Parabolicity and rigidity of spacelike hypersurfaces immersed in a Lorentzian Killing warped product

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Abstract. In this paper, we extend a technique due to Romero et al. establishing sufficient conditions to guarantee the parabolicity of complete spacelike hypersurfaces immersed into a Lorentzian Killing warped product whose Riemannian base has parabolic universal Riemannian covering. As applications, we obtain rigidity results concerning these hypersurfaces. A particular study of entire Killing graphs is also made.

Keywords: Lorentzian Killing warped product; complete spacelike hypersurfaces; parabolic spacelike hypersurfaces; entire Killing graphs

Classification: Primary 53C42; Secondary 53B30, 53C50

1. Introduction

The study of spacelike hypersurfaces immersed with constant mean curvature in a Lorentzian manifold has attracted the interest of a considerable group of geometers as evidenced by the amount of works that it has generated in the last decades. This is due not only to its mathematical interest but also to its relevance in General Relativity. For example, constant mean curvature spacelike hypersurfaces are particularly suitable for studying the propagation of gravitational radiation. See, for instance, [17], [23] for a summary of several reasons justifying this interest.

From the mathematical point of view, the study of the geometry of constant mean curvature spacelike hypersurfaces is mostly due to the fact that they exhibit nice Bernstein type properties, that is, geometric properties which guarantee the uniqueness of these hypersurfaces. In this context, several authors more recently have treated the problem of uniqueness for complete constant mean curvature spacelike hypersurfaces of the so-called generalized Robertson-Walker (GRW) spacetimes, that is, Lorentzian warped products with 1-dimensional negative definite base and Riemannian fiber. Along this branch, we point out the works of Romero, Rubio and Salamanca [20], [21], [22], where the authors studied complete spacelike hypersurfaces in GRW spacetimes whose Riemannian fiber has a parabolic universal Riemannian covering. We recall that a complete Riemannian

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manifold is said to be parabolic if it admits no nonconstant positive superharmonic function (see, for instance, [11] for a broad treatment). In this context, they obtained a new criterium to guarantee the parabolicity of complete spacelike hypersurfaces and, as application, they obtained several uniqueness results on complete maximal spacelike hypersurfaces. We also observe that, when the ambient space is a Lorentzian product space, Albujer and Alías [3], [4] obtained another very interesting rigidity results for complete maximal spacelike surfaces via the study of their parabolicity.

Here, our purpose is to study the rigidity of complete spacelike hypersurfaces immersed into a Lorentzian Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$, which is a particular model of Lorentzian manifold endowed with a globally defined timelike Killing vector field (for more details, see Section 2). In this setting, supposing that the Riemannian base M^n of $\overline{M}^{n+1}$ has parabolic universal covering, in Section 3 we extend the previously commented parabolicity criterium due to Romero et al. [20] (cf. Theorem 1 and Corollaries 1 and 2). Afterwards, under suitable constraints on the Ricci curvature of the ambient space, in Section 4 we apply our parabolicity criterion in order to establish rigidity results concerning complete spacelike hypersurfaces immersed into $\overline{M}^{n+1}$ (cf. Theorems 2 and 3 and Corollaries 3, 4 and 5). Furthermore, in Section 5 we make a particular study of entire Killing graphs constructed over the base M^n of $\overline{M}^{n+1}$ (cf. Theorems 4 and 5 and Corollaries 6 and 7).

Taking into account the characterization of Lorentzian Killing warped products given in Lemma 3.78 of [6], we observe that our results can be regarded as "dimensional dual" of those obtained by Romero, Rubio and Salamanca [20], [21], [22] in the context of the GRW spacetimes (see Remark 1). Moreover, we also note that constraints on the Ricci curvature of the ambient space are quite natural from the point of view of Physics. Indeed, we have in the current literature the following convergence conditions:

A Lorentzian manifold satisfies the *ubiquitous energy condition* if its Ricci tensor

$$\mathrm{Ric}(Z,Z) > 0,$$

for all timelike vector Z. This energy condition means the presence of matter (positive) at any point of the spacetime. A Lorentzian manifold obeys the *timelike convergence condition* if its Ricci tensor Ric satisfies

$$\mathrm{Ric}(Z,Z) \geq 0,$$

for all timelike vector Z. The physical meaning of the timelike convergence condition is that gravity, on average, attracts. A weaker energy condition is the *null convergence condition* which reads as

$$\mathrm{Ric}(Z,Z) \geq 0,$$

for any null vector Z. For more details concerning these convergence conditions see [8], [18].

Finally, we point out that the parabolicity is also important from the mathematical point of view since we have non-trivial examples of constant mean curvature spacelike graphs over $\mathbb{H}^2$, see [16] for an exposition regarding the maximality and completeness of these graphs.

2. Lorentzian Killing warped products

Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Lorentzian manifold endowed with a timelike Killing vector field K. Suppose that the distribution $\mathcal{D}$ orthogonal to K is of rank constant and integrable. We denote by $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ the flow generated by K, where M^n is an arbitrarily fixed spacelike integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected, and $\mathbb{I}$ is the maximal interval of definition. Without loss of generality, in what follows we will also consider $\mathbb{I} = \mathbb{R}$.

In this setting, $\overline{M}^{n+1}$ can be regarded as the *Lorentzian Killing warped product* $M^n \times_\rho \mathbb{R}_1$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle\ ,\ \rangle = \pi_M^*\left(\langle\ ,\ \rangle_M\right) - (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^*\left(dt^2\right), \tag{2.1}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M \times \mathbb{R}$ onto each factor, $\langle\ ,\ \rangle_M$ is the induced Riemannian metric on the Riemannian base M^n, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and the warping function $\rho \in C^\infty(M)$ is given by $\rho = |K| = \sqrt{-\langle K, K\rangle}$.

Let us consider a connected spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed into a Lorentzian Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$, which means that the metric induced on Σ^n via ψ is a Riemannian metric. Since K is a globally defined timelike vector field on $\overline{M}^{n+1}$, it follows that there exists a unique unitary timelike normal vector field N globally defined on Σ^n which is in the same time-orientation as K, that is, the angle function $\Theta = \langle N, K\rangle < 0$ on Σ^n. We will refer to that normal vector field N as the *future-pointing Gauss map* of the spacelike hypersurface Σ^n. Throughout this work, N will always denote the future-pointing Gauss map of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\overline{M}^{n+1}$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N \tag{2.2}$$

and

$$AX = -\overline{\nabla}_X N, \tag{2.3}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to its future-pointing Gauss map N.

For our purposes, we will also consider the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$. From the decomposition $K = K^{\top} - \Theta N$, where $(\ )^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n, we obtain

$$\nabla h = -\frac{1}{\rho^2} K^{\top} \quad \text{and} \quad |\nabla h|^2 = \frac{\Theta^2 - \rho^2}{\rho^4}. \tag{2.4}$$

3. A parabolicity criterium for spacelike hypersurfaces in $M^n \times_\rho \mathbb{R}_1$

We begin this section recalling that every connected manifold Σ has a universal covering, that is, there exists a simply connected manifold $\widetilde{\Sigma}$ (called a universal covering of Σ) and a smooth map $\kappa : \widetilde{\Sigma} \to \Sigma$ (called a covering map) such that each point $p \in \Sigma$ has a connected neighborhood U that is evenly covered by κ, that is, κ maps each component of $\kappa^{-1}(U)$ diffeomorphically onto U (for more details, see Appendix A of [19]). Moreover, if Σ is a Riemannian manifold, then it is possible to give $\widetilde{\Sigma}$ a Riemannian structure such that the covering map $\kappa : \widetilde{\Sigma} \to \Sigma$ is a local isometry. In this case, $\widetilde{\Sigma}$ is said to be a universal Riemannian covering of Σ (see p. 152 of [10]).

In [20], Romero, Rubio and Salamanca investigated the parabolicity of complete spacelike hypersurfaces in GRW spacetimes whose Riemannian fiber has a parabolic universal Riemannian covering. In this setting, they were able to guarantee the parabolicity of complete spacelike hypersurfaces, under suitable boundedness assumptions on the warping function and on the hyperbolic angle function of these hypersurfaces. Our aim in this section is just to obtain an extension of this parabolicity criterium to the context of Lorentzian Killing warped products. More precisely, we get the following

Theorem 1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n has parabolic universal Riemannian covering. If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a complete spacelike hypersurface such that the function $\eta := \frac{\Theta}{\rho}$ is bounded on it, then M^n is complete and Σ^n is parabolic.*

PROOF: The proof is based on two facts:

(i) parabolicity is invariant under a quasi-isometry (cf. [11], [13]);

(ii) if the universal Riemannian covering $\widetilde{\Sigma}$ of Σ^n is parabolic, then Σ^n is also parabolic.

Denoting $\pi = \pi_M \circ \psi : \Sigma^n \to M^n$, for any tangent vector $v \in T\Sigma$ we have

$$\langle v, v \rangle = \langle \pi_* v, \pi_* v \rangle_M - \rho^2 \langle h_* v, h_* v \rangle_{\mathbb{R}} \leq c \langle \pi_* v, \pi_* v \rangle_M,$$

where $c = \sup_\Sigma \eta^2 \geq 1$. In particular, by previous inequality we see that $\pi_{*,p} : T_p\Sigma \to T_{\pi(p)}M$ is an isomorphism for every $p \in \Sigma^n$. Then, from inverse function theorem we get that π is a local diffeomorphism and applying Lemma 7.3.3 of [10] (see also Lemma 8.8.1 of [14]) we can conclude that π is a covering map and that M^n is complete.

On the other hand, using the Cauchy-Schwartz inequality we see that

$$\langle \nabla h, v\rangle^2 \leq \langle \nabla h, \nabla h\rangle \langle v, v\rangle$$

and, consequently, since $h_* v = dh(v) = \langle \nabla h, v\rangle$, we have

$$\begin{aligned} \langle v, v\rangle &= \langle \pi_* v, \pi_* v\rangle_M - \rho^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \\ &= \langle \pi_* v, \pi_* v\rangle_M - \rho^2 \langle \nabla h, v\rangle^2 \\ &\geq \langle \pi_* v, \pi_* v\rangle_M - \rho^2 |\nabla h|^2 \langle v, v\rangle, \end{aligned}$$

that is,

$$\langle v, v\rangle \left(1 + \rho^2 |\nabla h|^2\right) \geq \langle \pi_* v, \pi_* v\rangle_M.$$

By definition of the function η and from (2.4) we get

$$\langle v, v\rangle \geq \frac{1}{\eta^2} \langle \pi_* v, \pi_* v\rangle_M.$$

From our hypothesis we conclude that

$$c^{-1} \langle \pi_* v, \pi_* v\rangle_M \leq \langle v, v\rangle \leq c \langle \pi_* v, \pi_* v\rangle_M. \tag{3.1}$$

So, let $\widetilde{\Sigma}$ be the universal Riemannian covering of Σ^n with projection $\pi_\Sigma : \widetilde{\Sigma} \to \Sigma^n$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \widetilde{\Sigma} \to M^n$ is a covering map. Now, if $\widetilde{M}$ is the universal Riemannian covering of M^n with projection $\widetilde{\pi} : \widetilde{M} \to M^n$, then there exists a diffeomorphism $\varphi : \widetilde{\Sigma} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \varphi = \pi_0$. Moreover, φ is a quasi-isometry. Indeed, if $v \in T\widetilde{\Sigma}$, we have from (3.1) that

$$\begin{aligned} \langle \varphi_* v, \varphi_* v\rangle_{\widetilde{M}} &= \langle \widetilde{\pi}_*(\varphi_* v), \widetilde{\pi}_*(\varphi_* v)\rangle_M \\ &= \langle (\pi_0)_* v, (\pi_0)_* v\rangle_M \\ &= \langle \pi_*((\pi_\Sigma)_* v), \pi_*((\pi_\Sigma)_* v)\rangle_M \\ &\leq c \langle (\pi_\Sigma)_* v, (\pi_\Sigma)_* v\rangle_\Sigma \\ &= c \langle v, v\rangle_{\widetilde{\Sigma}}. \end{aligned}$$

Analogously, we obtain $\langle \varphi_* v, \varphi_* v\rangle_{\widetilde{M}} \geq c^{-1} \langle v, v\rangle_{\widetilde{\Sigma}}$. Therefore, since the universal Riemannian covering of M^n is parabolic, it follows that the universal Riemannian covering of Σ^n is parabolic and, hence, Σ^n must be also parabolic. □

As a direct consequence of Theorem 1, we have the following corollaries.

Corollary 1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n is complete, simply connected and parabolic. If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a complete spacelike hypersurface such that η is bounded on it, then Σ^n is parabolic.*

Corollary 2. *Let $\overline{M}^{n+1} = M^n \times \mathbb{R}_1$ be a Lorentzian product whose Riemannian base M^n is complete and has parabolic universal Riemannian covering. If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a complete spacelike hypersurface whose angle function Θ is bounded, then Σ^n is parabolic.*

4. Rigidity results for spacelike hypersurfaces in $M^n \times_\rho \mathbb{R}_1$

In this section, we will apply Theorem 1 in order to obtain rigidity results for complete spacelike hypersurfaces in $M^n \times_\rho \mathbb{R}_1$. At this point, we will assume that these hypersurfaces are connected. Now, we present our first rigidity theorem.

Theorem 2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product with nonnegative Ricci curvature and whose Riemannian base M^n has parabolic universal Riemannian covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface with constant mean curvature such that its angle function Θ is bounded and the warping function ρ satisfies $\inf_\Sigma \rho > 0$. Then, Σ^n is totally geodesic. In addition, if Ric_M is positive at some point $p_0 \in \Sigma^n$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof: From Proposition 1 of [2] (see also Proposition 3.1 of [5]) we have the following formula

$$\Delta\Theta = \left(\overline{\mathrm{Ric}}(N,N) + |A|^2\right)\Theta, \tag{4.1}$$

where $|A|$ stands for the Hilbert-Schmidt norm of A. It follows that Θ is a bounded superharmonic function on Σ^n. From Theorem 1, Σ^n is parabolic and, thus Θ is constant on it. So, returning to (4.1) we obtain $|A|^2 = 0$, that is, Σ^n is totally geodesic. Now, we assert that ρ is constant. In fact, for any $X \in T\Sigma$, we can write

$$X = X^* - \frac{\langle X, K\rangle}{\rho^2} K,$$

where X^* denotes the orthogonal projection of X onto TM. Since Σ^n is totally geodesic, from Proposition 7.35 of [19], we have that

$$\begin{aligned} X(\Theta) &= \langle N, \overline{\nabla}_X K\rangle \\ &= \langle N, \overline{\nabla}_{X^*} K\rangle - \frac{\langle X, K\rangle}{\rho^2}\langle N, \overline{\nabla}_K K\rangle \\ &= \frac{1}{\rho}\langle X, \overline{\nabla}\rho\rangle\langle N, K\rangle - \frac{1}{\rho}\langle X, K\rangle\langle N, \overline{\nabla}\rho\rangle. \end{aligned}$$

Thus, from the least equation, we conclude that

$$\nabla\Theta = \frac{1}{\rho}\left(\Theta\overline{\nabla}\rho - \langle N, \overline{\nabla}\rho\rangle K\right) \in TM \oplus T\mathbb{R}_1.$$

Since Θ is constant, it follows that ρ is constant. In particular, $\overline{\mathrm{Ric}}(N,N) = \mathrm{Ric}_M(N^*, N^*)$, where N^* denotes the orthogonal projection of N onto TM. So, if Ric_M is positive at some point $p_0 \in \Sigma^n$, it follows again from (4.1) that

$\mathrm{Ric}_M(N^*, N^*)(p_0) = 0$ and, consequently, $N^*(p_0) = 0$. But, it is not difficult to see that

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_M = \frac{1}{\rho^2}\left(\frac{\Theta^2}{\rho^2} - 1\right), \tag{4.2}$$

which means that Σ^n is a slice of $\overline{M}^{n+1}$. □

Taking into account a classical result due to Ahlfors [1] and Blanc-Fiala-Huber [12] which asserts that a complete Riemannian surface of nonnegative Gaussian curvature must be parabolic, it is not difficult to see that we can reason as in the proof of Theorem 2 to get the following

Corollary 3. *Let $\overline{M}^3 = M^2 \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product with nonnegative Ricci curvature whose Riemannian base M^2 has nonnegative Gaussian curvature K_M. Let $\psi : \Sigma^2 \to \overline{M}^3$ be a complete spacelike surface with constant mean curvature such that its angle function Θ is bounded and the warping function ρ satisfies $\inf_\Sigma \rho > 0$. Then, Σ^2 is totally geodesic. In addition, if K_M is positive at some point $p_0 \in \Sigma^2$, then Σ^2 is a slice of $\overline{M}^3$.*

In our next result, the mean curvature H of the spacelike hypersurface is not supposed to be constant. In fact, we just assume that H does not change the sign.

Theorem 3. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n has parabolic universal Riemannian covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface such that η and h are bounded on it. If the mean curvature H and the function $\langle \nabla\rho, \nabla h\rangle$ have opposite signs, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof: Denoting by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $\overline{M}^{n+1}$ and Σ^n, respectively, we have from the decomposition $K = K^\top - \Theta N$ that

$$\overline{\nabla}_X K = \overline{\nabla}_X K^\top - X(\Theta)N - \Theta\overline{\nabla}_X N, \tag{4.3}$$

for every vector field X tangent to Σ^n. Then, using (2.2) and (2.3) into (4.3), we get

$$\nabla_X K^\top = (\overline{\nabla}_X K)^\top - \Theta AX. \tag{4.4}$$

Consequently, from (2.4) and (4.4) we obtain

$$\begin{aligned}\nabla_X \nabla h &= X\left(-\frac{1}{\rho^2}\right)K^\top - \frac{1}{\rho^2}\nabla_X K^\top \\ &= \frac{2X(\rho)}{\rho^3}K^\top - \frac{1}{\rho^2}(\overline{\nabla}_X K)^\top + \frac{1}{\rho^2}\Theta AX \\ &= \frac{2\langle\nabla\rho, X\rangle}{\rho^3}K^\top - \frac{1}{\rho^2}(\overline{\nabla}_X K)^\top + \frac{1}{\rho^2}\Theta AX.\end{aligned} \tag{4.5}$$

Since K is a Killing vector field, taking a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(\Sigma)$, from (4.5) we get

$$
\begin{aligned}
\Delta h &= \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla h, E_i \rangle \\
&= \sum_{i=1}^{n} \frac{2}{\rho^3} \langle \nabla \rho, E_i \rangle \langle K^\top, E_i \rangle - \sum_{i=1}^{n} \frac{1}{\rho^2} \langle (\overline{\nabla}_{E_i} K)^\top, E_i \rangle \\
&\quad + \sum_{i=1}^{n} \frac{1}{\rho^2} \Theta \langle A E_i, E_i \rangle \\
&= \frac{2}{\rho^3} \langle \nabla \rho, K^\top \rangle - \sum_{i=1}^{n} \frac{1}{\rho^2} \langle \overline{\nabla}_{E_i} K, E_i \rangle - \frac{nH\Theta}{\rho^2} \\
&= -\frac{2}{\rho} \langle \nabla \rho, \nabla h \rangle - \frac{nH\Theta}{\rho^2}\,.
\end{aligned}
\tag{4.6}
$$

Hence, taking into account our hypothesis on H and $\langle \nabla \rho, \nabla h \rangle$, from (4.6) we conclude that Δh does not change the sign. Therefore, since Theorem 1 guarantees the parabolicity of Σ^n, h must be constant and, consequently, Σ^n is a slice of $\overline{M}^{n+1}$. □

Using once more [1], [12], from Theorem 3 we get

Corollary 4. *Let $\overline{M}^3 = M^2 \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^2 has nonnegative Gaussian curvature. Let $\psi : \Sigma^2 \to \overline{M}^3$ be a complete spacelike surface such that η and h are bounded on it. If the mean curvature H and the function $\langle \nabla \rho, \nabla h \rangle$ have opposite signs, then Σ^2 is a slice of $\overline{M}^3$.*

We recall that a spacelike hypersurface Σ^n is said *maximal* if its mean curvature vanishes identically on it. In this setting, from Theorem 3 we also obtain the following

Corollary 5. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n has parabolic universal Riemannian covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete maximal spacelike hypersurface such that η and h are bounded on it. If the function $\langle \nabla \rho, \nabla h \rangle$ does not change sign, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Remark 1. In each of the exact solutions to Einstein's equations which are presented as warped product manifolds, the warped product decomposition emerges as a natural mathematical expression of assumed physical symmetries. Moreover, formulas for warped product curvatures (see Chapter 7 of [19]) indicate that any semi-Riemannian manifold $(\overline{N}, g)$ must possess certain measures of symmetry and flatness in order to be (locally or globally) isometric to a warped $B \times_f F$. In this context, geometric conditions on a semi-Riemannian manifold $(\overline{N}, g)$ which are necessary and sufficient to ensure that $(\overline{N}, g)$ is locally isometric to a warped product $B \times_f F$ are very important. For instance, Lemma 3.78 of [6] guarantees that, under our constraints, the Lorentzian manifold $\overline{M}^{n+1}$ can be regarded as the Lorentzian Killing warped product. This fact shows the "character dual" of our case in relation to the GRW spacetimes.

5. Entire Killing graphs in $M^n \times_\rho \mathbb{R}_1$

According to [7], we define the *entire Killing graph* $\Sigma(u)$ associated to a smooth function $u \in C^\infty(M)$ as the hypersurface given by

$$\Sigma(u) = \{\Psi(x, u(x)) : x \in M^n\} \subset M^n \times_\rho \mathbb{R}_1.$$

The metric induced on M^n from the Lorentzian metric (2.1) via $\Sigma(u)$ is given by

$$\langle\ ,\ \rangle_u = \langle\ ,\ \rangle_M - \rho^2 du^2. \tag{5.1}$$

Moreover, $\Sigma(u)$ is spacelike if, and only if, $\rho^2|Du|_M^2 < 1$, where Du denotes the gradient of a function u with respect to the metric $\langle\ ,\ \rangle_M$ of M^n. Indeed, if $\Sigma(u)$ is spacelike, then

$$0 < \langle Du, Du\rangle_u = \langle Du, Du\rangle_M - \rho^2\langle Du, Du\rangle_M^2$$

and, hence, we conclude that $\rho^2|Du|_M^2 < 1$. Conversely, if $\rho^2|Du|_M^2 < 1$ and X is a vector field tangent to $\Sigma(u)$, we obtain from Cauchy-Schwarz inequality,

$$\langle X, X\rangle_u = \langle X^*, X^*\rangle_M - \rho^2\langle Du, X^*\rangle_M^2 \geq \langle X^*, X^*\rangle_M(1 - \rho^2|Du|_M^2),$$

where X^* is the orthogonal projection of X onto TM. Thus, $\langle X, X\rangle_u \geq 0$ and $\langle X, X\rangle_u = 0$ if, and only if, $X = 0$.

The function $g : M^n \times \mathbb{R}_1 \to \mathbb{R}$ given by $g(x,t) = u(x) - t$ is such that $\Sigma(u) = \Psi(g^{-1}(0))$. Thus, for each vector field X tangent to $M^n \times_\rho \mathbb{R}_1$, we have

$$X(g) = X^*(g) - \frac{1}{\rho^2}\langle X, \partial_t\rangle\partial_t(g) = \langle \frac{1}{\rho^2}\partial_t + Du, X\rangle.$$

Thus,

$$\overline{\nabla} g = \frac{1}{\rho^2}\partial_t + Du$$

is a normal vector field on $g^{-1}(0)$ and, consequently,

$$N_0 = \Psi_*(\overline{\nabla} g) = \frac{1}{\rho^2}K + \Psi_*(Du)$$

is a normal timelike vector field on $\Sigma(u)$. Since,

$$|N_0| = \frac{(1 - \rho^2|Du|_M^2)^{1/2}}{\rho},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\rho(1-\rho^2|Du|_M^2)^{1/2}}(K + \rho^2\Psi_*(Du))$$

defines the future-pointing Gauss map of $\Sigma(u)$ such that its angle function is given by

$$\Theta = \langle N, K\rangle = -\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}} < 0. \tag{5.2}$$

Moreover, for each vector field X tangent to M^n, the shape operator A of $\Sigma(u)$ with respect to N is given by

$$\begin{aligned} AX = &-\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}} D_X Du - \frac{\rho^3\langle D_X Du, Du\rangle}{(1-\rho^2|Du|_M^2)^{3/2}} Du \\ &- \frac{\rho^2\langle D\rho, X\rangle|Du|_M^2}{(1-\rho^2|Du|_M^2)^{3/2}} Du - \frac{\langle D\rho, X\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} Du \\ &- \frac{\langle Du, X\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} D\rho, \end{aligned} \tag{5.3}$$

where D denotes the Levi-Civita connections in M^n.

So, it follows from (5.3) that the mean curvature H_u of a spacelike entire Killing graph $\Sigma(u)$ is given by

$$nH_u = \operatorname{Div}\left(\frac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \frac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}},$$

where Div stands for the divergence operator on M^n with respect to the metric $\langle\ ,\ \rangle_M$. In particular, an entire Killing graph $\Sigma(u)$ is maximal if, and only if, the function $u \in C^\infty(M)$ satisfies the following elliptic partial differential equation of divergence form

$$\begin{cases} \operatorname{Div}\left(\dfrac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \dfrac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} = 0, \quad \text{in} \quad M^n \\ \\ \rho^2|Du|_M^2 < 1. \end{cases}$$

We also note that, since

$$N^* = N - N^\perp = \frac{\rho\Psi_*(Du)}{(1-\rho^2|Du|_M^2)^{1/2}},$$

we have that

$$|N^*|_M^2 = \frac{\rho^2|Du|_M^2}{1-\rho^2|Du|_M^2}, \tag{5.4}$$

and, consequently, we get from (4.2) and (5.4) the following relation

$$|\nabla h|^2 = \frac{|Du|_M^2}{1-\rho^2|Du|_M^2}\,. \tag{5.5}$$

In this context, we obtain the following rigidity result.

Theorem 4. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product with nonnegative Ricci curvature whose Riemannian base M^n is complete and has parabolic universal Riemannian covering. Let $\Sigma(u)$ be an entire Killing graph in $\overline{M}^{n+1}$ with constant mean curvature, such that $\rho|_{\Sigma(u)}$ is bounded and $\inf_{\Sigma(u)} \rho > 0$. Suppose that, for some positive constant $\alpha < 1$,*

$$\sup_{\Sigma(u)} \rho^2|Du|_M^2 \leq \alpha. \tag{5.6}$$

Then, $\Sigma(u)$ is totally geodesic. In addition, if Ric_M is positive at some point $p_0 \in \Sigma(u)$, then $\Sigma(u)$ is a slice of $\overline{M}^{n+1}$.

Proof: In order to apply Theorem 2, we must show that Θ is bounded and that $\Sigma(u)$ is complete. For the first fact, from (5.6) we have

$$\Theta \geq -\frac{\rho}{(1-\alpha)^{1/2}} \geq -\frac{\sup_{\Sigma(u)}\rho}{(1-\alpha)^{1/2}}\,,$$

that is, Θ is bounded.

Now, let X be any vector field tangent to $\Sigma(u)$. From the Cauchy-Schwarz inequality we get

$$\langle X,X\rangle_u = \langle X^*,X^*\rangle_M - \rho^2\langle Du,X^*\rangle_M^2 \geq (1-\rho^2|Du|_M^2)\langle X^*,X^*\rangle_M.$$

Hence, again from (5.6), we obtain

$$\langle X,X\rangle_u \geq (1-\alpha)\langle X^*,X^*\rangle_M.$$

This implies that

$$L_u(\gamma) \geq (1-\alpha)^{1/2}L_M(\gamma^*), \tag{5.7}$$

where $L_u(\gamma)$ stands for the length of a curve γ on $\Sigma(u)$ with respect to the induced metric (5.1) and $L_M(\gamma^*)$ denotes the length of the projection γ^* of γ onto M^n with respect to its metric $\langle\ ,\ \rangle_M$.

Here, let us recall that a curve $\alpha : [0,+\infty) \to \Sigma$ on a Riemannian manifold Σ is said to be a divergent curve if for any compact set K there exist $t \in [0,+\infty)$ with $\alpha(t) \notin K$. It is well known that, as a consequence of the Hopf-Rinow Theorem, the following holds: a Riemannian manifold is complete if and only if the length of any divergent curve is unbounded (see p. 153 of [10] or p. 113 of [15]).

Finally, since projections onto M^n of divergent curves on $\Sigma(u)$ give divergent curves on M^n and as we assume that the metric $\langle\ ,\ \rangle_M$ is complete, we conclude by equation (5.7) that the induced metric (5.1) is also complete. Therefore, we are in position to apply Theorem 2 to finish the proof. □

As a consequence of Theorem 4 we obtain the following result.

Corollary 6. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product with nonnegative Ricci curvature whose base M^n is complete and has parabolic universal Riemannian covering and positive Ricci curvature. Suppose that the warping function ρ is bounded on M^n and $\inf_M \rho > 0$. Let $u \in C^\infty(M)$ be a solution to the problem*

$$\begin{cases} \operatorname{Div}\left(\dfrac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \dfrac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} = 0, \quad in \quad M^n \\ \\ \sup_M \rho^2|Du|_M^2 < 1. \end{cases}$$

Then, u is constant on M^n.

Reasoning as in Theorem 4, we can apply Theorem 3 to get the following

Theorem 5. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n is complete and has parabolic universal Riemannian covering. Let $\Sigma(u)$ be an entire Killing graph in $\overline{M}^{n+1}$ of a bounded function $u \in C^\infty(M)$ such that for some positive constant $\alpha < 1$,*

$$\sup_{\Sigma(u)} \rho^2|Du|_M^2 \leq \alpha.$$

If the mean curvature H_u and the function $\langle \nabla\rho, \nabla h\rangle$ have opposite sings, then u is constant on M^n.

Proof: In fact, in this case the function $\eta = -\frac{1}{(1-\rho^2|Du|_M^2)^{1/2}}$ is bounded on $\Sigma(u)$. Therefore, the result follows from Theorem 3. □

We close our paper stating the following consequence of Theorem 5.

Corollary 7. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a Lorentzian Killing warped product whose Riemannian base M^n is complete and has parabolic universal Riemannian covering. Let $u \in C^\infty(M)$ be a bounded solution to the problem*

$$\begin{cases} \operatorname{Div}\left(\dfrac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \dfrac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} = 0, \quad in \quad M^n \\ \\ \sup_M \rho^2|Du|_M^2 < 1. \end{cases}$$

If the height function h of the entire Killing graph associated to u is such that $\langle \nabla\rho, \nabla h\rangle$ does not change sign, then u is constant on M^n.

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