On a bifurcation problem arising in cholesteric liquid crystal theory

Carlo Greco

Abstract. In a cholesteric liquid crystal the director field $n(x, y, z)$ tends to form a right-angle helicoid around a twist axis in order to minimize the internal energy; however, a fixed alignment of the director field at the boundary (strong anchoring) can give rise to distorted configurations of the director field, as oblique helicoid, in order to save energy. The transition to this distorted configurations depend on the boundary conditions and on the geometry of the liquid crystal, and it is known as Freedericksz transition (without external fields).

We consider the classical situation of a thin layer between two glass sheet assuming the Oseen-Frank model for the energy, and that the director field depend only on the direction z orthogonal to the layer; then we focus on two kinds of boundary conditions: the planar case and the orthogonal case.

In the first, we impose that $n(0) = (1, 0, 0)$, $n(d) = (\cos\alpha, \sin\alpha, 0)$ (where $z = 0$ and $z = d > 0$ are, respectively, the bottom and the top of the layer), and search for the couples (d, α) such that oblique helicoid appear. In the case $K_1 > 0$, $K_2 = K_3 = 1$ for the elastic constants of the Oseen-Frank energy, we completely characterize these couples.

In the second case it is a classical result that oblique helicoid bifurcates from the trivial solution $n(z) = (0, 0, 1)$ for suitable values of d; then we study the exact number of these nontrivial solutions and their stability.

Keywords: bifurcation; time map; cholesteric liquid crystals; Freedericksz transition

Classification: 34B15, 82D30

1. Introduction and statement of results

A configuration of a liquid crystal is given by a function $n : \Omega \to S^2$, where $\Omega \subset \mathbb{R}^3$ is the region in which lies the material, and S^2 is the unit sphere of $\mathbb{R}^3$; for $(x, y, z) \in \Omega$, the unit-length vector $n(x, y, z)$ is the optical axis at the point (x, y, z). We assume the Oseen-Frank model, so that the energy density $W(n, \nabla n)$ of the cholesteric liquid crystal is:

$$2\, W(n, \nabla n) = K_1(\operatorname{div} n)^2 + K_2(n \cdot \operatorname{curl} n + \tau)^2 + K_3\|n \times \operatorname{curl} n\|^2,$$

where K_1, K_2, K_3, are the elastic splay, twist, and bend constants; the possible equilibrium configurations, under a strong anchoring condition on the boundary

DOI 10.14712/1213-7243.2015.206

of Ω, are critical points of the energy functional (see [4]):

$$E(n) = \int_\Omega W(n, \nabla n)\, dx.$$

The constant τ is $\neq 0$ in cholesteric (namely chiral nematics) liquid crystals, and it is called free wave number; if the twist axis is, for instance, the z axis, then $(\cos(\tau z), \sin(\tau z), 0)$ is an equilibrium configuration. We assume $\tau > 0$, which gives right-handed helicoid with period $P = 2\pi/\tau$; however, since in a liquid crystal we identify n and $-n$, the actual period is given by the so called half-pitch, namely $P/2$.

The Euler-Lagrange equations for the functional $E(n)$ are:

$$\begin{aligned} K_1\nabla(\operatorname{div} n) &- K_2(A\cdot \operatorname{curl} n + \operatorname{curl}(A\cdot n)) \\ &+ K_3(B\times \operatorname{curl} n - \operatorname{curl}(B\times n)) - 2K_2\tau(\operatorname{curl} n - An) + \lambda n = 0 \end{aligned}$$

where $A = n\cdot \operatorname{curl} n$, $B = n\times \operatorname{curl} n$, and λ is the Lagrange multiplier due to constraint $\|n\| = 1$:

$$\lambda = K_1\left(\|\nabla n\|^2 - \|\operatorname{curl} n\|^2\right) + 2\left(K_2A^2 + K_3\|B\|^2\right) + (K_1 - K_3)\operatorname{div} B.$$

In this paper we consider the particular case in which Ω is a layer $0 < z < d$ between two glass sheet, and suppose that the optical axis depends only on the direction z orthogonal to the layer: $n(x, y, z) = n(z)$. The glass surfaces at $z = 0$ and at $z = d$ can be suitably treated in order to impose various kinds of boundary conditions. If we set:

$$n(z) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta),$$

where $\varphi = \varphi(z)$ and $\theta = \theta(z)$ are the Euler angles of the optical axis $n(z)$, the Euler-Lagrange equations become:

$$(1)\qquad \begin{cases} 2f(\theta)\theta'' + f'(\theta)\theta'^2 - (2K_2\tau + (K_3 - 2g(\theta))\varphi')\,\varphi'\sin(2\theta) = 0 \\ g(\theta)(\cos\theta)\varphi'' + 2\,(K_2\tau + (K_3 - 2g(\theta))\varphi')\,\theta'\sin\theta = 0 \end{cases}$$

where the functions $f(\theta)$ and $g(\theta)$ are the following:

$$f(\theta) = K_1\cos^2\theta + K_3\sin^2\theta, \qquad g(\theta) = K_2\cos^2\theta + K_3\sin^2\theta.$$

With these notations, the energy functional is:

$$(2)\qquad E(n) = E(\varphi, \theta) = \int_0^d \left(f(\theta)\theta'^2 + \cos^2\theta\left(g(\theta)\varphi'^2 - 2\tau K_2\varphi'\right)\right) dz.$$

Notice that, in general, a minimum $n(z)$ of $E(\varphi, \theta)$ does not minimize the energy over the larger set of the configurations depending also on x and y; for some cases in which this happen, see [1], [2].

In the first case that we want to study we impose planar boundary conditions:

$$n(0) = (1,0,0), \qquad n(d) = (\cos\alpha, \sin\alpha, 0), \tag{3}$$

namely:

$$\theta(0) = \theta(d) = 0, \qquad \varphi(0) = 0, \varphi(d) = \alpha.$$

Clearly the problem (1)–(3) admits the trivial solution $\theta(z) = 0$, $\varphi(z) = \alpha z/d$:

$$n(z) = (\cos(\alpha z/d), \sin(\alpha z/d), 0);$$

the question is if there are non trivial solutions. More precisely, we search for *positive* solutions, namely with $\theta(z) > 0$ and $\varphi'(z) > 0$ for $0 < z < d$, which ensures right-handed helicoids. We can consider τ as fixed, in fact, by a scaling of z we see that the problem (1)–(3) depends indeed only on d/τ; then we choose d as bifurcation parameter. Notice moreover that a non trivial solution with $\alpha > \pi$ means that the optical axis $n(z)$ carries out a half turn at $z_0 < d$, but $n(z_0)$ is not planar, since $\theta(z_0) > 0$; in the case $0 < \alpha < \pi$, $n(z)$ becomes planar at $z = d$ before the half turn.

For nematic liquid crystals, namely in the case $\tau = 0$, this problem has been studied in [5] (see also [6], or [8], Chapter 3) for all possible values of the constants K_i. The cholesteric case is more complicated, and in this paper we study the case $K_2 = K_3 = 1$.

Let us denote by $\mathcal{R}$ the set of the couples (d, α), with $d > 0$, $\alpha > 0$, strictly between the straight line $\alpha_1(d) = \tau d + \pi$ and the hyperbola $\alpha_2(d) = \tau d + \sqrt{K_1\pi^2 + \tau^2 d^2}$. Clearly, if $K_1 \geq 1$, we have $\alpha_1(d) < \alpha_2(d)$; whereas if $0 < K_1 < 1$, $\alpha_1(d)$ and $\alpha_2(d)$ intersect at the point $P = (\pi\sqrt{1-K_1}/\tau, \pi(1 + \sqrt{1-K_1}))$. In this case, we set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1$ is the bounded component of $\mathcal{R}$. Then we have the following result.

Theorem 1. *Let $K_2 = K_3 = 1$; then there exists exactly one non trivial solution of* (1)–(3) *if and only if $(d, \alpha) \in \mathcal{R}$. Moreover, if $0 < K_1 < 1$ and $(d, \alpha) \in \mathcal{R}_1$, then this solution minimizes the energy $E(n)$ over the functions depending only on z. If $(d, \alpha) = P$, there are infinitely many non trivial solutions of* (1)–(3).

Let us consider now the case of orthogonal (homeotropic) conditions:

$$n(0) = n(d) = (0,0,1). \tag{4}$$

In terms of Euler angles, we must have $\theta(0) = \theta(d) = \pi/2$, while $\varphi(0)$ and $\varphi(d)$ are indeterminate. The trivial solution for the problem (1)–(4) is the constant solution $n(z) = (0,0,1)$; we search for non trivial solutions with $0 < \theta(z) < \pi/2$ for $0 < z < d$. It is well known (see [9], [10]) that non trivial solutions bifurcate from the constant solution when d becomes large with respect to the threshold $2d_0 = \pi K_3/\tau K_2$. In this paper we want to provide an analytical proof of the exact number of these solutions, and their stability.

As we shall see later, the number of such non trivial solutions depends on the behavior of the function

$$Z(m) = \int_m^{\frac{\pi}{2}} \frac{\sqrt{g(m)}\sqrt{f(\theta)g(\theta)}}{\tau K_2\sqrt{K_3}\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta, \tag{5}$$

where $0 < m < \pi/2$ is the minimum of $\theta(z)$. More precisely, $Z(m)$ is the time-map for a suitable Cauchy problem, and a solution $0 < m < \pi/2$ of the equation $Z(m) = d/2$ gives rise to a non trivial solution $\theta_m(z)$, with minimum m. It is easy to see that

$$\lim_{m\to 0} Z(m) = +\infty, \qquad \lim_{m\to\pi/2} Z(m) = d_0. \tag{6}$$

We prove the following theorems.

Theorem 2. *Let us suppose that $K_1 - 3(K_3 - K_2) < 0$; then the function $Z(m)$ is strictly decreasing from $+\infty$ to a minimum value $d^* < d_0$, and strictly increasing from d^* to d_0.*

Theorem 3. *Let us suppose that $K_1 - 3(K_3 - K_2) \geq 0$; then, two possibilities can occur, namely either $Z(m)$ is strictly decreasing from $+\infty$ to d_0, or it has exactly one local minimum d^* and one local maximum d^{**}, with $d_0 < d^{**}$.*

Clearly from Theorems 2 and 3 follows an exact multiplicity result for the number of non trivial solutions of the problem (1)–(4). The numbers d^* and d^{**} cannot be explicitly calculated; however, in the case $K_1 - 3(K_3 - K_2) \geq 0$, we give later sufficient conditions in order to ensure the strict decreasing of $Z(m)$.

Finally, as for the linear stability of the non trivial solutions, we use the well known relation with the monotonicity of the time-map $Z(m)$ (see, for instance, [7], Chapter IV), and we prove the following theorem.

Theorem 4. *Let $0 < m < \pi/2$ be such that $Z(m) = d/2$; then, if $Z'(m) < 0$, the corresponding non trivial solution $\theta_m(z)$ is stable; if $Z'(m) > 0$, $\theta_m(z)$ is unstable.*

2. Planar boundary conditions

In this section we shall prove Theorem 1; we assume that $K_2 = K_3 = 1$, and so the equations (1) become:

$$\begin{cases} 2f(\theta)\theta'' + f'(\theta)\theta'^2 + (\varphi' - 2\tau)\,\varphi'\sin(2\theta) = 0 \\ \varphi''\cos\theta + 2\,(\tau - \varphi')\,\theta'\sin\theta = 0 \end{cases} \tag{7}$$

where $f(\theta) = K_1\cos^2\theta + \sin^2\theta$; the boundary conditions are:

$$\theta(0) = \theta(d) = 0, \qquad \varphi(0) = 0, \varphi(d) = \alpha. \tag{8}$$

As we mentioned, we search for *positive* solutions, namely with $\theta(z) > 0$ for $0 < z < d$, and $\varphi'(z) > 0$ for $0 \le z \le d$. To this end we convert the problem (7)–(8) into a Cauchy problem.

Lemma 1. *Let us consider the Cauchy problem:*

$$(9) \qquad \begin{cases} 2f(\theta)\theta'' + f'(\theta)\theta'^2 + (\varphi' - 2\tau)\,\varphi'\sin(2\theta) = 0 \\ \varphi''\cos\theta + 2\,(\tau - \varphi')\,\theta'\sin\theta = 0 \\ \theta(0) = M \qquad \theta'(0) = 0 \\ \varphi(0) = 0 \qquad \varphi'(0) = \frac{\lambda}{\cos^2 M} + \tau. \end{cases}$$

Then, the problem (7)–(8) *has a positive solution if and only if there exists* $0 < M < \pi/2$ *and* $\lambda > \tau\cos M$ *such that* (9) *has a solution* $(\theta_1(z), \varphi_1(z))$ *with* $\theta_1(d/2) = 0$, $\varphi_1(d/2) = \alpha/2$, *and* $\theta_1(z) > 0$ *for* $0 \le z < d/2$.

Proof: The proof is standard. Let us suppose that $\theta(z)$, $\varphi(z)$ is a positive solution of (7)–(8). By multiplying the second equation for $\cos\theta$ and integrating, we have $\cos^2(\theta)(\varphi' - \tau) = \lambda$, where λ is some constant, and since $\cos^2\theta \neq 0$, $\varphi' = \lambda/\cos^2\theta + \tau$. Inserting this in the first equation, multiplying it by θ' and integrating, we have also $f(\theta)\theta'^2 + h(\theta) = \text{constant}$, where $h(\theta)$ is the function

$$h(\theta) = \frac{\lambda^2}{\cos^2\theta} - \tau^2\sin^2\theta.$$

We must have $\theta'(0)^2 = \theta'(d)^2$, and since $\theta(z)$ is positive, $\theta'(d) = -\theta'(0)$. Then $\theta(z)$ is symmetric with respect to $z = d/2$, since $\theta(d - z)$ satisfies the same Cauchy problem of $\theta(z)$. The function $\varphi(z)$ is symmetric with respect to the point $(d/2, \alpha/2)$. Set $M = \theta(d/2)$; then $0 < M < \pi/2$ and $f(\theta)\theta'^2 = h(M) - h(\theta)$, so M is the maximum of $\theta(z)$, and the level set $h(\theta) \le M$ must be the interval $[-M, M]$. It is easy to check that this condition is satisfied if and only if $h(0) < h(M)$, namely if $\lambda^2 > \tau^2\cos^2 M$. We observe now that $\lambda < -\tau\cos M$ implies $\varphi'(d/2) < 0$, while we suppose $\varphi' > 0$, so we must have $\lambda > \tau\cos M$. Then, the shifted functions $\theta_1(z) = \theta(z + d/2)$, $\varphi_1(z) = \varphi(z + d/2) - \alpha/2$ satisfy the Cauchy problem (9) with $0 < M < \pi/2$ and $\lambda > \tau\cos M$. If conversely $0 < M < \pi/2$, $\lambda > \tau\cos M$ and θ_1 and φ_1 are solutions of (9) as required (clearly this can happen because of the condition $\lambda > \tau\cos M$), then we can continue these solutions for $-d/2 < z < 0$ and, respectively, $d/2 < z \le d$, by symmetry, and then the functions $\theta(z) = \theta_1(z + d/2)$ and $\varphi = \varphi_1$ are solutions of (7)–(8). □

Now, let $0 < M < \pi/2$ and $\lambda > \tau\cos M$; the Cauchy problem (9) has a solution well defined for all $z \ge 0$, and $\theta(z)$ is periodic and strictly decreasing from M to zero in some interval $[0, z_0]$; let us denote by $Z_\lambda(M)$ this first zero z_0 of $\theta(z)$, so that $Z_\lambda(M)$ is a time-map for (9). We can calculate $Z_\lambda(M)$ as follows.

Since $f(\theta)\theta'^2 = h(M) - h(\theta)$, and $\theta'(z) < 0$ for $0 < z \le Z_\lambda(M)$, we have

$$(10) \qquad -\frac{\sqrt{f(\theta(z))}\,\theta'(z)}{\sqrt{h(M) - h(\theta(z))}} = 1$$

for $0 < z \le Z_\lambda(M)$, so that, integrating in this interval, we have:

$$Z_\lambda(M) = \int_0^M \frac{\sqrt{f(\theta)}}{\sqrt{h(M) - h(\theta)}}\, d\theta,$$

or, more explicitly:

$$Z_\lambda(M) = \int_0^M \frac{\sqrt{f(\theta)}\cos M \cos\theta}{\sqrt{\sin^2 M - \sin^2\theta}\sqrt{\lambda^2 - \tau^2\cos^2 M\cos^2\theta}}\, d\theta.$$

We set now $\Phi_\lambda(M) = \varphi(Z_\lambda(M))$; since

$$\varphi(z) = \int_0^z \left(\frac{\lambda}{\cos^2\theta(z)} + \tau\right) dz,$$

using (10) we have

$$\Phi_\lambda(M) = \int_0^M \frac{\lambda\sqrt{f(\theta)}\cos M}{\cos\theta\sqrt{\sin^2 M - \sin^2\theta}\sqrt{\lambda^2 - \tau^2\cos^2 M\cos^2\theta}}\, d\theta + \tau Z_\lambda(M).$$

For $0 < M < \pi$ fixed, let us consider the function $\gamma_M(\lambda) = (Z_\lambda(M), \Phi_\lambda(M))$; we think $\gamma_M(\lambda)$ as a curve on the plane Z–Φ, parameterized by $\lambda > \tau\cos M$; then, from Lemma 1, we have a non trivial positive solution of (7)–(8) if and only if there exists a curve $\gamma_M(\lambda)$ passing through the point $(d/2, \alpha/2)$. So, we study the region of the plane Z–Φ filled by the curves $\gamma_M(\lambda)$.

Clearly $Z_\lambda(M)$ and $\Phi_\lambda(M)$ are strictly decreasing with respect to λ, and $Z_\lambda(M) \to +\infty$, $\Phi_\lambda(M) \to +\infty$ for $\lambda \to \tau\cos M$. For $\lambda \to +\infty$ we have $Z_\lambda(M) \to 0$ and $\Phi_\lambda(M) \to \Phi_\infty(M)$, where

$$\Phi_\infty(M) = \int_0^M \frac{\sqrt{f(\theta)}\cos M}{\cos\theta\sqrt{\sin^2 M - \sin^2\theta}}\, d\theta,$$

so $\gamma_M(\lambda)$ has the limit-point $(0, \Phi_\infty(M))$ for $\lambda \to \infty$.

Lemma 2. *The function $M \to \Phi_\infty(M)$ is strictly increasing if $K_1 < 1$, strictly decreasing if $K_1 > 1$, and equal to $\pi/2$ for $K_1 = 1$. Moreover*

$$\lim_{M\to 0}\Phi_\infty(M) = \frac{\pi}{2}\sqrt{K_1}, \quad \lim_{M\to\pi/2}\Phi_\infty(M) = \frac{\pi}{2}\,.$$

Proof: If $K_1 = 1$ we have $f(\theta) = 1$, so it is easy that $\Phi_\infty(M) = \pi/2$ for every M. If $K_1 \ne 1$, the function $\Phi_\infty(M)$ can be explicitly calculated in terms of elliptic functions. In fact, after the change of the integration variable $\sin\theta = \sin M \sin x$, we have

$$\Phi_\infty(M) = \frac{\cos M}{\sqrt{K_1}} \int_0^{\frac{\pi}{2}} \frac{1}{(1-\sin^2 M \sin^2 x)\sqrt{1-\frac{K_1-1}{K_1}\sin^2 M \sin^2 x}}\, dx$$
$$+ \frac{\cos M}{\sqrt{K_1}} \int_0^{\frac{\pi}{2}} \frac{K_1 - 1}{\sqrt{1-\frac{K_1-1}{K_1}\sin^2 M \sin^2 x}}\, dx,$$

namely,

$$\Phi_\infty(M) = \frac{\cos M}{\sqrt{K_1}} \Pi\left(\sin^2 M, \frac{K_1-1}{K_1}\sin^2 M\right)$$
$$+ \frac{(K_1-1)\cos M}{\sqrt{K_1}} K\left(\frac{K_1-1}{K_1}\sin^2 M\right),$$

where $\Pi(n,m)$ and $K(m)$ are the complete elliptic integrals

$$\Pi(n,m) = \int_0^{\frac{\pi}{2}} \frac{1}{(1-n\sin^2 x)\sqrt{1-m\sin^2 x}}\, dx,$$

and

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-m\sin^2 x}}\, dx.$$

Then, using the derivatives of elliptic functions (see, for instance, [3]), we have

$$\partial_M \Phi_\infty(M) = \frac{\sqrt{K_1}}{\sin M}\left(E\left(\frac{(K_1-1)\sin^2 M}{K_1}\right) - K\left(\frac{(K_1-1)\sin^2 M}{K_1}\right)\right)$$

where, as usual,

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1-m\sin^2 x}\, dx.$$

Now, since $E(x) < K(x)$ for $0 < x < 1$, while $E(x) > K(x)$ for $x < 0$, the first part of lemma is proved. With regard to the limits, the first one is obvious. We shall prove the second directly; in fact, since $\sqrt{f(\theta)} \geq \sin\theta$, we have

$$\Phi_\infty(M) \geq \int_0^M \frac{\sin\theta\cos M}{\cos\theta\sqrt{\sin^2 M - \sin^2\theta}}\, d\theta = M,$$

and since $\sin^2\theta/\sqrt{f(\theta)}$ is increasing, we have

$$\begin{aligned}\Phi_\infty(M) &\le \int_0^M \frac{K_1\cos\theta\cos M}{\sqrt{\sin^2 M-\sin^2\theta}\sqrt{f(\theta)}}\,d\theta \\ &\quad+\int_0^M \frac{\sin^2 M\cos M}{\cos\theta\sqrt{\sin^2 M-\sin^2\theta}\sqrt{f(M)}}\,d\theta \\ &\le \int_0^M \frac{K_1\cos\theta\cos M}{\sqrt{\sin^2 M-\sin^2\theta}\sqrt{f(\theta)}}\,d\theta+\frac{\pi}{2}\frac{\sin^2 M}{\sqrt{f(M)}}\,.\end{aligned}$$

Clearly, the last integral tend to zero for $M\to\pi/2$, so, by using these inequalities, we get the second limit of the lemma. □

We observe now that

$$\lim_{M\to 0} Z_\lambda(M)=\frac{\pi}{2}\frac{\sqrt{K_1}}{\sqrt{\lambda^2-\tau^2}},\qquad \lim_{M\to 0}\Phi_\lambda(M)=\frac{\pi}{2}\frac{\sqrt{K_1}(\lambda+\tau)}{\sqrt{\lambda^2-\tau^2}}$$

(the convergence is actually uniform for λ far away from τ). Clearly the curve $\gamma_0(\lambda)=(Z_\lambda(0),\Phi_\lambda(0))$ is a hyperbola in the Z–Φ plane, of equation

$$\phi=\tau Z+\frac{1}{2}\sqrt{K_1\pi^2+4\tau^2Z^2}. \tag{11}$$

Moreover it is not difficult to check that

$$\frac{\partial_\lambda\Phi_\lambda(M)}{\partial_\lambda Z_\lambda(M)}=\tau\left(1+\frac{\tau\cos^2 M}{\lambda}\right),$$

which gives the tangent line to γ_M. Since this derivative is increasing from τ to $\tau(1+\cos M)$ when λ goes from $+\infty$ to $\tau\cos M$, γ_M is convex as function of Z. Let us denote now by $\mathcal{S}$ the region of the plane Z–Φ (more precisely with $Z>0$ and $\Phi>0$) strictly between the hyperbola and the line

$$\Phi=\tau Z+\frac{\pi}{2}\,. \tag{12}$$

If $0<K_1<1$, then (11) and (12) intersect at the point

$$Q=(\pi\sqrt{1-K_1}/2\tau,\pi(1+\sqrt{1-K_1})/2),$$

and we set $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$, where $\mathcal{S}_1$ is the bounded component of $\mathcal{S}$. Clearly $(d,\alpha)\in\mathcal{R}\Leftrightarrow(d/2,\alpha/2)\in\mathcal{S}$, where $\mathcal{R}$ is the region in Theorem 1. In the following lemma we show that the curves γ_M fill the region $\mathcal{S}$.

Lemma 3. *For every $P\in\mathcal{S}$, there exist $0<M<\pi/2$ and $\lambda>\tau\cos M$ such that $P=(Z_\lambda(M),\Phi_\lambda(M))$.*

PROOF: Let $P_0 = (Z_0, \Phi_0) \in \mathcal{S}$; since $\lambda \to Z_\lambda(M)$ is strictly decreasing from $+\infty$ to zero, there exists $\lambda(M)$ such that $Z_{\lambda(M)}(M) = Z_0$; moreover the hyperbola intersects $Z = Z_0$ for λ_0 such that $Z_0 = \pi\sqrt{K_1}/2\sqrt{\lambda_0^2 - \tau^2}$. We claim that

$$\lim_{M\to 0} \Phi_{\lambda(M)}(M) = \frac{\pi}{2} \frac{\sqrt{K_1}(\lambda_0 + \tau)}{\sqrt{\lambda_0^2 - \tau^2}} \,. \tag{13}$$

In fact, we observe that

$$\frac{\pi}{2} \frac{\sqrt{K_1} \cos M}{\sqrt{\lambda^2 - \tau^2 \cos^4 M}} < Z_\lambda(M) < \frac{\pi}{2} \frac{\sqrt{K_1} \cos M}{\sqrt{\lambda^2 - \tau^2}}$$

and then $\lambda_1(M) < \lambda(M) < \lambda_2(M)$, where $\lambda_1(M)$ and $\lambda_2(M)$ are defined by

$$\frac{\pi}{2} \frac{\sqrt{K_1} \cos M}{\sqrt{\lambda_1(M)^2 - \tau^2 \cos^4 M}} = Z_0 \quad \text{and} \quad \frac{\pi}{2} \frac{\sqrt{K_1} \cos M}{\sqrt{\lambda_2(M)^2 - \tau^2}} = Z_0$$

respectively. Clearly for $M \to 0$ we have $\lambda_1(M) \to \lambda_0$ and $\lambda_2(M) \to \lambda_0$, so that $\lambda(M) \to \lambda_0$ and the claim is proved. We observe now that

$$\lim_{M\to \pi/2} \Phi_{\lambda(M)}(M) = \tau Z_0 + \frac{\pi}{2} \,. \tag{14}$$

In fact, from the convexity of γ_M, we have

$$\tau Z_0 + \Phi_\infty(M) < \Phi_{\lambda(M)}(M) < \tau \left(1 + \frac{\tau \cos^2 M}{\lambda(M)}\right) Z_0 + \Phi_\infty(M).$$

Recalling that $\tau < \tau(1+\tau \cos^2 M/\lambda(M)) < \tau(1+\cos M)$, and Lemma 2, passing to the limit for $M \to \pi/2$ we get (14). From (13) and (14), we have $\Phi_{\lambda(M)}(M) = \Phi_0$ for some $0 < M < \pi/2$, and the lemma is proved. □

In the next lemma we prove that γ_M is confined in $\mathcal{S}$.

Lemma 4. *Let M be such that $0 < M < \pi/2$; if $K_1 \geq 1$, then γ_M does not intersect the hyperbola* (11) *and the line* (12). *If $0 < K_1 < 1$, then γ_M intersects* (11) *and* (12) *only at the point $Q = (\pi\sqrt{1-K_1}/\tau, \pi(1+\sqrt{1-K_1}))$.*

PROOF: Let $0 < M < \pi/2$, and suppose first $K_1 \geq 1$. Clearly γ_M does not intersect (12) because of the convexity. Now, let us suppose that γ_M intersects the hyperbola in a point (Z_0, Φ_0) for some $\lambda > \tau \cos M$. Then we must have $\Phi_\lambda(M) = \Phi_0$, and then

$$\int_0^M \frac{\lambda\sqrt{f(\theta)} \cos M}{\cos\theta\sqrt{\sin^2 M - \sin^2\theta}\sqrt{\lambda^2 - \tau^2\cos^2 M \cos^2\theta}}\, d\theta = \frac{1}{2}\sqrt{K_1\pi^2 + 4\tau^2 Z_0^2} \,.$$

The integral is strictly less than

$$\frac{\lambda}{\cos^2 M}\int_0^M \frac{\sqrt{f(\theta)}\cos M\cos\theta}{\sqrt{\sin^2 M-\sin^2\theta}\sqrt{\lambda^2-\tau^2\cos^2 M\cos^2\theta}}\,d\theta=\frac{\lambda}{\cos^2 M}Z_0,$$

so that we have

$$\frac{1}{2}\sqrt{K_1\pi^2+4\tau^2 Z_0^2}<\frac{\lambda}{\cos^2 M}Z_0$$

at the contact point. On the other hand, for derivatives we have

$$\tau\left(1+\frac{\tau\cos^2 M}{\lambda}\right)\geq\tau+\frac{2Z_0\tau^2}{\sqrt{K_1\pi^2+4\tau^2 Z_0^2}} \tag{15}$$

since γ_M goes through the hyperbola from the bottom to the top at the (first) contact point. Then we get $\lambda/\cos^2 M<\lambda/\cos^2 M$, which is impossible.

Let us suppose now $0<K_1<1$. Then (11) and (12) intersect exactly at the point $Q=(\pi\sqrt{1-K_1}/\tau,\pi(1+\sqrt{1-K_1}))$. Let us consider a curve γ_M; for $\lambda=\tau\cos M/\sqrt{1-K_1}$, we have

$$\lambda^2-\tau^2\cos^2 M\cos^2\theta=\frac{\tau^2\cos^2 M}{1-K_1}f(\theta),$$

so the integrals in $Z_\lambda(M)$ and in $\Phi_\lambda(M)$ will simplify and we get

$$\gamma_M(\lambda)=(Z_\lambda(M),\Phi_\lambda(M))=Q.$$

In other words, for $\lambda=\tau\cos M/\sqrt{1-K_1}$, the curve $\gamma_M(\lambda)$ goes through the point Q. We claim that $\gamma_M(\lambda)$ is confined in $\mathcal{S}_2$ if $\lambda<\tau\cos M/\sqrt{1-K_1}$, while is confined in $\mathcal{S}_1$ if $\lambda>\tau\cos M/\sqrt{1-K_1}$. In fact, by the convexity, γ_M does not intersect the line (12) at any point different from Q.

Now, let us suppose that $\gamma_M(\lambda)$ intersects (11) at a point $P_0=(Z_0,\Phi_0)\neq Q$ for some $\lambda<\tau\cos M/\sqrt{1-K_1}$; then (15) holds true at this contact point, and we can obtain an absurd arguing as above.

If we have instead $\lambda>\tau\cos M/\sqrt{1-K_1}$, there exists a second contact point at some λ_1 with $\tau\cos M/\sqrt{1-K_1}<\lambda_1<\lambda$, such that (15) holds (with $\lambda=\lambda_1$), and we can again argue as above. □

Lemma 5. *Let us suppose that $M_1<M_2$, and that there exists $\lambda_0>\tau\cos M_1$, $\mu_0>\tau\cos M_2$ such that $Z_{\lambda_0}(M_1)=Z_{\mu_0}(M_2)$. Then*

$$\frac{\cos^2 M_1}{\lambda_0}>\frac{\cos^2 M_2}{\mu_0}.$$

Proof: Arguing by contradiction, let us suppose that

$$\frac{\cos^2 M_1}{\lambda_0}\leq\frac{\cos^2 M_2}{\mu_0}. \tag{16}$$

Since $\mu_0 > \tau \cos M_2$, from (16) follows that

$$\lambda_0 > \frac{\tau \cos^2 M_1}{\cos M_2}. \tag{17}$$

We observe now that, after the change of the integration variable $\sin\theta = \sin M \sin x$, we have

$$Z_\lambda(M) = \int_0^{\frac{\pi}{2}} \frac{\sqrt{K_1 - (K_1-1)\sin^2 M \sin^2 x}\,\cos M}{\sqrt{\lambda^2 - \tau^2 \cos^2 M \left(1 - \sin^2 M \sin^2 x\right)}}\, dx,$$

and let us consider the function

$$a(M) = \int_0^{\frac{\pi}{2}} \frac{\sqrt{K_1 - (K_1-1)\sin^2 M \sin^2 x}}{\sqrt{\frac{\lambda_0^2 \cos^2 M}{\cos^4 M_1} - \tau^2 \left(1 - \sin^2 M \sin^2 x\right)}}\, dx.$$

Clearly $a(M_1) = Z_{\lambda_0}(M_1)$; moreover $a(M)$ is well defined on $[M_1, M_2]$, in fact, if $M \in [M_1, M_2]$, then from (17) we have $\lambda_0^2 \cos^2 M / \cos^4 M_1 > \tau^2$. We observe now that, for all $K_1 > 0$, $a(M)$ is strictly increasing in the interval $[M_1, M_2]$. In fact, it is easy to calculate $a'(M)$, and to check that the sign of $a'(M)$ depends on

$$\frac{\lambda_0^2 \left(1 + K_1 + (K_1 - 1)\cos(2x)\right)}{2\cos^4 M_1} - \tau^2 \sin^2 x,$$

which is positive for all $K_1 > 0$, since $\lambda_0 > \tau \cos M_1$. Set, for brevity: $\mu_1 = \lambda_0 \cos^2 M_2 / \cos^2 M_1$; then

$$Z_{\mu_0}(M_2) = Z_{\lambda_0}(M_1) = a(M_1) < a(M_2) = Z_{\mu_1}(M_2).$$

Since the function $\lambda \to Z_\lambda(M_2)$ is strictly decreasing, we must have $\mu_0 > \mu_1$, so that

$$\frac{\cos^2 M_1}{\lambda_0} > \frac{\cos^2 M_2}{\mu_0},$$

a contradiction, and the lemma is proved. □

Now we have the following lemma.

Lemma 6. *If $M_1 \neq M_2$, then, if $K_1 \geq 1$, γ_{M_1} and γ_{M_2} do not intersect; if $0 < K_1 < 1$, then they intersect only at the point Q.*

Proof: Suppose first that $K_1 \geq 1$; let $M_1 < M_2$, and, arguing by contradiction, let us suppose that $(Z_{\lambda_1}(M_1), \Phi_{\lambda_1}(M_1)) = (Z_{\mu_1}(M_2), \Phi_{\mu_1}(M_2))$ for some $\lambda_1 > \tau \cos M_1$ and some $\mu_1 > \tau \cos M_2$. Then there exist $\lambda_0 > \lambda_1$ and $\mu_0 > \mu_1$ such that $Z_{\lambda_0}(M_1) = Z_{\mu_0}(M_2)$, and

$$\frac{\Phi_{\lambda_1}(M_1) - \Phi_\infty(M_1)}{\Phi_{\mu_1}(M_2) - \Phi_\infty(M_2)} = \frac{\tau\left(1 + \tau \cos^2 M_1/\lambda_0\right)}{\tau\left(1 + \tau\cos^2 M_2/\mu_0\right)}.$$

Since (from Lemma 2) $\Phi_\infty(M_1) \geq \Phi_\infty(M_2)$, we get

$$\frac{\cos^2 M_1}{\lambda_0} \leq \frac{\cos^2 M_2}{\mu_0},$$

which is impossible because of Lemma 5.

Let us suppose now $0 < K_1 < 1$; then the curves $\gamma_M(\lambda)$ intersect at the point $Q = (\pi\sqrt{1-K_1}/\tau, \pi(1+\sqrt{1-K_1}))$ for $\lambda = \tau\cos M/\sqrt{1-K_1}$ (see Lemma 4). Moreover, about the derivative of γ_M at the point Q, we have

$$\tau\left(1+\frac{\tau\cos^2 M}{\lambda}\right) = \tau\left(1+\sqrt{1-K_1}\cos M\right).$$

Now, let $M_1 < M_2$, and let us suppose that γ_{M_1} and γ_{M_2} intersect at a point $P_0 = (Z_{\lambda_0}(M_1), \Phi_{\lambda_0}(M_1)) = (Z_{\mu_0}(M_2), \Phi_{\mu_0}(M_2))$, with $P_0 \neq Q$. Then, if $P_0 \in \mathcal{S}_2$, namely if $\lambda_0 < \tau\cos M_1/\sqrt{1-K_1}$ and $\mu_0 < \tau\cos M_2/\sqrt{1-K_1}$, we can assume that there are not other intersections for $\lambda_0 < \lambda < \tau\cos M_1/\sqrt{1-K_1}$ and $\mu_0 < \mu < \tau\cos M_2/\sqrt{1-K_1}$; then, since the derivative of γ_{M_2} at Q is less than that of γ_{M_1}, we have

$$\frac{\cos^2 M_1}{\lambda_0} \leq \frac{\cos^2 M_2}{\mu_0}, \tag{18}$$

at the contact point P_0, which is not possible from Lemma 5.

Similarly, if $P_0 \in \mathcal{S}_1$, namely if we have $\lambda_0 > \tau\cos M_1/\sqrt{1-K_1}$, and $\mu_0 > \tau\cos M_2/\sqrt{1-K_1}$, recalling again the derivatives at Q, we can assume that for $\tau\cos M_1/\sqrt{1-K_1} < \lambda < \lambda_0$ and $\tau\cos M_2/\sqrt{1-K_1} < \mu < \mu_0$ there are not other intersections, and then we must have (18), and we can conclude as above. $\square$

We are now in the position to prove Theorem 1.

Proof of Theorem 1: From the previous lemmas, it is clear that the problem (1)–(3) has a non trivial positive solution if and only if $(d/2, \alpha/2) \in \mathcal{S}$, namely $(d, \alpha) \in \mathcal{R}$, and this non trivial solution is unique. In the case $0 < K_1 < 1$, the problem (1)–(3), with $d = 2\pi\sqrt{1-K_1}/\tau$, and $\alpha = \pi(1+\sqrt{1-K_1})$ has infinitely many non trivial positive solutions, corresponding to the infinitely many curves γ_M passing through the point Q.

Now, let $(d, \alpha) \in \mathcal{R}$, and let us consider the trivial solution $\theta_0(z) = 0$ and $\varphi_0(z) = \alpha z/d$, namely:

$$n_0(z) = \left(\cos\left(\frac{\alpha z}{d}\right), \sin\left(\frac{\alpha z}{d}\right), 0\right);$$

if we set $\theta_\epsilon(z) = \epsilon\sin(\pi z/d)$, then the second variation of the functional $j(\epsilon) = E(\varphi_0, \theta_\epsilon)$ (see (2)) gives:

$$j''(0) = \frac{1}{d}\left(2\tau d\alpha - \alpha^2 + K_1\pi^2\right),$$

which is negative if (d, α) is above the hyperbola $2\tau d\alpha - \alpha^2 + K_1\pi^2 = 0$, namely for $K_1 < 1$ and $(d, \alpha) \in \mathcal{R}_1$, so the theorem is proved. □

3. Orthogonal boundary conditions

Let us consider equations (1), and search for solutions $\theta(z)$, $\varphi(z)$ such that $0 < \theta(z) < \pi/2$ for $0 < z < d$, $\theta(z) = \pi/2$ for $z = 0$ and $z = d$. Then the director field $n(z)$ is orthogonal to the layer at the boundary, namely (4) is satisfied, and we can leave $\varphi(0)$ and $\varphi(d)$ indeterminate. Since from the second equation we have again $\cos^2(\theta)(g(\theta)\varphi' - K_2\tau) = \lambda =$ constant, from the boundary conditions follows that $\lambda = 0$, so (1) reduces to equation

$$2f(\theta)\theta'' + f'(\theta)\theta'^2 - \frac{K_3K_2^2\tau^2}{g(\theta)^2}\sin(2\theta) = 0, \tag{19}$$

while $\varphi'(z) = K_2\tau/g(\theta(z))$. Arguing as in the previous section, it is easy to see that this problem is equivalent to the Cauchy problem

$$\begin{cases} 2f(\theta)\theta'' + f'(\theta)\theta'^2 - \frac{K_3K_2^2\tau^2}{g(\theta)^2}\sin(2\theta) = 0 \\ \theta(0) = m \qquad \theta'(0) = 0 \end{cases} \tag{20}$$

in the sense that, if $0 < m < \pi/2$, and $\theta_1(z)$ is a solution of (20) such that $m \le \theta_1(z) < \pi/2$ for $0 \le z < d/2$, and $\theta_1(d/2) = \pi/2$, then the function θ obtained from θ_1 as in the proof of Lemma 1 satisfies the original boundary value problem, and vice versa.

Let us denote by $Z(m)$ the first $z > 0$ such that the solution $\theta(z)$ of (20) reach $\pi/2$. Using the fact that $f(\theta)\theta'^2 + (K_2^2\tau^2/K_3)\sin^2\theta =$ constant, we can calculate $Z(m)$, and we obtain the function (5). As mentioned in Section 1, we are interested in the exact number of solutions of $Z(m) = d/2$. To this end, we first calculate $Z'(m)$.

Lemma 7. *Set $a(\theta) = g(\theta)\sqrt{f(\theta)g(\theta)}/\sin\theta$. Then*

$$Z'(m) = \frac{\tan m}{\tau K_2\sqrt{K_3}\sqrt{g(m)}}\int_m^{\frac{\pi}{2}} \frac{a'(\theta)\cos\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\,d\theta.$$

Proof: We can write $Z(m)$ as:

$$Z(m) = \frac{\sqrt{g(m)}}{\tau K_2\sqrt{K_3}}\int_m^{\frac{\pi}{2}} \frac{a(\theta)\sin\theta}{g(\theta)\sqrt{\sin^2\theta - \sin^2 m}}\,d\theta;$$

by integration by parts, we have

$$Z(m) = \frac{a(m)}{\tau K_2K_3} + \frac{1}{\tau K_2K_3}\int_m^{\frac{\pi}{2}} a'(\theta)\arctan\left(\frac{\sqrt{g(m)}\cos\theta}{\sqrt{K_3}\sqrt{\sin^2\theta - \sin^2 m}}\right)d\theta.$$

The derivative with respect to m of the last integral is

$$\int_m^{\frac{\pi}{2}} a'(\theta)\frac{\sqrt{K_3}\tan m\cos\theta}{\sqrt{g(m)}\sqrt{\sin^2\theta-\sin^2 m}}\,d\theta-\frac{\pi}{2}a'(m),$$

and so we get the lemma. □

Since we are interested in the sign of $Z'(m)$ and other similar functions that we shall see later, we prove the following lemma.

Lemma 8. *Let $r(\theta)$ be continuous for $0<\theta\le\pi/2$, and suppose that, for some $0<\theta_0<\pi/2$, $r(\theta)$ is strictly increasing from $-\infty$ to 0 for $0<\theta\le\theta_0$, and strictly positive for $\theta_0<\theta<\pi/2$. Then, the function*

$$F(m)=\int_m^{\frac{\pi}{2}}\frac{r(\theta)\cos\theta}{\sqrt{\sin^2\theta-\sin^2 m}}\,d\theta$$

has exactly one zero θ_1 with $0<\theta_1<\theta_0$.

Proof: Clearly there exists θ_1 with $0<\theta_1<\theta_0$ such that $F(\theta_1)=0$ and $F(m)>0$ for $\theta_1<m<\pi/2$. We claim that $F(m)<0$ for $0<m<\theta_1$. In fact, let $0<m<\theta_1$; observing that $\sqrt{\sin^2\theta-\sin^2\theta_1}<\sqrt{\sin^2\theta-\sin^2 m}$ for $\theta_0<\theta<\pi/2$, and using the fact that $F(\theta_1)=0$, we have:

$$\int_{\theta_0}^{\frac{\pi}{2}}\frac{r(\theta)\cos\theta}{\sqrt{\sin^2\theta-\sin^2 m}}\,d\theta<-\int_{\theta_1}^{\theta_0}\frac{r(\theta)\cos\theta}{\sqrt{\sin^2\theta-\sin^2\theta_1}}\,d\theta.$$

Then

$$\begin{aligned}F(m)<\int_m^{\theta_1}&\frac{r(\theta)\cos\theta}{\sqrt{\sin^2\theta-\sin^2 m}}\,d\theta\\&+\int_{\theta_1}^{\theta_0}r(\theta)\left(\frac{\cos\theta}{\sqrt{\sin^2\theta-\sin^2 m}}-\frac{\cos\theta}{\sqrt{\sin^2\theta-\sin^2\theta_1}}\right)d\theta,\end{aligned}$$

and since $r(\theta)$ is increasing in $[m,\theta_0]$, we get

$$\begin{aligned}F(m)<r(\theta_1)&\left(\int_m^{\theta_0}\frac{\cos\theta}{\sqrt{\sin^2\theta-\sin^2 m}}\,d\theta-\int_{\theta_1}^{\theta_0}\frac{\cos\theta}{\sqrt{\sin^2\theta-\sin^2\theta_1}}\,d\theta\right)\\&=r(\theta_1)\log\frac{\sin\theta_1\left(\sqrt{\sin^2\theta_0-\sin^2 m}+\sin\theta_0\right)}{\sin m\left(\sqrt{\sin^2\theta_0-\sin^2\theta_1}+\sin\theta_0\right)},\end{aligned}$$

and the lemma is proved, because of $m<\theta_1$, and $r(\theta_1)<0$. □

We can now prove Theorem 2.

Proof of Theorem 2: We can write $a(\theta) = p(\cot^2\theta)$, where $a(\theta)$ is the function in Lemma 7, and

$$p(x) = \frac{(K_1 x + K_3)^{1/2} (K_2 x + K_3)^{3/2}}{(x+1)^{3/2}}.$$

We have

$$p'(x) = \frac{(K_2 x + K_3)^{\frac{1}{2}}}{2 (K_1 x + K_3)^{1/2} (x+1)^{5/2}} q(x),$$

where

$$q(x) = K_1 K_2 x^2 + 2K_1(2K_2 - K_3)x + K_3(K_1 - 3(K_3 - K_2)).$$

Clearly, since we suppose $K_1 - 3(K_3 - K_2) < 0$, $q(x)$ has exactly one positive zero, and so it is easy to see that $a'(\theta)$ is in the same conditions of the function $r(\theta)$ in Lemma 8, and we get the theorem. □

Let us suppose now that $K_1 - 3(K_3 - K_2) \geq 0$; then, if the quadratic polynomial $q(x)$ above is positive for $x > 0$, $Z(m)$ is strictly decreasing from $+\infty$ to $d_0 = \pi K_3 / 2\tau K_2$, and we have exactly one non trivial solution of the problem (1)–(4) if and only if d is greater than the threshold d_0. If $q(x)$ has two positive zeros, $Z(m)$ can still be strictly increasing, or not. If $Z(m)$ is not strictly increasing, then $Z'(m) = 0$ in exactly two points, as we shall prove in Theorem 3.

For future reference, we observe that $q(x)$ has two positive zeroes if:

$$K_1 > 3K_2, \quad \text{and} \quad \frac{4K_1K_2}{K_1 + 3K_2} < K_3 < \frac{K_1}{3} + K_2. \tag{21}$$

We need now a lemma about the derivatives of $Z(m)$.

Lemma 9. *Let $a(\theta)$ be as in Lemma 7, and set $b(\theta) = a'(\theta)/\sin\theta$, $c(\theta) = b'(\theta)/\sin\theta$; moreover, let us denote by $A(m)$, $B(m)$ and $C(m)$ the integrals:*

$$\int_m^{\frac{\pi}{2}} \frac{a'(\theta)\cos\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta, \int_m^{\frac{\pi}{2}} \frac{b'(\theta)}{\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta, \int_m^{\frac{\pi}{2}} \frac{c'(\theta)\cos\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta,$$

respectively. Then, we have:

$$A'(m) = \frac{\sin(2m)}{2} B(m), \quad B'(m) = (\tan m) C(m).$$

Proof: We can write, by integration by parts:

$$A(m) = \int_m^{\frac{\pi}{2}} \frac{b(\theta)\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta = -\int_m^{\frac{\pi}{2}} b'(\theta)\sqrt{\sin^2\theta - \sin^2 m}\, d\theta,$$

and we can calculate $A'(m)$ immediately. In a similar way we have:

$$B(m) = \int_m^{\frac{\pi}{2}} \frac{c(\theta)\sin\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\, d\theta$$
$$= \frac{\pi}{2} c(m) + \int_m^{\frac{\pi}{2}} c'(\theta) \arctan\left(\frac{\cos\theta}{\sqrt{\sin^2\theta - \sin^2 m}}\right) d\theta,$$

and we get $B'(m) = (\tan m)C(m)$. □

We can now give the proof of Theorem 3.

Proof of Theorem 3: Let us suppose that $Z(m)$ is not strictly decreasing, so (21) holds. Since $a'(\pi/2) = 0$, and $a''(\pi/2) = K_3(K_1 - 3(K_3 - K_2)) > 0$, from Lemma 7, $Z(m)$ is strictly decreasing on a left neighborhood of $\pi/2$, and, because of (6), it has a local minimum at m_1 and a local maximum at $m_2 > m_1$, with $Z(m_2) > d_0$. We claim that $Z(m)$ is strictly decreasing for $0 < m < m_1$; for if not, again because of (6), $Z'(m)$ should have at least two other zeros for $0 < m < m_1$, and so, by Lemma 7 and Lemma 9, the function $C(m)$ should have at least two zeros for $0 < m < \pi/2$. On the other hand, we can calculate explicitly $c''(\theta)$, and we obtain:

$$c''(\theta) = \frac{K_2^2\, p(\cot^2\theta)}{\sqrt{1+\cot^2\theta}\,\left((k + h\cot^2\theta)(k+\cot^2\theta)\right)^{7/2}}, \tag{22}$$

where we have set $K_1 = hK_2$ and $K_3 = kK_2$, and $p(x) = \sum_{i=0}^{11} c_i x^i$ is a suitable polynomial. Looking at the coefficients of $p(x)$, which are listed in the Appendix, we see that $c_2 \cdots c_{11}$ are positive, while $c_0 < 0$. Then $p(x)$ is a convex function for $x > 0$ with $p(0) < 0$, and has only one zero for $x > 0$. This implies that $c'(\theta)$ satisfies the same assumptions as the function $r(\theta)$ in Lemma 8, and so $C(m)$ has only one zero. This is a contradiction, and the theorem is proved. □

We prove now Theorem 4 adapting the Proposition 4.1.3 of [7] to our context.

Proof of Theorem 4: We set

$$H(x) = \frac{K_2^2\tau^2}{g(x)}\cos^2 x,$$

then we can rewrite (20) as:

$$\begin{cases} 2f(\theta)\theta'' + f'(\theta)\theta'^2 + H'(\theta) = 0 \\ \theta(0) = m \qquad \theta'(0) = 0 \end{cases} \tag{23}$$

and we denote by $\theta_m(z)$ its solution. Calculating the derivative with respect to m of the identity $2f(\theta_m)\theta_m'' + f'(\theta_m)\theta_m'^2 + H'(\theta_m) = 0$ we have:

$$(f(\theta_m)w')' + \frac{1}{2}\left(2f'(\theta_m)\theta_m'' + f''(\theta_m)\theta_m'^2 + H''(\theta_m)\right) w = 0,$$

where we have set $w(z) = \partial_m(\theta_m(z))$, and the prime $'$ is the derivative with respect to z. Clearly we have $w(0) = \partial_m(\theta_m(0)) = \partial_m m = 1$, and since $\theta_m(Z(m)) = \pi/2$, by derivation with respect to m we get

$$w(Z(m)) = -\theta_m'(Z(m))Z'(m). \tag{24}$$

We set now

$$\begin{aligned} p(z) &= f(\theta_m(z)) \\ q(z) &= \tfrac{1}{2}\left(2f'(\theta_m(z))\theta_m''(z) + f''(\theta_m(z))\theta_m'^2(z) + H''(\theta_m(z))\right) \end{aligned}$$

and consider the eigenvalues sequence $(\mu_i)_i$ of the Sturm-Liouville problem:

$$\begin{cases} (p(z)v'(z))' + (q(z)+\mu)v(z) = 0 \\ v(-Z(m)) = v(Z(m)) = 0. \end{cases}$$

In order to prove the first part of the theorem, we suppose that $Z'(m) > 0$, and we must show that $\mu_1 < 0$. In fact, from (24), we have $w(Z(m)) < 0$ because of $\theta_m'(Z(m)) > 0$. Then there exists z_0 such that $w(z) > 0$ for $-z_0 < z < z_0$, and $w(\pm z_0) = 0$. Let $v_1(z)$ be the (positive) eigenfunction for μ_1; since $(p(z)w')' + q(z)w = 0$, multiplying by v_1 and integrating on $[-z_0, z_0]$, we get

$$2v_1(z_0)p(z_0)w'(z_0) + \int_{-z_0}^{z_0} (p(z)v_1'(z))'w(z)\,dz + \int_{-z_0}^{z_0} q(z)w(z)v_1(z)\,dz = 0.$$

On the other hand, we have $(p(z)v_1'(z))' + (q(z)+\mu_1)v_1(z) = 0$, so that

$$2v_1(z_0)p(z_0)w'(z_0) - \mu_1 \int_{-z_0}^{z_0} v_1(z)w(z)\,dz = 0.$$

Since $p(z_0) > 0$, $v_1(z_0) > 0$ and $w'(z_0) < 0$, we have $\mu_1 < 0$, and the first result is achieved.

Suppose now that $Z'(m) < 0$, and let $w(z)$ be as above. Then from (24) we have $w(Z(m)) > 0$. We claim that $w(z) > 0$ for $-Z(m) < z < Z(m)$. In fact, from (23) follows $f(\theta_m)\theta_m'^2 + H(\theta_m) =$ constant, so

$$f(\theta_m)\theta_m'^2 + H(\theta_m) = H(m).$$

Computing the derivative with respect to m, we have:

$$f'(\theta_m)\theta_m'^2 w + 2f(\theta_m)\theta_m' w' + H'(\theta_m)w = H'(m).$$

If the claim is not true, there exists $0 < z_0 < Z(m)$ such that $w(z) > 0$ for $z_0 < z < Z(m)$, and $w(z_0) = 0$, $w'(z_0) \geq 0$. Then

$$2f(\theta_m(z_0))\theta_m'(z_0)w'(z_0) = H'(m) < 0,$$

so that $\theta_m'(z_0) < 0$; but this is impossible since θ_m is strictly increasing from m to $\pi/2$ on $[0, Z(m)]$, and the claim is proved. Now, multiplying the equation

$(p(z)w')' + q(z)w = 0$ by v_1, and integrating over $[-Z(m), Z(m)]$, we get:

$$\mu_1 \int_{-Z(m)}^{Z(m)} v_1 w\, dz = -2p(Z(m))v_1'(Z(m))w(Z(m)) > 0,$$

and since $w(z) > 0$, we have $\mu_1 > 0$, and the proof of the theorem is completed. □

4. Appendix

In the proof of Theorem 3 we have used the polynomial $p(x) = \sum_{i=0}^{11} c_i x^i$, and we have claimed that the coefficients c_2–c_{11} are positive, while $c_0 < 0$. In fact, the explicit calculation of the coefficients (which are easy to get by using a software for algebraic manipulations) show that c_6–c_{11} are positive for all values of $h > 0$ and $k > 0$:

$$\begin{aligned}
c_{11} &= 90h^4;\\
c_{10} &= 3h^3(95h + 103k + 99hk);\\
c_9 &= 3h^2(124k^2 + hk(327 + 341k) + h^2(107 + 316k + 111k^2));\\
c_8 &= 3hk^2(396h + 56k + 413hk) + 3h^3k(370 + 1089k + 385k^2)\\
&\quad + 3h^4(48 + 361k + 359k^2 + 42k^3);\\
c_7 &= 12hk^3(46 + 49k) + 3h^2k^2(452 + 1317k + 469k^2) + 6h^4(4 + 81k + 211k^2 + 69k^3)\\
&\quad + 3h^3k(166 + 1250k + 1245k^2 + 147k^3);\\
c_6 &= 3h^4k(28 + 189k + 173k^2) + 3hk^3(219 + 644k + 245k^2)\\
&\quad + 3h^2k^2(203 + 1513k + 1491k^2 + 175k^3) + 3h^3k(28 + 560k + 1483k^2 + 483k^3).
\end{aligned}$$

Moreover, since we have set $K_1 = hK_2$ and $K_3 = kK_2$, from (21) we have

$$h > 3, \quad \text{and} \quad \frac{4h}{h+3} < k < \frac{h}{3} + 1, \tag{25}$$

so $2h - 3k > 0$, $h - 3k + 3 > 0$, $k > 2$, and we get that, if (21) holds, then c_3–c_5 are positive, in fact:

$$\begin{aligned}
c_5 &= 3h^4k^2(35 + 72k) + 21h^2k^4(h^2 - k^2) + 3h^3k^2(98 + 673k + 637k^2)\\
&\quad + 3hk^3(103 + 777k + 805k^2 + 119k^3) + 3h^2k^2(35 + 664k + 1757k^2 + 525k^3);\\
c_4 &= 63h^2k^5(2h - 3k) + h^2k^5(h - 3k + 3)(h + 3k + 2) + (k - 2)k^4(5h^4 + 3h^2k^2 + 3h^2k + 3)\\
&\quad + h^4k^3(45 + k) + h^3k^3(390 + 873k + 10k^2) + 3h^2k^3(115 + 755k + 659k^2)\\
&\quad + 6k^5 + hk^3(60 + 1077k + 3033k^2 + 1113k^3 + 56k^4) + k^7(h - 3);\\
c_3 &= 27k^4(h^3 - k^3) + h^3k^4(3k^2 + 10k + 183) + 18h^2k^6(2h - 3k) + 53h^2k^5(h - 3k + 3)\\
&\quad + h^2k^4(576k + 420) + 3hk^4(5k^4 + 31k^3 + 501k^2 + 469k + 70) + 11k^5(k - 2)\\
&\quad + k^5(3k^3 + 34k + 1).
\end{aligned}$$

The coefficient c_2 is the following:

$$\begin{aligned}
c_2 &= 11^{-1}(3 + h)^{-1}(4hk^5(15831 + 3567h - 1089h^2 + 176h^3) + 3141hk^5((3 + h)k - 4h))\\
&\quad + 288hk^5(3 + h - 3k) + 11^{-1}hk^5(117 - 11h)^2(k - 2) + 3(h - 2)k^6 + 16h^2k^6(3 + h - 3k)\\
&\quad + 6k^7(2h - 3k) + 3k^7(3 + h)^{-1}(5h^2(10 + k) + 6(3 + k^2) + h(156 + 15k + 2k^2)),
\end{aligned}$$

and it is positive because (21) implies $k(h+3) > 4h$ and moreover it is easy to check that $15831 + 3567h - 1089h^2 + 176h^3 > 0$ for $h > 0$.

Finally, the coefficients c_1 and c_0 are:

$$c_1 = 3k^6(4h^3 - h^2(13k+5) + h(6k^2 + 8k + 54) + 3(1-k)^2(k+1));$$

$$c_0 = -3k^7((h-3)(h-k+1) + 2h(k-2) + k(h-3k+3)),$$

and clearly $c_0 < 0$, as claimed.

References

[1] Bedford S., *Global minimisers of cholesteric liquid crystal systems*, arXiv:1411.3599v2, 2014.

[2] Bedford S., *Analysis of local minima for constrained minimization problems*, arXiv:1411.3595, 2014.

[3] Byrd P.F., Friedman M.D., *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer, New York, 1971.

[4] Hardt R., Kinderlehrer D., Lin F.H., *Existence and partial regularity of static liquid crystal configurations*, Commun. Math. Phys. **105** (1986), 547–570.

[5] Leslie F.M., *Distorted twisted orientation patterns in nematic liquid crystals*, Pramana Suppl. **1** (1975), 41–55.

[6] Leslie F.M., *Some topics in equilibrium theory of liquid crystals*, in Theory and Applications of Liquid Crystals, J. Ericksen and D. Kinderlehrer, Eds., Springer, New York, 1987, pp. 211–234.

[7] Schaaf R., *Global Solution Branches of Two-point Boundary Value Problems*, Lecture Notes in Mathematics, 1458, Springer, New York, 1990.

[8] Stewart I.W., *The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction*, Taylor & Francis, London and New York, 2004.

[9] Zel'dovic B.Ya., Tabiryan N.V., *Freedericksz transition in cholesteric liquid crystals without external fields*, JETP Lett. **34** (1981), no. 8, 406–408.

[10] Zel'dovic B.Ya., Tabiryan N.V., *Equilibrium structure of a cholesteric with homeotropic orientation on the walls*, Sov. Phys. JETP **56** (1982), no. 3, 563–566.

Politecnico di Bari, v. Orabona, 4, 70126 Bari, Italy

E-mail: carlo.greco@poliba.it

(*Received* July 12, 2016)