Compactness theorems for the Bakry-Emery Ricci tensor on semi-Riemannian manifolds

M.S. Santos

Abstract. In this manuscript we provide new extensions for the Myers theorem in weighted Riemannian and Lorentzian manifolds. As application we obtain a closure theorem for spatial hypersurfaces immersed in some time-like manifolds.

Keywords: Bakry-Emery Ricci curvature tensor; closure theorem; Riccati equation

Classification: 53C20

1. Introduction

Given a semi-Riemannian manifold (M^n, g) and a smooth function f on M^n, the *weighted manifold* M^n_f associated to M^n and f is the triple $(M^n, g, d\mu = e^{-f} d\nu)$, where $d\nu$ is the volume element of M. In the literature it is also called *smooth metric space*. The analysis of the weighted geometry in M_f is deeply related with a family of Ricci tensors introduced by Bakry and Emery in [1] given by

$$\mathrm{Ric}\,_f^k = \mathrm{Ric}\, + \mathrm{Hess}\, f - \frac{1}{k} df \otimes\, df,$$

where $k \in (0, \infty)$.

In the Riemannian context, the classical Myers compactness theorem [15] was extended for weighted manifolds in many ways. See for instance, [2], [5], [11], [12], [13], [14], [16], [17], [21], [24]. Exploring a Riccati inequality obtained from the Bochner formula we get the following improved version of weighted Myers theorem in the spirit of the Sprouse theorem in [20].

Theorem 1.1. *Let M^n_f be a weighted complete Riemannian manifold. Then for any $\delta > 0$, and $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each geodesic γ emanating from p, the $\mathrm{Ric}\,_f^k$ *curvature satisfies*

$$\int_0^\infty \max\left\{(n-1)a - \mathrm{Ric}\,_f^k(\gamma', \gamma'), 0\right\} dt < \epsilon(n, a, \delta) \tag{1.1}$$

then M is compact with $\mathrm{diam}\,(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

DOI 10.14712/1213-7243.2015.177

Recently, the geometry of weighted Lorentzian manifold has been subject of great interest. For instance in [4], J. Case has shown that certain aspects of the Bakry-Emery comparison theory can be adapted to a Lorentzian manifold, allowing him to prove a Bakry-Emery version of the Hawking-Penrose singularity theorem for general relativity. This result leads to a Hawking-Penrose theorem for scalar-tensor gravitation theories. In this same context, we can mention Rupert and Woolgar in [19], Woolgar in [22] and Galloway and Woolgar in [9].

Motivated by the ideas of Yun in [23] we obtain an extension of Theorem 1.1 for *weighted globally hyperbolic space-time* (see Section 2.2). The main tool in our approach is a Raychaudhuri type inequality raised by Case in [4]. Precisely, we obtain the following result:

Theorem 1.2. *Let M^n_f be a weighted globally hyperbolic space-time. Then for any $\delta > 0$, $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each future directed time-like geodesic γ emanating from p, with $l(\gamma) = \sup\{t \geq 0, d(p, \gamma(t)) = t\}$ the Ric^k_f curvature satisfies

$$\int_0^{l(\gamma)} \max\left\{(n-1)a - \mathrm{Ric}^k_f(\gamma', \gamma'), 0\right\} \, dt < \epsilon(n, a, \delta)$$

then the time-like diameter satisfies $\mathrm{diam}\,(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta.$

Theorem 1.1 can be applied in the context of space-time manifolds. Namely, suppose that $\bar{M}^{n+1}$ is a space-time and M^n is a spatial hypersurface of $\bar{M}$. It is an interesting problem to inquire under which conditions M is compact. In this sense, the works of Galloway in [8] and Galloway and Frankel in [7] are very successful. Recently, Cavalcante, Oliveira and the author [5] generalized the closure theorems of [7] and [8] in the weighted case. Here, using Theorem 1.1 we are able to improve Theorem 4.3 of [5] as follows:

Theorem 1.3. *Let M^n be a spatial hypersurface in $\bar{M}^{n+1}_f$ and assume that M is complete in the induced metric. Then for any $\delta > 0$, $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point $p \in M$ such that along each geodesic γ in M emanating from p, the condition

$$\int_0^{\infty} \max\left\{(n-1)a - \overline{\mathrm{Ric}}^k_f(X, X) - \langle v(X), X\rangle H_f + \langle a(X), X\rangle, 0\right\} dt < \epsilon(n, a, \delta)$$

is satisfied, where $X = \gamma'$, then M is compact and $\mathrm{diam}\,(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta.$

2. Weighted Myers theorems

2.1 The Riemannian case. Given a weighted manifold M^n_f the *weighted Laplacian operator* can be defined by $\Delta_f u = \Delta u - \langle \nabla f, \nabla u\rangle$, for functions u of class

C^2 on M. It is a remarkable fact that the weighted Laplacian satisfies a Bochner type inequality (see [10] or [21]).

Fixed $p \in M$ let $r(x) = d(x,p)$ denote the distance function and let $m_f(r)$ be the weighted Laplacian of the distance function. From the Bochner inequality for the weighted Laplacian one can verify that m_f satisfies the following Riccati inequality (see Appendix A of [21]):

$$m_f' + \frac{m_f^2}{n+k-1} \leq -\mathrm{Ric}_f^k(\gamma',\gamma'), \quad \text{for all} \ \ t \geq 0, \tag{2.1}$$

where γ is a geodesic emanating from p and m_f' stands for the derivative of m_f with respect to r. The inequality (2.1) is the main tool in the proof of our first result.

Proof of Theorem 1.1: For any small positive $\epsilon < a^2$ to be determined later, consider the following sets

$$E_1 = \left\{t \in [0,\infty); \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)) \geq (n-1)(a-\sqrt{\epsilon})\right\}$$

and

$$E_2 = \left\{t \in [0,\infty); \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)) < (n-1)(a-\sqrt{\epsilon})\right\}.$$

From the inequality (2.1) we have on E_1

$$\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}} \leq -1. \tag{2.2}$$

On the other hand, on E_2 we have

$$\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}} \leq \frac{a-\sqrt{\epsilon}-\mathrm{Ric}_f^k(\gamma',\gamma')/(n-1)}{a-\sqrt{\epsilon}}. \tag{2.3}$$

Now, using the assumption (1.1) on the Bakry-Emery Ricci tensor we get

$$\begin{aligned}
\epsilon &> \int_0^\infty \max\left\{(n-1)a - \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)), 0\right\} \\
&> \int_{E_2} \left\{(n-1)a - \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t))\right\} \, dt \\
&> \int_{E_2} \left\{(n-1)a - (n-1)(a-\sqrt{\epsilon})\right\} \, dt \\
&= \mu(E_2)(n-1)\sqrt{\epsilon}.
\end{aligned}$$

That is, we have

$$\mu(E_2) < \frac{\sqrt{\epsilon}}{n-1}, \tag{2.4}$$

where μ is the Lebesgue measure on $\mathbb{R}$.

Using inequalities (2.2)−(2.4) we obtain

$$\begin{aligned}
\int_0^r \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt &\leq \int_{[0,r]\cap E_1} \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt \\
&\quad + \int_{[0,r]\cap E_2} \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt \\
&\leq -\mu\{[0,r]\cap E_1\} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})} \\
&\leq -r + \mu\{[0,r]\cap E_2\} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})} \\
&\leq -r + \frac{\sqrt{\epsilon}}{n-1} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})}\,.
\end{aligned}$$

Define $\tau(\epsilon) = \frac{\sqrt{\epsilon}}{n-1} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})}$. The integral of the left hand side can be computed explicitly and therefore we get

$$\arctan\left(\frac{m_f(r)}{(n+k-1)a(\epsilon)}\right) \leq a(\epsilon)(-r+\tau(\epsilon)) + \frac{\pi}{2}\,,$$

where $a(\epsilon) = \sqrt{\frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}$.

So we get

$$m_f(r) \leq -(n+k-1)a(\epsilon)\cot(a(\epsilon)(-r+\tau(\epsilon))),$$

for any r such that $\tau(\epsilon) < r < \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$.

In particular, $m_f(\gamma(r))$ goes to $-\infty$ as $r \to (\frac{\pi}{a(\epsilon)} + \tau(\epsilon))^+$. It implies that γ cannot be minimal beyond $\frac{\pi}{a(\epsilon)} + \tau(\epsilon)$. Otherwise m_f would be a smooth function at $r = \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$. Taking ϵ explicitly so that $\frac{\pi}{a(\epsilon)} + \tau(\epsilon) = \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$ and using the completeness of M we have the desired result. □

2.2 The Lorentzian case. Now let us discuss the Lorentzian version of the Theorem 1.1. Let M be a time-oriented Lorentzian manifold. Given $p \in M$ we set

$$J^+(p) = \{q \in M : \text{ there exist a future pointing causal curve from } p \text{ to } q\},$$

called the *causal future* of p. The *causal past* $J^-(p)$ is defined similarly. We say that M is *globally hyperbolic* if the set $J(p,q) := J^+(p) \cap J^-(q)$ is compact for all p and q joined by a causal curve (see [3]). Mathematically, global hyperbolicity often plays a role analogous to geodesic completeness in Riemannian geometry.

Let $\gamma : [a,b] \longrightarrow M$ be a future-directed time-like unit-speed geodesic. Given $\{E_1, E_2, \ldots, E_n\}$ an orthonormal frame field along γ, and for each $i \in \{1, \ldots, n\}$

we let J_i be the unique Jacobi field along c such that $J_i(a) = 0$ and $J_i'(0) = E_i$. Denote by A the matrix $A = [J_1 J_2 \dots J_n]$, where each column is just the vector for J_i in the basis defined by $\{E_i\}$. In this situation, we have that $A(t)$ is invertible if and only if $\gamma(t)$ is not conjugate to $\gamma(a)$.

Now we define $B_f = A'A^{-1} - \frac{1}{n-1}(f \circ \gamma)'E$ wherever A is invertible, where $E(t)$ is the identity map on $(\gamma'(t))^{\perp}$. The *f-expansion function* is a smooth function defined by $\theta_f = \operatorname{tr} B_f$ (see [4, Definition 2.6]). Note that, if $|\theta_f| \to \infty$ as $t \to t_0$, where $t_0 \in [a, b]$, then $\gamma(t_0)$ is conjugate to $\gamma(a)$.

Recently, Case in [4] obtained the following relation between the Bakry-Emery Ricci tensor and the *f-expansion function θ_f*:

Lemma 2.1. *Under the above notations,*

$$\theta_f' \leq -\operatorname{Ric}_f^k(\gamma', \gamma') - \frac{\theta_f^2}{k+n-1}. \tag{2.5}$$

The inequality (2.5) is called (k, f)-*Raychaudhuri inequality*. This inequality is a generalization of the well known Raychaudhuri inequality (see for instance [6]).

The *distance* between two time-like related points is the supremum of lengths of causal curves joining the points. It follows that the distance between any two time-like related points in a globally hyperbolic space-time is the length of such a maximal time-like geodesic. The *time-like diameter*, $\operatorname{diam}(M)$, of a Lorentzian manifold is defined to be the supremum of distances $d(p, q)$ between points of M.

Proof of Theorem 1.2: Let $\epsilon(n, a, \delta)$ be the explicit constant in the previous theorem. Assume by contradiction that there are two points p and q with $d(p, q) > \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$. On the other hand, since M is globally hyperbolic, there exists a maximal time-like geodesic γ joining p and q such that $\ell(\gamma) = d(p, q)$. Following the steps of the proof of Theorem 1.1 using the (k, f)-Raychaudhuri inequality (2.5) we get

$$\lim_{t \to t_0^+} \theta_f(t) = -\infty,$$

where $t_0 = \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$. So, we conclude that γ cannot be maximal beyond $\frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$ which is a contradiction. □

An immediate consequence of Theorem 1.2 is the Lorentzian version of the original weighted Myers theorem obtained by Qian in [16]. Namely:

Corollary 2.2. Let M_f^n be a weighted globally hyperbolic space-time. Let a be a positive constant and assume that

$$\operatorname{Ric}_f^k(v, v) \geq (n-1)a,$$

for all unit time-like vector field $v \in TM$. Then the time-like diameter satisfies $\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}}$.

3. Application

Let $\bar{M}^{n+1}$ be an $(n+1)$-dimensional space-time manifold. Let M^n be a spatial hypersurface immersed in $\bar{M}^{n+1}$. That is, the metric induced in M^n by $\bar{M}^{n+1}$ is a Riemannian metric. Then the unit tangent vectors to the future directed time-like geodesics orthogonal to M define a smooth unit time-like vector field $\mathbf{u}$ in a neighborhood of M. For the sake of simplicity we will omit the dimension super index and we always use a *bar* for geometric objects related to $\bar{M}$.

Let X be a vector field tangent to M. Extend X along the flow lines generated by $\mathbf{u}$, and suppose that X is invariant under the flow generated by $\mathbf{u}$, i.e, $[X, \mathbf{u}] = 0$, where $[,]$ is the Lie bracket. The *velocity* and *acceleration* of X along the flow generated by $\mathbf{u}$ are given respectively by

$$v(X) = \overline{\nabla}_{\mathbf{u}} X \quad \text{and} \quad a(X) = \overline{\nabla}_{\mathbf{u}} \overline{\nabla}_{\mathbf{u}} X.$$

The second fundamental form of M as a hypersurface of $\bar{M}$ is defined by $b(X) = -\overline{\nabla}_X \mathbf{u}$. Let H denote its *mean curvature* function. We point out that $H = -\overline{\mathrm{div}}\mathbf{u}$, that is, the averaged *Hubble expansion parameter* at points of M in relativistic cosmology (see [18, §3.3.1] or [6, p. 161]).

The *f-mean curvature* or *weighted mean curvature* of M is defined by

$$H_f = H - (\overline{\nabla} f)^{\perp},$$

where $(\overline{\nabla} f)^{\perp}$ is a normal projection of $\overline{\nabla} f$ on M.

We will use the following technical lemma obtained in [7] (see also [5]).

Lemma 3.1. *Under the above conditions, let $\gamma(s)$ be a normalized geodesic in M and set $X = \gamma'$. Then*

$$\begin{aligned} \mathrm{Ric}(X,X) = \overline{\mathrm{Ric}}(X,X) - \langle a(X), X\rangle + \langle v(X), X\rangle H + \langle v(X), X\rangle^2 \\ + \sum_{j=2}^{n} \langle v(X), e_j\rangle^2, \end{aligned} \tag{3.1}$$

where $\{e_1 = X, e_2, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Finally, we are in position to prove our closure theorem.

Proof of Theorem 1.3: Taking into account that X is invariant under the flow, we get $v(X) = -b(X)$. Then

$$\begin{aligned} \overline{\mathrm{Hess}}\, f(X,X) &= \langle \overline{\nabla}_X \overline{\nabla} f, X\rangle \\ &= \mathrm{Hess}\, f(X,X) + \langle \overline{\nabla} f, \mathbf{u}\rangle \langle b(X), X\rangle \\ &= \mathrm{Hess}\, f(X,X) - \langle \overline{\nabla} f, \mathbf{u}\rangle \langle v(X), X\rangle. \end{aligned} \tag{3.2}$$

From equations (1), (3.1) and (3.2) we get easily

$$\mathrm{Ric}_f^k(X,X) \geq \overline{\mathrm{Ric}}_f^k(X,X) - \langle a(X), X\rangle + \langle v(X), X\rangle H_f,$$

for all $k > 0$. Thus

$$(n-1)a - \mathrm{Ric}\,_f^k(X,X) \le (n-1)a - \overline{\mathrm{Ric}}\,_f^k(X,X) + \langle a(X), X\rangle - \langle v(X), X\rangle H_f.$$

Applying Theorem 1.1 we have the desired result. $\square$

Acknowledgment. The results in this paper are part of the author's doctoral thesis. While the author was at Universidade Federal de Alagoas, he was fully supported by CAPES-Brazil. He wants to thank M. Cavalcante for the valuable suggestions and comments.

References

[1] Bakry D., Emery E., *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., 1123, Springer, Berlin, 1985, pp. 177–206.

[2] Bakry D., Ledoux M., *Sobolev inequalities and Myers diameter theorem for an abstract Markov generator*, Duke Math. J. **81** (1996), no. 1, 252–270.

[3] Beem J., Ehrlich P., Easley K., *Global Lorentzian Geometry*, 2nd edn., Marcel Dekker, New York, 1996.

[4] Case J., *Singularity theorems and the lorentzian splitting theorem for the Bakry-Emery-Ricci tensor*, J. Geom. Phys. **60** (2010), no. 3, 477–490.

[5] Cavalcante M.P., Oliveira J.Q., Santos M.S., *Compactness in weighted manifolds and applications*, Results Math. **68** (2015), 143–156.

[6] Frankel T., *Gravitation Curvature. An Introduction to Einstein's Theory*, W.H. Freeman and Co., San Francisco, Calif., 1979.

[7] Frankel T., Galloway G., *Energy density and spatial curvature in general relativity*, J. Math. Phys. **22** (1981), no. 4, 813–817.

[8] Galloway G.J., *A generalization of Myers theorem and an application to relativistic cosmology*, J. Differential Geom. **14** (1979), 105–116.

[9] Galloway G.J., Woolgar E., *Cosmological singularities in Bakry-Émery space-times*, preprint, 2013.

[10] Ledoux M., *The geometry of Markov diffusion generators*, Ann. Fac. Sci. Toulouse Math. **9** (2000), no. 2, 305–366.

[11] Limoncu M., *The Bakry-Emery Ricci tensor and its applications to some compactness theorems*, Math. Z. **271** (2012), 715–722.

[12] Limoncu M., *Modifications of the Ricci tensor and applications*, Arch. Math. (Basel) **95** (2010), 191–199.

[13] Lott J., *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. **78** (2003), no. 4, 865–883.

[14] Morgan F., *Myers' theorem with density*, Kodai Math. J. **29** (2006), no. 3, 454–461.

[15] Myers S.B., *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941) 401–404.

[16] Qian Z., *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford **48** (1997), 235–242.

[17] Rimoldi M., *A remark on Einstein warped products*, Pacific J. Math. **252** (2011), no. 1, 207–218.

[18] Ringström H., *On the Topology and Future Stability of the Universe*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2013.

[19] Rupert M., Woolgar E., *Bakry-Émery black holes*, Classical Quantum Gravity **31** (2014), no. 2, 025008.

[20] Sprouse S., *Integral curvature bounds and bounded diameter*, Comm. Anal. Geom. **8** (2000), 531–543.

[21] Wei G., Wylie W., *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405.

[22] Woolgar E., *Scalar-tensor gravitation and the Bakry-Emery-Ricci tensor*, Classical Quantum Gravity **30** (2013) 085007.

[23] Yun J.-G., *A note on the generalized Myers theorem*, Bull. Korean Math. Soc. **46** (2009), no. 1, 61–66.

[24] Zhang S., *A theorem of Ambrose for Bakry-Emery Ricci tensor*, Ann. Global Anal. Geom. **45** (2014), no. 3, 233–238.

Universidade Federal do Ceará, Campus Russas, CE, CEP 62900-000, Brazil

E-mail: marciosantos@ufc.br

(*Received* February 2, 2016)