On Hattori spaces

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Abstract. For a subset A of the real line $\mathbb{R}$, Hattori space $H(A)$ is a topological space whose underlying point set is the reals $\mathbb{R}$ and whose topology is defined as follows: points from A are given the usual Euclidean neighborhoods while remaining points are given the neighborhoods of the Sorgenfrey line. In this paper, among other things, we give conditions on A which are sufficient and necessary for $H(A)$ to be respectively almost Čech-complete, Čech-complete, quasicomplete, Čech-analytic and weakly separated (in Tkacenko sense). Some of these results solve questions raised by V.A. Chatyrko and Y. Hattori.

Keywords: Hattori space; Čech-complete space; Čech-analytic space; neighborhood assignment; Sorgenfrey line; scattered set; weakly separated space

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1. Introduction

Following a well-known process on linearly ordered sets that leads to generalized ordered spaces (see, for instance, Faber [8]), Hattori [11] has recently considered a poset of topologies having the reals $\mathbb{R}$ as the underlying point set by matching each $A \subset \mathbb{R}$ to a topological space, denoted here by $H(A)$. The space $H(A)$ is defined as follows: a basis of neighborhoods for $x \in A$ is given by the usual Euclidean neighborhoods of x and a basis for $x \notin A$ is given by the right open intervals $[x, y[$, $x < y$, $y \in \mathbb{R}$. Beside [11], the spaces $H(A)$ were studied by Chatyrko and Hatori in [3] where several properties of $H(A)$ (shared by all A or not) are established. In particular, for any $A \subset \mathbb{R}$, $H(A)$ is regular, hereditarily Lindelöf, hereditarily separable and Baire space. Moreover, if A is closed in $\mathbb{R}$, then $H(A)$ is homeomorphic to the Sorgenfrey line $\mathbb{S}$ if and only if A is countable. Recall that a basis of the space $\mathbb{S}$ is given by right open intervals $[x, y[$, $x, y \in \mathbb{R}$. It is also proved in [3, Proposition 2.3] that $H(A)$ is 0-dimensional and nowhere locally compact for every A such that $\mathbb{R} \setminus A$ is countable and dense in $\mathbb{R}$.

In this paper, we continue the work of Chatyrko and Hattori by proving additional information about the spaces $H(A)$, $A \subset \mathbb{R}$. We show that $H(A)$ is weakly separated in the sense of Tkacenko (the definition is given below) if and only if A is left scattered. (It is well-known and easy to see that the Sorgenfrey line is weakly separated.) We also have that $H(A)$ is almost Čech-complete iff A is a residual subset of $\mathbb{R}$. Several completeness type properties of $H(A)$ among them

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quasi-completeness (in Creede sense), Čech-analycity, Čech-completeness and being p-space (in Arhangel'skiĭ sense) are shown to be equivalent to the countability of $\mathbb{R} \setminus A$. In particular, $H(A)$ is homeomorphic to the usual space of the irrationals if and only if $\mathbb{R} \setminus A$ is countable and dense in $\mathbb{R}$, answering another question in [3].

We previously thought that $H(A)$ is homeomorphic to $\mathbb{S}$ if and only if the closure of A in $\mathbb{S}$ is countable. However there was a gap in our proof pointed out by the referee. We learned from the referee that J. Kulesza recently proved in an independent work [12] that $H(A)$ is homeomorphic to $\mathbb{S}$ if and only if A is scattered. We include here a proof (extracted from the submitted version of our paper) of the "only if" part of Kulesza's theorem. Some results of Kulesza are quiet similar to the results in the present paper.

The remaining of this paper is organized as follows. Section 2 concerns spaces $H(A)$ that are topologically near to the Sorgenfrey line, roughly speaking. In Section 3, we study the interplay between A, as a subspace of $\mathbb{R}$, and various completeness type properties of $H(A)$. Section 2 includes also some properties of $H(A)$ that are crucial in this paper, as for example the fact that $H(A)$ is always hereditarily Baire. We refer to [7] for undefined terms.

2. When $H(A)$ is close to $\mathbb{S}$

It is well-known that the Sorgenfrey line is hereditarily Baire. In what follows we shall use the fact that $H(A)$ is hereditarily Baire for each $A \subset \mathbb{R}$. Baireness of the space $H(A)$ has been proved in [3].

Proposition 2.1. *For any $A \subset \mathbb{R}$, the space $H(A)$ is hereditarily Baire.*

PROOF: Since $H(A)$ is regular and first countable, it suffices by [5] to show that each closed countable subspace of $H(A)$ is scattered. Let F be a closed countable subspace of $H(A)$. Since F is closed in $\mathbb{S}$, it is a G_δ-set in $\mathbb{R}$, so it is scattered in $\mathbb{R}$ hence scattered in $H(A)$. □

Proposition 2.2. *Let X be a subspace of $H(A)$ and $f : X \to H(B)$ be a continuous function. Then $f(X \cap A) \setminus B$ is countable.*

PROOF: Since the subspace $X \cap A$ of $H(A)$ is second countable, $f(X \cap A)$ has a countable network, and so is $f(X \cap A) \setminus B$. Note that $H(B)$ and $\mathbb{S}$ induce the same topology on $\mathbb{R} \setminus B$, in particular on $f(X \cap A) \setminus B$. To conclude, recall the well-known fact that every subspace of the Sorgenfrey line having a countable network is countable (see e.g. [13, Lemma 2.10]). □

It follows from Proposition 2.2 that if $H(A)$ is homeomorphic to $\mathbb{S}$, then A is countable, a result observed in [3] under a descriptive condition on A. Indeed, since $\mathbb{S} = H(\emptyset)$, if $f : H(A) \to \mathbb{S}$ is an homeomorphism then $f(A)$ is countable and so is A.

A space X is said to be *totally imperfect* if each compact subspace of X is countable. Recall that every Polish totally imperfect space is countable. It is also well-known that $\mathbb{S}$ is totally imperfect.

Proposition 2.3. *Let $A \subset \mathbb{R}$. Then $H(A)$ is totally imperfect if and only if A is totally imperfect.*

Proof: Suppose that A is totally imperfect. Let K be a compact subspace of $H(A)$. Then the topology of K in $H(A)$ is the Euclidean topology, so the function $x \in K \subset \mathbb{R} \to x \in H(A)$ is continuous. It follows from Proposition 2.2 that $K \setminus A$ is countable, thus $K \cap A$ is a G_δ-set of K hence it is a Polish space. Since A is totally imperfect, $K \cap A$ is totally imperfect hence countable.

The converse is obvious. □

A *neighborhood assignment* of a space X is a collection $(V_x)_{x\in X}$ of sets such that each V_x is a neighborhood (not necessarily open) of x in X. Following [15], a Hausdorff space X is said to be *weakly separated* if there is a neighborhood assignment $(V_x)_{x\in X}$ of X such that for every $x, y \in X$, if $(x, y) \in V_y \times V_x$ then $x = y$. A typical example of weakly separated space is the Sorgenfrey line. As we shall show in Theorem 2.7, weakly separatedness of Hattori spaces is more subtle. In particular, it follows from Theorem 2.7 that for each somewhere dense subset A of $\mathbb{R}$, the corresponding Hattori space $H(A)$ is not weakly separated, hence cannot be homeomorphic to $\mathbb{S}$, which answers a question in [3].

Proposition 2.4. *Let X be a weakly separated space and B a second category subspace of $H(A)$. Then for every continuous function $f : B \to X$, there is a nonempty open subset W of B such that for each $x \in W \cap A$, there is $\varepsilon > 0$ such that $f(]x-\varepsilon, x] \cap B) \subset \{f(x)\}$.*

Proof: Let $(V_x)_{x\in X}$ be any neighborhood assignment of X. Following an idea from [6], for $n \geq 1$, let

$$F_n = \{x \in B : (y \in B \,\text{and}\, x \leq y < x + \frac{1}{n}) \Rightarrow f(y) \in V_{f(x)}\}.$$

Then $B = \bigcup_{n\geq 1} F_n$ (by continuity of f). Since B is a second category space, there are $n \in \mathbb{N}$ and a nonempty open subset W of B such that $W \subset \overline{F_n}$ (in this proof, all closures are in B). Let $x \in W \cap A$. There is $0 < \varepsilon < \frac{1}{n}$ such that $]x-\varepsilon, x+\varepsilon[\cap B \subset W$ and $f(]x-\varepsilon, x+\varepsilon[\cap B) \subset V_{f(x)}$. Put $V =]x-\varepsilon, x[$ and let $y \in V \cap F_n$. Then $y < x < y+\varepsilon \leq y+\frac{1}{n}$, hence $f(x) \in V_{f(y)}$. Assume now that X is weakly separated by $(V_x)_{x\in X}$. Since $f(y) \in V_{f(x)}$, it follows that $f(x) = f(y)$, therefore $f(V \cap F_n) \subset \{f(x)\}$. Since f is continuous and

$$V \cap B \subset V \cap \overline{F_n} \subset \overline{V \cap F_n},$$

it follows that $f(]x-\varepsilon, x] \cap B) \subset \{f(x)\}$. □

Let (X, τ) be a topological space and let $\mathcal{V} = (V_x)_{x\in X}$ be an *assignment* of X, that is, a collection of subsets of X (not necessarily related to τ) such that for each $x \in X$, $x \in V_x$. We shall say that a subspace A of X is $\mathcal{V}$*-scattered* if for each nonempty subset F of A, there are $x \in F$ and an open set $U \subset X$ such that $x \in U \cap F \subset V_x$. The $\mathcal{V}$*-derivative* of A is defined by transfinite induction as follows: $A^0 = A$, $A^{\beta+1}$, where β is an ordinal, is the set of $x \in A^\beta$ such that for each neighborhood U of x in (X,τ), $U \cap A^\beta \not\subset V_x$. If α is a limit ordinal, then $A^\alpha = \bigcap_{\beta<\alpha} A^\beta$. As for the classical Cantor-Bendixson derivative, A is $\mathcal{V}$-scattered if and only if there is an ordinal, denoted $rk_{\mathcal{V}}(A)$ (or simply $rk(A)$), such that $A^{rk(A)} = \emptyset$, and for each $\beta < \gamma < rk(A)$, $A^\beta \setminus A^\gamma \neq \emptyset$. Moreover, for each $x \in A$, there exist a unique $\alpha_x < rk(A)$ (the rank of x) and a neighborhood U_x of x in (X,τ) such that $x \in U_x \cap A^{\alpha_x} \subset V_x$.

Lemma 2.5. *Let X be a space and $\mathcal{V} = (V_x)_{x\in X}$ an assignment of X. Let $A \subset X$ be such that the subspaces A and $X \setminus A$ of X are $\mathcal{V}$-scattered. Then, there is a neighborhood assignment $(W_x)_{x\in X}$ of X such that if $(x,y) \in W_y \times W_x$ and $\chi_A(x) \neq \chi_A(y)$ then $(x,y) \in \overline{V_y} \times \overline{V_x}$.*

Proof: Put $B = X \setminus A$, and consider the $\mathcal{V}$-derivative sequences $(A^\alpha)_{\alpha<rk(A)}$ and $(B^\alpha)_{\alpha<rk(B)}$ of A and B respectively, as described above. We shall define W_x for $x \in A$, W_x is defined analogously when $x \in B$. Let α_x be the rank of x and choose an open neighborhood U_x of x in X such that $x \in U_x \cap A^{\alpha_x} \subset V_x$. If $x \in \bigcap_{\alpha<rk(B)} \overline{B^\alpha}$, let $W_x = U_x$. If $x \notin \bigcap_{\alpha<rk(B)} \overline{B^\alpha}$, let $\beta_x = \min\{\beta < rk(B) : x \notin \overline{B^\beta}\}$ and define $W_x = U_x \setminus \overline{B^{\beta_x}}$.

Let $(x,y) \in W_y \times W_x$ be such that $\chi_A(x) \neq \chi_A(y)$, say $x \in A$ and $y \notin A$. Then $x \in W_y \cap A^{\alpha_x}$, hence $y \in \overline{A^{\alpha_x}}$, because otherwise β_y would be defined and satisfies $\beta_y \leq \alpha_x$, thus $W_y \cap A^{\alpha_x} \subset W_y \cap A^{\beta_y} = \emptyset$, which is impossible. Since $U_x \cap A^{\alpha_x} \subset V_x$ and $y \in U_x$, it follows that

$$y \in U_x \cap \overline{A^{\alpha_x}} \subset \overline{U_x \cap A^{\alpha_x}} \subset \overline{V_x}.$$

Similarly, $x \in \overline{V_y}$.

□

Proposition 2.6. *Let (X,τ) be a space which is weakly separated by a neighborhood assignment $\mathcal{V} = (V_x)_{x\in X}$ of (X,τ). Let τ_1 be a topology on X such that each V_x is τ_1-closed and suppose that there is $A \subset X$ such that the subspaces A and $X \setminus A$ of (X,τ_1) are $\mathcal{V}$-scattered and weakly separated. Then (X,τ_1) is weakly separated.*

Proof: Since the subspaces A and $X \setminus A$ of (X,τ_1) are weakly separated, there exists a neighborhood assignment $(U_x)_{x\in X}$ of (X,τ_1) such that $x = y$ whenever $(x,y) \in U_y \times U_x$ and $\chi_A(x) = \chi_A(y)$. By Lemma 2.5, there is also a neighborhood assignment $(W_x)_{x\in X}$ of (X,τ_1) such that $(x,y) \in V_y \times V_x$ whenever $(x,y) \in W_y \times W_x$ and $\chi_A(x) \neq \chi_A(y)$. Then $(U_x \cap W_x)_{x\in X}$ is a neighborhood assignment of (X,τ_1) satisfying $x = y$ whenever $(x,y) \in (U_y \cap W_y) \times (U_x \cap W_x)$. □

Now we are ready to determinate for what $A \subset \mathbb{R}$, Hattori space $H(A)$ is weakly separated. Recall that a subset $A \subset \mathbb{R}$ is said to be *left scattered* if A is scattered as a subspace of the left Sorgenfrey line (the reals with the topology generated by the intervals $]x,y]$, $x,y \in \mathbb{R}$). Right scattered sets are defined analogously. Every left scattered set is countable and every left and right scattered set is scattered (as a subspace of $\mathbb{R}$).

Theorem 2.7. *The space $H(A)$ is weakly separated if and only if A is left scattered.*

Proof: Suppose that $H(A)$ is weakly separated. Let $E \subset A$ be a nonempty set. Since $H(A)$ is hereditarily Baire (Proposition 2.1), Proposition 2.4 applied to $B = \overline{E}$ (closure in $H(A)$) and the identity mapping $x \in B \to x \in H(A)$ gives us an open set $W \subset B$ such that $W \cap E \neq \emptyset$ and for each $x \in W \cap E$, there is $\varepsilon > 0$ such that $]x - \varepsilon, x] \cap E = \{x\}$. Clearly, every $x \in W \cap E$ is left isolated in E.

Conversely, suppose that A is left scattered and let us show that $H(A)$ is weakly separated by applying Proposition 2.6. Let $X = \mathbb{R}$, τ be the Sorgenfrey topology and τ_1 the topology of $H(A)$. Put $V_x = [x, +\infty[$, $x \in \mathbb{R}$. The space (X, τ) is weakly separated by the neighborhood assignment $\mathcal{V} = (V_x)_{x\in\mathbb{R}}$ and each V_x is closed in (X, τ_1). Clearly, the subspace $\mathbb{R} \setminus A$ of (X, τ_1) is $\mathcal{V}$-scattered and weakly separated. Moreover, the subspace A of (X, τ_1) is $\mathcal{V}$-scattered if (and only if) for each nonempty subspace B of A, there are $x \in B$ and $\varepsilon > 0$, such that $]x-\varepsilon, x+\varepsilon[\cap B \subset [x, +\infty[$. This means exactly that A is left scattered as assumed. Finally, since $H(A)$ is a T_1 space and A is countable (because left scattered), the subspace A of $H(A)$ is weakly separated. It follows from Proposition 2.6 that $H(A)$ is weakly separated. □

It is proved in [3] that for every closed countable subspace A of $\mathbb{R}$, $H(A)$ is homeomorphic to $\mathbb{S}$. The authors ask if their result remains true if A has countable closure in $\mathbb{R}$. The following gives a complete answer and is due to Kulesza [12, Theorem 6]:

Theorem 2.8. *H(A) is homeomorphic to $\mathbb{S}$ if and only if A is scattered.*

We shall give an alternative proof of the necessity condition (Proposition 2.11 below) of Kulesza's theorem based on Ščepin's concept of capacity space [14].

Let (X, τ) be topological space. A function $\phi : X \times \tau \to [0, +\infty[$ is said to be a *precapacity* if the following conditions hold:

(i) for each $x \in X$ and $U, V \in \tau$ such that $V \subset U$, $\phi(x, U) \leq \phi(x, V)$,
(ii) for each $x \in X$ and each totally ordered collection $(U_i)_{i\in I} \subset \tau$, we have $\phi(x, \bigcup_{i\in\mathbb{N}} U_i) = \inf_{i\in I} \phi(x, U_i)$.

A *capacity* (in Ščepins's sense) on X is a precapacity ϕ satisfying the additional conditions:

(iii) for each $x \in X$ and $U \in \tau$, $x \in \overline{U}$ iff $\phi(x, U) = 0$,
(iv) for each $U \in \tau$, the function $\phi(\cdot, U) : X \to \mathbb{R}$ is continuous.

It is well-known and easy to check that the function $\phi : \mathbb{S} \times \tau \to [0, +\infty[$ defined by $\phi(x, U) = \min\{1, d(x, U \cap [x, +\infty[)\}$, where d is the usual metric (with the convention $d(x, \emptyset) = 1$), is a Ščepin capacity on the Sorgenfrey line.

The following is essentially proved by Bennett and Lutzer in [2, Lemma 2.6]. As usual, if $(X,<)$ is an ordered set and $x \in X$, then $]x,\rightarrow[$ stands for $\{y \in X : x < y\}$ ($[x,\rightarrow[$, $]\leftarrow,x[$ are defined analogously).

Proposition 2.9. *Let $(X,\tau,<)$ be a generalized ordered space having a precapacity $\phi : X \times \tau \to [0,+\infty[$ such that for each $y \in X$, $\phi(\cdot,]\leftarrow,y[)$ is continuous at each $x \in [y,\rightarrow[$. Then the function $f_\phi : X \to \mathbb{R}$, defined by $f_\phi(x) = \phi(x,]\leftarrow,x[)$, is upper semicontinuous.*

Proof: Let $\delta > 0$ and let $x \in X$ be such that $\phi(x,]\leftarrow,x[) < \delta$. If

$$]\leftarrow,x[= \bigcup_{y<x}]\leftarrow,y[,$$

then by (ii) there is $y < x$ such that $\phi(x,]\leftarrow,y[) < \delta$. Since $\phi(\cdot,]\leftarrow,y[)$ is continuous at x, there is a neighborhood V of x in (X,τ) such that $\phi(t,]\leftarrow,y[) < \delta$ for every $t \in V$. Let $W =]y,\rightarrow[\cap V$. Then W is a neighborhood of x in (X,τ) and for each $t \in W$, by (i), $\phi(t,]\leftarrow,t[) \leq \phi(t,]\leftarrow,y[) < \delta$.

If $]\leftarrow,x[\neq \bigcup_{y<x}]\leftarrow,y[$, then $[x,\rightarrow[$ is a neighborhood of x in (X,τ). In this case, we proceed as above taking $y = x$ and $W = [x,\rightarrow[\cap V$. $\square$

The function f_ϕ from 2.9 is also used in the next lemma.

Lemma 2.10. *If $H(A)$ has a capacity ϕ, then $A = f_\phi^{-1}(\{0\})$. In particular, A is G_δ set in $H(A)$ (equivalently, in $\mathbb{R}$).*

Proof: From the definition of the topology of $H(A)$, we have for each $x \in H(A)$, $x \in \overline{]\leftarrow,x[}$ iff $x \in A$. In other words, by (iii), $f_\phi^{-1}(\{0\}) = A$. Since f_ϕ is upper semicontinuous (Proposition 2.9), $f_\phi^{-1}(\{0\})$ is a G_δ set in $H(A)$. Since points of A in $H(A)$ are given the Euclidean neighborhoods, A is a G_δ set in $\mathbb{R}$. $\square$

Proposition 2.11. *Let $A \subset \mathbb{R}$ be such that $H(A)$ is homeomorphic to $\mathbb{S}$. Then A is scattered.*

Proof: Since $\mathbb{S}$ is homeomorphic to $H(A)$ and has a capacity, $H(A)$ has a capacity too. It follows from Lemma 2.10 that A is G_δ set in $\mathbb{R}$ and we know from Proposition 2.2 that A is countable. Consequently, A is scattered. $\square$

To conclude this section, let us mention that when A is left scattered, the identity function $x \in H(A) \to x \in \mathbb{S}$ is G_δ-measurable. This is a consequence of the following by taking $B = \emptyset$.

Proposition 2.12. *Let $A, B \subset \mathbb{R}$ be such that every right discrete subset of $A \setminus B$ is left scattered (equivalently, scattered). Then every open set $O \subset H(B)$ is G_δ and F_σ in $H(A)$.*

Proof: Let $O \subset H(B)$ be an open set. Since O is open in $\mathbb{S}$, it is an F_σ set in $\mathbb{R}$ hence F_σ in $H(A)$. To show that O is a G_δ set of $H(A)$, for each $x \in (O \cap A) \setminus B$, let $\varepsilon_x > 0$ be such that $[x, x+\varepsilon_x[\subset O$. The set

$$F = \Big(\bigcup_{x \in O\cap A\setminus B} [x, x+\varepsilon_x[\Big) \setminus \bigcup_{x \in O\cap A\setminus B}]x, x+\varepsilon_x[$$

is a right discrete subset of $A \setminus B$, hence scattered. The set $O \setminus F$ is a subset of the interior W of O in $H(A)$. Indeed, let $x \in O \setminus F$. If $x \in B$ then O is a neighborhood of x in $\mathbb{R}$, hence it is a neighborhood of x in $H(A)$. If $x \notin A$, then O is a neighborhood of x in $\mathbb{S}$ hence in $H(A)$ too. If $x \in A \setminus B$ then since $x \notin F$, there is $y \in A \setminus B$ such that $x \in]y, y+\varepsilon_y[\subset O$.

We have $O = W \cup (O \cap F)$ and since $O \cap F$ is G_δ in $H(A)$ (in fact in $\mathbb{R}$), O is the union of an open set and G_δ set in $H(A)$. □

3. Completeness type properties of $H(A)$

There are various completeness or generalized metrizability properties which can be defined in terms of the existence of certain kinds of sequences of covers. One of the lowest is due to Creede [4]: a space X is said to be *quasicomplete* if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X such that the following holds: if $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, are such that $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$, then each sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n \in \bigcap_{i\le n} U_i$ for each $n \in \mathbb{N}$, has a cluster point in X. Every p-space (in Arhangelskiĭ sense) is quasicomplete. Recall that a Tychonoff space X is said to be *Čech-complete* if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X which is complete, that is, every filter $\mathcal{F}$ on X such that $\mathcal{F} \cap \mathcal{U}_n \neq \emptyset$ for each $n \in \mathbb{N}$, has a cluster point in X. The space X is *almost Čech-complete* if X has a dense Čech-complete subspace.

Proposition 3.1. *Let X be a dense subset of $\mathbb{R}$ and let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of collections of open subsets of $H(A)$ with the property that if $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, are such that $(\bigcap_{n\in\mathbb{N}} U_n) \setminus A \neq \emptyset$, then each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ satisfying $x_n \in \bigcap_{i\le n} U_i$ for each $n \in \mathbb{N}$, has a cluster point in $H(A)$. Let O_n be the union of the interiors in $\mathbb{R}$ of all members of $\mathcal{U}_n$. Then*

1) *$\bigcap_{n\in\mathbb{N}} O_n \subset A$, and if $A \subset \bigcup \mathcal{U}_n$ for each $n \in \mathbb{N}$, then $A = \bigcap_{n\in\mathbb{N}} O_n$ (in particular, A is a G_δ set in $\mathbb{R}$),*
2) *if $X = \mathbb{R}$ and each $\mathcal{U}_n$ is a cover of $H(A)$, then $\mathbb{R} \setminus A$ is countable.*

In particular, if $H(A)$ is quasicomplete (e.g., p-space), then $\mathbb{R} \setminus A$ is countable.

Proof: 1) For each $U \in \mathcal{U}_n$, let $\mathring{U}$ be the interior of U in $\mathbb{R}$ for the Euclidean topology, so that $O_n = \bigcup\{\mathring{U} : U \in \mathcal{U}_n\}$. Clearly, if $A \subset \bigcup \mathcal{U}_n$ for each $n \in \mathbb{N}$, then $A \subset \bigcap_{n\in\mathbb{N}} O_n$. Suppose by contradiction that there exists $x \in (\bigcap_{n\in\mathbb{N}} O_n) \setminus A$. Choose a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ without cluster point in $H(A)$ such that $\lim x_n = x$ in $\mathbb{R}$. For each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{U}_n$ such that x belongs to $\mathring{U}_n$. Let $k_n \in \mathbb{N}$ be such that $\{x_m : m \ge k_n\} \subset \bigcap_{i\le n} U_i$. Since $x \in (\bigcap_{n\in\mathbb{N}} U_n) \setminus A$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point in $H(A)$, a contradiction.

2) By 1) it is possible to write $\mathbb{R} \setminus A = \bigcup_{n\in\mathbb{N}} F_n$, where each F_n is closed in $H(A)$. Let $n \in \mathbb{N}$ and let us show that F_n is countable. Observe that the

subspace F_n of $H(A)$ is quasicomplete, hence, being Lindelöf and submetrizable, it is metrizable by a result of Gittings [10, Corollary 4.2]. But the topology of F_n is the topology inherited from the Sorgenfrey line, hence F_n is countable. □

Clearly, if the subspace A of $\mathbb{R}$ is residual in $\mathbb{R}$, then the space $H(A)$ is almost Čech-complete. The converse is also true, which answers the question posed at the end of Chatyrko and Hattori paper [3] . Indeed, suppose $H(A)$ has a dense Čech-compete subspace X and let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open collections of $H(A)$ such that the sequence $\{U \cap X : U \in \mathcal{U}_n\}$, $n \in \mathbb{N}$, is a complete sequence of covers of X. For each $n \in \mathbb{N}$, the open subset O_n (notation of Proposition 3.1) of $\mathbb{R}$ is dense in $\mathbb{R}$, and we have $\bigcap_{n\in\mathbb{N}} O_n \subset A$ by Proposition 3.1.

We refer the reader to [9] for the definition and basic properties of analytic spaces and Čech-analytic spaces.

Proposition 3.2. *If $H(A)$ is Čech-analytic, then $\mathbb{R} \setminus A$ is countable.*

Proof: Suppose that $H(A)$ is Čech-analytic. Since $H(A)$ is hereditarily Lindelöf, it is K-analytic (see [9]). Furthermore, being submetrizable, $H(A)$ is analytic which implies that its subspace $\mathbb{R} \setminus A$ has a countable network. Consequently, as in the proof of Proposition 2.2, $\mathbb{R} \setminus A$ is countable. □

Proposition 3.3. *If $\mathbb{R} \setminus A$ is countable, then $H(A)$ is a Polish space.*

Proof: For each $a \in \mathbb{R} \setminus A$, let $(\mathbb{R}, \tau_a)$ be the space given by the topological sum $]\leftarrow, a[\oplus[a, \rightarrow[$, where each factor is endowed with the Euclidean topology. Then $(\mathbb{R}, \tau_a)$ is a Polish space (as a topological sum of two Polish spaces). Moreover, the topology τ_a is finer than the Euclidean topology. Suppose now that $\mathbb{R} \setminus A$ is countable. Then the topology τ generated by $\bigcup_{a\in\mathbb{R}\setminus A} \tau_a$ is a Polish topology. Clearly, τ is the topology of $H(A)$ hence $H(A)$ is Polish. □

An alternative way to show that $H(A)$ is Polish if $\mathbb{R} \setminus A$ is countable is to use the following fact: For each $A \subset \mathbb{R}$, there exists a compactification $K(A)$ of the space $H(A)$ such that $K(A) \setminus H(A)$ is homeomorphic to $\mathbb{R} \setminus A$ endowed with the topology induced by the left Sorgenfrey topology. The space $K(H)$ is a variant of the Alexandrov double arrow space and is defined as follows[1]. To simplify we replace $\mathbb{R}$ by the interval $]0,1[$ and consider that $A \subset]0,1[$. Put $B =]0,1[\setminus A$ and let $K(A) = ([0,1] \times \{0\}) \cup (B \times \{1\})$. Consider the topology on $K(A)$ defined as follows: Let $x \in B$. A basis of neighborhoods for $(x,1)$ is given by sets of the form $((]y,x[) \times \{0\}) \cup (]y,x] \cap B) \times \{1\})$, where $y < x$. A basis of neighborhoods for $(x,0)$ are sets of the form $(([x,y[) \times \{0\}) \cup (]x,y[\cap B) \times \{1\})$, where $x < y$. If $x \in A \cup \{0,1\}$, then a basis of neighborhoods for $(x,0)$ is given by sets of the form $(V \times \{0\}) \cup ((V \cap B) \times \{1\})$, where V is an Euclidean neighborhood of x in $[0,1]$. Note that $H(A)$ is homeomorphic to the subspace $]0,1[\times\{0\}$ of $K(A)$ and the subspace $B \times \{1\}$ of $K(A)$ is homeomorphic to the subspace B of the left Sorgenfrey line.

[1] We realized that this compactification has already been considered by van Mill in [13].

It is easy to check that the subspace $]0,1[\times\{0\}$ of $K(A)$ is open in $K(A)$ if and only if the following conditions hold.

(i) For each $x \in A$, there exists $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\cap B = \emptyset$, that is, B is closed in $\mathbb{R}$.

(ii) For each $x \in B$, there exists $y > x$ such that $]x, y[\cap B = \emptyset$, that is B is a discrete subspace of $\mathbb{S}$.

It is also easy to check that for each $x \in H(A)$, the space $H(A)$ is locally connected at x iff $H(A)$ is locally compact at x iff there exists a neighborhood V of x in $H(A)$ such that $V \cap (\mathbb{R} \setminus A) \subset \{x\}$.

Lemma 3.4. *The space $K(A)$ is compact.*

Proof: Clearly $K(A)$ is a Lindelöf space, hence to show that $K(A)$ is compact it suffices to show that $K(A)$ is countably compact. Let $((x_n, \varepsilon_n))_{n\in\mathbb{N}} \subset K(A)$. We may assume that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence for the Euclidean topology and let $x = \lim x_n$. If $x \in A \cup \{0,1\}$, then $((x_n, \varepsilon_n))_{n\in\mathbb{N}}$ converges in $K(A)$ to $(x, 0)$. If $x \in B$ and $((x_n, \varepsilon_n))_{n\in\mathbb{N}}$ does not converge to $(x, 0)$ in $K(A)$, then $x_n < x$ for infinitely many $n \in \mathbb{N}$, hence $((x_n, \varepsilon_n))_{n\in\mathbb{N}}$ converges in $K(A)$ to $(x, 1)$. □

The following answers Question 3.1 in [3]. It is obtained by combination of Propositions 3.1 and 3.2 (or 3.3). We give a proof based on Lemma 3.4.

Proposition 3.5. *The space $H(A)$ is Čech-complete if and only if $\mathbb{R} \setminus A$ is countable.*

Proof: We replace $\mathbb{R}$ by $]0,1[$ and consider the compactification $K(A)$ of $H(A)$ from Lemma 3.4. Suppose that $]0,1[\setminus A$ is countable. Since the set $B = [0,1] \setminus A$ is countable, $H(A)$ is a G_δ subspace of $K(A)$. It follows from Lemma 3.4 that $H(A)$ is Čech-complete (see [7]).

Conversely, since the subspace B of the left Sorgenfrey line is homeomorphic to the subspace $B \times \{1\}$ of $K(A)$, if $H(A)$ is Čech-complete then by Lemma 3.4, B is a K_σ set, hence B and $]0,1[\setminus A$ are countable. □

It is proved in [3] that if $\mathbb{R} \setminus A$ is countable and dense in $\mathbb{R}$, then $H(A)$ is 0-dimensional and nowhere locally compact. It is easy to see that, conversely, if $H(A)$ is nowhere locally compact, then $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$. Combining this result with Proposition 3.5 and the well-know fact that every separable completely metrizable 0-dimensional and nowhere locally compact space is homeomorphic to the irrationals [1], we obtain the following statement answering Question 2.2 in [3]:

Proposition 3.6. *Let $A \subset \mathbb{R}$. Then $\mathbb{R} \setminus A$ is countable and dense in $\mathbb{R}$ if and only if $H(A)$ is homeomorphic to the irrationals.*

Using Lemma 3.4 in the same way as in the proof of Proposition 3.5, we obtain the following:

Proposition 3.7. *$H(A)$ is locally compact if and only if $\mathbb{R} \setminus A$ is closed in $\mathbb{R}$ and discrete in $\mathbb{S}$.*

Proof: We again replace $\mathbb{R}$ by $]0,1[$. Since $H(A)$ is dense in $K(A)$, the space $H(A)$ is locally compact if and only if it is open in $K(A)$, which is equivalent to say that A is open in $\mathbb{R}$ and $\mathbb{R}\setminus A$ is discrete in $\mathbb{S}$ (see the comment after the definition of the space $K(A)$). □

The following question is posed in [3]: what is the space $H(A)$ if $\mathbb{R}\setminus A$ is countable and closed in $\mathbb{R}$? Of course, A is open in $\mathbb{R}$ if and only if $H(A)$ is locally compact at every $x\in A$, but in general there is no topological property of $H(A)$, independent from A, which is equivalent to the fact that $\mathbb{R}\setminus A$ is countable and closed in $\mathbb{R}$. Here are some examples related to this question.

Examples 3.8. 1) Let $S=\{\frac{1}{n}:n>0\}\cup\{0\}$, $T=\{\frac{1}{n}:n>0\}\cup\{-\frac{1}{n}:n>0\}$, $A=\mathbb{R}\setminus S$ and $B=\mathbb{R}\setminus T$. The set $\mathbb{R}\setminus A$ is compact (hence closed) but $\mathbb{R}\setminus B$ is not closed in $\mathbb{R}$, yet $H(A)$ and $H(B)$ are homeomorphic. To show that $H(A)$ and $H(B)$ are homeomorphic, for each integer $n>0$, put $I_n=[\frac{1}{2n},\frac{1}{2n-1}[$, $J_n=[\frac{1}{2n+1},\frac{1}{2n}[$, $L_n=[-\frac{1}{n},-\frac{1}{n+1}[$ and choose homeomorphisms $h_n:[\frac{1}{n+1},\frac{1}{n}[\to I_n$, $g_n:L_n\to J_n$ and

$$h:]-\infty,-1[\cup[1,+\infty[\to]-\infty,0[\cup[1,+\infty[.$$

All these spaces are endowed with the Euclidean topology. Consider the function $g:H(B)\to H(A)$ defined by $g(0)=0$, $g(x)=h(x)$ if $x\in]+\infty,-1[\cup[1,+\infty[$, $g(x)=h_n(x)$ if $x\in[\frac{1}{n+1},\frac{1}{n}[$ and $g(x)=g_n(x)$ if $x\in L_n$. Then g is a homeomorphism.

2) Let $U=\{0\}\cup\{-\frac{1}{n}:n>0\}$ and $C=\mathbb{R}\setminus U$. Then $H(A)$ and $H(C)$ are not homeomorphic, yet U and T are two convergent sequences in $\mathbb{R}$. To show that $H(A)$ and $H(C)$ are not homeomorphic, only observe that $H(C)$ is locally compact (since U is closed in $\mathbb{R}$ and discrete in $\mathbb{S}$, see Proposition 3.7) but $H(A)$ is not locally compact (at the point 0).

3) The spaces $H(\mathbb{R}\setminus\mathbb{Z})$ and $H(\mathbb{R}\setminus\mathbb{N})$ are not homeomorphic. Indeed, if $h:H(\mathbb{R}\setminus\mathbb{N})\to H(\mathbb{R}\setminus\mathbb{Z})$ is a homeomorphism, then there is $n\in\mathbb{Z}$ such that $h(]-\infty,0[)=]n,n+1[$. It follows that $]n,n+1[$ is closed in $H(\mathbb{R}\setminus\mathbb{N})$ which is not true. In fact, it can be shown (see Proposition 3.9 below) that $H(\mathbb{R}\setminus\mathbb{Z})$ is homeomorphic to $H(A)$ if and only if $\mathbb{R}\setminus A$ is closed in $\mathbb{R}$, discrete in $\mathbb{S}$ and unbounded from above.

We end this section by the following, the proof of which is left to the interested reader. It implies that there are exactly $\aleph_0$ non homeomorphic locally compact Hattori spaces.

Proposition 3.9. *Let $\mathcal{F}$ be the set of all open sets $A\subset R$ such that $\mathbb{R}\setminus A$ is discrete in $\mathbb{S}$ and let $\mathcal{F}_0$ be the set of $A\in\mathcal{F}$ such that $\mathbb{R}\setminus A$ is bounded from above. Then, for every $A\in\mathcal{F}$ and $B\subset\mathbb{R}$, the following are equivalent:*

1) *$H(A)$ and $H(B)$ are homeomorphic,*
2) *$B\in\mathcal{F}$, $|\mathbb{R}\setminus A|=|\mathbb{R}\setminus A|$ and $B\in\mathcal{F}_0$ if and only if $A\in\mathcal{F}_0$.*

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