T" is necessary in Theorem 2.1 (Example 4.4).

Example 4.1. Let $T = \mathbb{Q}[[X]] = \mathbb{Q} + XT$ be the formal power series ring over the field $\mathbb{Q}$ and let $R = \mathbb{Z} + XT$. Then:

(1) R is a G-domain by Theorem 2.2 since T is a local G-domain and $XT \subseteq R$;
(2) R is a coherent domain by [8, Theorem 3] and is not Noetherian by [4, Theorem 4];
(3) T is not finitely generated as a ring over R by Corollary 2.4;
(4) $\mathbb{Z} \to R$ is a faithfully flat ring extension and $\mathbb{Z}$ is not a G-domain.

Example 4.2. Let $T = \mathbb{R}[X]_{(X)} = \mathbb{R} + XT$, where X is an indeterminate over $\mathbb{R}$, and let $R = \mathbb{Z} + XT$. Then:

(1) R is a G-domain by Theorem 2.2 since T is a local G-domain and $XT \subseteq R$;
(2) R is not a coherent domain ([8, Theorem 3]);
(3) $\mathbb{Z} \to R$ is a faithfully flat ring extension and $\mathbb{Z}$ is not a G-domain.

Example 4.3. Let A be a G-domain which is not a field, $K = qf(A)$, and let $R := A \propto K$ be the trivial ring extension of A by K. Then:

(1) R is a G-ring by Corollary 3.2 since A is a G-domain;
(2) R is not a coherent ring since $R(0,1)$ is a finitely generated ideal which is not finitely presented as shown by the exact sequence of R-modules:

$$0 \to 0 \propto K \to R \xrightarrow{u} R(0,1) \to 0$$

where $u(a,e) = (a,e)(0,1) = (0,a)$ (since $0 \propto K$ is not a finitely generated ideal of R).

Example 4.4. Let A be a non G-domain, $K = qf(A)$, $T = K \propto K$ be the trivial ring extension of K by K, and let $R := A \propto K$ be the trivial ring extension of A by K. Then:

(1) T is a G-ring since it is a total ring;
(2) R is not a G-ring by Corollary 3.2 since A is not a G-domain;
(3) $(0,1)T = 0 \propto K \subseteq R$.

Acknowledgment. I would like to thank the referee for a careful reading of this manuscript.

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(*Received* December 20, 2015, *revised* July 25, 2016)