On an affirmative answer to Y. Tanaka's and Y. Ge's problem

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Abstract. In this paper, we give an affirmative answer to the problem posed by Y. Tanaka and Y. Ge (2006) in *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math. **32** (2006) no. 1, 99–117.

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1. Introduction and preliminaries

One of the central problems in general topology is to establish relationships between various topological spaces and metric spaces by means of various maps. Some characterizations for certain quotient compact images of metric spaces are obtained by means of σ-strong networks ([1], [2], [5], [9]), [11]). In 2006, Y. Tanaka and Y. Ge gave some characterizations around sequence-covering quotient compact images of metric spaces that are obtained in terms of symmetric spaces ([9]). Also, the authors posed the following question.

Question 1.1 ([9, Question 3.5]). *Let X be a space satisfying one of the following properties.*

(1) *X is a sequence-covering quotient compact, σ-image of a metric space.*
(2) *X is a sequence-covering quotient π, σ-image of a metric space.*
(3) *X is a Cauchy symmetric, $\aleph$-space.*

Then, is X a strongly g-developable space?

In this paper, we give an affirmative answer to Question 1.1.

Throughout this paper, all spaces are T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, we denote

$$(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\};$$
$$\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

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For a sequence $\{x_n\}$ converging to x, we say that $\{x_n\}$ is *eventually* in P, if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P, if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.2. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is a *network at x* in X, if $x \in P$ for every $P \in \mathcal{P}$, and whenever $x \in U$ with U is open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is a *cs-network* for X [9], if each sequence S converging to a point $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a *cs-cover* [12], if every convergent sequence is eventually in some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is *point-countable* [4], if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.

(5) $\mathcal{P}$ is *locally finite* [4], if for each $x \in X$, there exists a neighborhood V of x such that V meets only finitely many members of $\mathcal{P}$.

Definition 1.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following (1) and (2) for every $x \in X$.

(1) $\mathcal{P}_x$ is a network at x.

(2) If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

$\mathcal{P}$ is a *weak base* for X [3], if for $G \subset X$, G is open in X if and only if for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base* at x.

Definition 1.4. Let (X, d) be a symmetric space.

(1) X is *Cauchy symmetric* [7], if every convergent sequence is d-Cauchy.

(2) For each $x \in X$ and $n \in \mathbb{N}$, let $S_n(x) = \{y \in X : d(x, y) < 1/n\}$.

(3) Let $P \subset X$, we put $d(P) = \sup\{d(x, y) : x, y \in P\}$.

Remark 1.5. (1) Let X be a symmetric space. Then $\{S_n(x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x in X for all $x \in X$.

(2) X is Cauchy symmetric if and only if for each $x \in X$, $d\big(S_n(x)\big)$ converges to 0, see [7].

Definition 1.6. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$.

(1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X [5], if $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each point $x \in X$.

(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-locally finite strong network* for X [9], if it is a σ-strong network and each $\mathcal{P}_n$ is locally finite.

Definition 1.7 ([9])**.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally finite strong network for a space X.

(1) $\mathcal{P}$ is a *σ-locally finite strong weak base*, if $\mathcal{P}$ is a weak base.

(2) $\mathcal{P}$ is a *σ-locally finite strong network consisting of cs-covers*, if each $\mathcal{P}_n$ is a *cs*-cover.

For some undefined or related concepts, we refer the reader to [4], [8] and [9].

2. Main results

Theorem 2.1. *The following are equivalent for a space X.*

(1) *X is a sequence-covering quotient compact, σ-image of a metric space.*
(2) *X is a sequence-covering quotient π, σ-image of a metric space.*
(3) *X is a Cauchy symmetric, $\aleph$-space.*
(4) *X is a strongly g-developable space.*

Proof: (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). It follows from Proposition 3.4 [9].

(3) $\Longrightarrow$ (4) Let X be a Cauchy symmetric and $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a σ-locally finite cs-network for X. We can assume that each $\mathcal{U}_n$ is closed under finite intersections and $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ for all $n \in \mathbb{N}$. So, $\mathcal{U}$ is closed under finite intersections. For each $x \in X$, let $\mathcal{V}_x = \{V_n(x) : n \in \mathbb{N}\}$ be a decreasing weak neighborhood base at x in X, and put

$$\mathcal{P}_x = \big\{P \in \mathcal{U} : V_n(x) \subset P \text{ for some } n \in \mathbb{N}\big\}.$$

Claim. For each U open in X and $x \in U$, there exists $P \in \mathcal{P}_x$ such that $P \subset U$.

In fact, conversely assume that there exists U open in X and $x \in U$ such that $P \not\subset U$ for all $P \in \mathcal{P}_x$. Let

$$\{P \in \mathcal{P}_x : x \in P \subset U\} = \{P_m(x) : m \in \mathbb{N}\}.$$

Then $V_n(x) \not\subset P_m(x)$ for all $n, m \in \mathbb{N}$, so choose $x_{n,m} \in V_n(x) - P_m(x)$. For $n \geq m$, we denote $x_{n,m} = y_k$ with $k = m + n(n-1)/2$. Because $\mathcal{V}_x$ is a decreasing weak neighborhood base at x, the sequence $\{y_k : k \in \mathbb{N}\}$ converges to the point x in X. Thus, there exist $m, i \in \mathbb{N}$ such that

$$\{x\} \bigcup \{y_k : k \geq i\} \subset P_m(x) \subset U.$$

Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

Then we have

(1) $\mathcal{P}_x$ is a network at x in X. Let U be an open subset of X and $x \in U$. Then there exists $P \in \mathcal{P}_x$ such that $P \subset U$ by the Claim.

(2) Let $P_1, P_2 \in \mathcal{P}_x$ and $P = P_1 \cap P_2$. Hence, there exist $n, m \in \mathbb{N}$ such that $V_m(x) \subset P_1$ and $V_n(x) \subset P_2$. If we put $k = \max\{m, n\}$, then $V_k(x) \in \mathcal{V}_x$ and $V_k(x) \subset P \in \mathcal{U}$. Thus, $P \in \mathcal{P}_x$ and $P \subset P_1 \cap P_2$.

(3) Let U be an open subset of X. By the Claim, there exists $P \in \mathcal{P}_x$ such that $P \subset U$. Conversely, if $U \subset X$ satisfies that for each $x \in U$ there exists $P \in \mathcal{P}_x$ with $P \subset U$, then for each $x \in U$, there exists $n \in \mathbb{N}$ such that $V_n(x) \subset U$. Because $\mathcal{V}_x$ is a weak neighborhood at x for all $x \in X$, U is open in X.

Therefore, $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a weak base for X and $\mathcal{P} \subset \mathcal{U}$.

Now, for each $n \in \mathbb{N}$, put $\mathcal{P}_n = \mathcal{U}_n \cap \mathcal{P}$. Then, $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Since $\mathcal{U}$ is a σ-locally finite cs-network, $\mathcal{P}$ is a σ-locally finite weak base.

Next, for each $m, n \in \mathbb{N}$, put

$$\begin{aligned}\mathcal{Q}_{m,n}(x) &= \Big\{P \in \mathcal{P}_m \cap \mathcal{P}_x : S_m(x) \subset P \text{ and } d(P) < \frac{1}{n}\Big\};\\ A_{m,n} &= \{x \in X : \mathcal{Q}_{m,n}(x) = \emptyset\};\\ B_{m,n} &= X - A_{m,n};\\ \mathcal{Q}_{m,n} &= \bigcup\{\mathcal{Q}_{m,n}(x) : x \in B_{m,n}\};\\ \mathcal{F}_{m,n} &= \mathcal{Q}_{m,n} \bigcup\{A_{m,n}\}.\end{aligned}$$

Then, each $\mathcal{F}_{m,n}$ is locally finite. Furthermore, we have

(i) *Each $\mathcal{F}_{m,n}$ is a cs-cover for X.*

Let $x \in X$ and $S = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to x in X, then

Case 1. If $x \in B_{m,n}$, then there is $P \in \mathcal{Q}_{m,n}(x)$ such that $S_m(x) \subset P$. Hence, S is eventually in $P \in \mathcal{F}_{m,n}$.

Case 2. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is finite, then S is eventually in $A_{m,n} \in \mathcal{F}_{m,n}$.

Case 3. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is infinite, then we can assume that

$$S \cap B_{m,n} = \{x_{i_k} : k \in \mathbb{N}\}.$$

Since X is Cauchy symmetric and S converges to x, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_i, x_j) < \frac{1}{m} \text{ and } d(x, x_i) < \frac{1}{m} \text{ for every } i, j \geq n_0.$$

Now, we pick $k_0 \in \mathbb{N}$ such that $i_{k_0} \geq n_0$. Because

$$d(x_{i_{k_0}}, x) < \frac{1}{m} \text{ and } d(x_{i_{k_0}}, x_i) < \frac{1}{m} \text{ for every } i \geq n_0,$$

it implies that S is eventually in $S_m(x_{i_{k_0}})$. Furthermore, since $x_{i_{k_0}} \in B_{m,n}$, $S_m(x_{i_{k_0}}) \subset P$ for some $P \in Q_{m,n}(x_{i_{k_0}})$. Hence, $P \in \mathcal{F}_{m,n}$ and S is eventually in P.

Therefore, each $\mathcal{F}_{m,n}$ is a *cs*-cover for X.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ *is a network at x.*

Let $x \in U$ with U open in X. Then, $S_n(x) \subset U$ for some $n \in \mathbb{N}$. Since X is Cauchy symmetric, there exists $j \in \mathbb{N}$ such that $d\big(S_j(x)\big) < 1/n$. Furthermore, we have $P \subset S_j(x)$ for some $P \in \mathcal{P}_x$. Indeed, since $\mathcal{P}$ is point-countable, we can put

$$\mathcal{P}_x = \{P_n(x) : n \in \mathbb{N}\}.$$

On the other hand, because $\mathcal{P}$ is a weak base, we can choose sequence $\{n_i : i \in \mathbb{N}\}$ such that $\{P_{n_i}(x) : i \in \mathbb{N}\}$ is a decreasing network at x. Then, there exists $i \in \mathbb{N}$ such that $P_{n_i}(x) \subset S_n(x)$. Thus, $P \in \mathcal{P}_k$ for some $k \in \mathbb{N}$.

Because P is a sequential neighborhood at x, there exists $i \in \mathbb{N}$ such that $S_i(x) \subset P$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - P$. Hence, $\{x_n\}$

converges to x. Then, there exists $m \in \mathbb{N}$ such that $x_n \in P$ for every $n \geq m$. This is a contradiction.

Denote $m = \max\{i, k\}$, then

$$S_m(x) \subset S_i(x) \subset P \in \mathcal{P}_k \subset \mathcal{P}_m.$$

Since $d(P) < 1/n$, it implies that $P \in \mathcal{F}_{m,n}$. Then, we have $\mathtt{St}(x, \mathcal{F}_{m,n}) \subset S_n(x)$. It follows that $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ is a network at x.

Finally, we write

$$\{\mathcal{F}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{H}_n : n \in \mathbb{N}\},$$

and for each $n \in \mathbb{N}$, put

$$\mathcal{G}_n = \bigwedge\{\mathcal{H}_i : i \leq n\}.$$

Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-locally finite strong network consisting of cs-covers for X. $\square$

Remark 2.2. By Theorem 2.1, we get an affirmative answer to Question 1.1.

References

[1] An T.V., Tuyen L.Q., *On an affirmative answer to S. Lin's problem*, Topology Appl., **158** (2011), 1567–1570.

[2] An T.V., Tuyen L.Q., *On π-images of separable metric spaces and a problem of Shou Lin*, Mat. Vesnik **64** (2012), no. 4, 297–302.

[3] Arhangel'skii A.V., *Mappings and spaces*, Russian Math. Surveys **21** (1966), no. 4, 115–162.

[4] Engelking R., *General Topology*, PWN-Polish Scientific Publishers, Warsaw, 1977.

[5] Ikeda Y., Liu C., Tanaka Y., *Quotient compact images of metric spaces, and related matters*, Topology Appl. **122** (2002), no. 1-2, 237–252.

[6] Lin S., Tanaka Y., *Point-countable k-networks, closed maps, and related results*, Topology Appl. **59** (1994), 79–86.

[7] Tanaka Y., *Symmetric spaces, g-developable spaces and g-metrizable spaces*, Math. Japon. **36** (1991), 71–84.

[8] Tanaka Y., *Theory of k-networks II*, Questions Answers Gen. Topology **19** (2001), 27–46.

[9] Tanaka Y., Ge Y., *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math. **32** (2006) no. 1, 99–117.

[10] Tanaka Y., Li Z., *Certain covering-maps and k-networks, and related matters*, Topology Proc. **27** (2003), no. 1, 317–334.

[11] Tuyen L.Q., *A new characterization of spaces with locally countable sn-networks*, Mat. Vesnik **65** (2013), no. 1, 8–13.

[12] Yan P., *On strong sequence-covering compact mappings*, Northeast. Math. J. **14** (1998), 341–344.

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