The prime ideals intersection graph of a ring

M.J. Nikmehr, B. Soleymanzadeh

Abstract. Let R be a commutative ring with unity and $U(R)$ be the set of unit elements of R. In this paper, we introduce and investigate some properties of a new kind of graph on the ring R, namely, the prime ideals intersection graph of R, denoted by $G_p(R)$. The $G_p(R)$ is a graph with vertex set $R^* - U(R)$ and two distinct vertices a and b are adjacent if and only if there exists a prime ideal $\mathfrak{p}$ of R such that $a, b \in \mathfrak{p}$. We obtain necessary and sufficient conditions on R such that $G_p(R)$ is disconnected. We find the diameter and girth of $G_p(R)$. We also determine all rings whose prime ideals intersection graph is a star, path, or cycle. At the end of this paper, we study the planarity and outerplanarity of $G_p(R)$.

Keywords: the prime ideals intersection graph of a ring; clique number; planar graph

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1. Introduction

In 1988, Beck in [7] assigned a graph to a commutative ring, namely, the zero-divisor graph of the ring. He studied the interplay between ring-theoretic and graph-theoretic properties. After that, a lot of work was done in this area of research. Several other graph structures were defined on rings. Recently, many researchers have obtained ring-theoretic properties in terms of graph-theoretic properties by a suitable assignment of graph structures on some elements of a ring, for example, the zero-divisor graph, the total graph, and the intersection graph [1], [2], [3], [4], [5], [7], and [12].

The field of graph theory and ring theory both benefit from the study of algebraic concepts using graph-theoretic concepts. Usually, when one assigns a graph to an algebraic structure, some problems in ring theory might be more easily solved. There exist numerous interesting algebraic problems arising from the translation of some graph-theoretic parameters and properties such as diameter, girth, clique number, planarity, and so on. The main goal of this paper is the study of the prime ideals intersection graph of a ring using graph-theoretic concepts.

Throughout this paper, all rings are commutative and have unity. Let R be a ring. By R^*, $\mathrm{U}(R)$, and $\mathrm{Z}(R)$, we mean the set of non-zero elements of R, the set

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of unit elements of R, and the set of zero-divisors of R, respectively. A ring R is said to be *local* if it has a unique maximal ideal. A local ring with maximal ideal m is denoted by (R, m). A ring R is said to be *zero-dimensional* if every prime ideal is a maximal ideal. The ring R is said to be *reduced* if it has no non-zero nilpotent elements. The set of nilpotent elements of R is denoted by $\mathrm{Nil}(R)$. The set of maximal ideals of R is denoted by $\mathrm{Max}(R)$, and the intersection of all maximal ideas of R is called the *Jacobson radical* of R and is denoted by $\mathrm{J}(R)$. The set of minimal prime ideals of R is denoted by $\mathrm{Min}(R)$.

Let G be a graph with the vertex set $V(G)$. The *degree* of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $\deg(v)$. If $\deg(v) = 0$, then v is called an *isolated vertex*. If u and v are two adjacent vertices of G, then we write $u — v$. The maximum degree and the minimum degree of the graph G are $\Delta(G)$ and $\delta(G)$, respectively. The set of vertices adjacent to vertex v of the graph G is called the neighborhood of v and denoted by $\mathrm{N}(v)$. The *complete graph* of order n, denoted by K_n, is a graph with n vertices in which every two distinct vertices are adjacent. The *path graph* and *cycle graph* with n vertices are denoted by P_n and C_n, respectively. By the *null graph*, we mean the graph with no vertices. Recall that a graph G is *connected* if there is a path between every two distinct vertices. A *tree* is a connected graph which does not contain a cycle. A *star* is a tree consisting of one vertex adjacent to all the others. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. A graph G is a *bipartite graph* if $V(G)$ is the union of two disjoint independent sets. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face. For every pair of distinct vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y and if there is no such a path, we define $d(x, y) = \infty$. The *diameter* of G, $\mathrm{diam}(G)$, is the supremum of the set $\{d(x, y) : x \text{ and } y \text{ are distinct vertices of } G\}$. The *girth* of a graph G with a cycle is the length of a shortest cycle and is denoted by $\mathrm{gr}(G)$. A graph with no cycles has infinite girth. A *clique* of G is a complete subgraph of G, and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G.

We know that the behavior of the prime ideals of a commutative ring reflects many properties of a ring. Accordingly, we define a new kind of graph based on the prime ideals of a ring, namely, the *prime ideals intersection graph* of a ring. The *prime ideals intersection graph* of a ring R is denoted by $G_p(R)$. The graph $G_p(R)$ is a graph with vertex set $R^* - \mathrm{U}(R)$ and two distinct vertices a and b are adjacent if and only if there exists a prime ideal $\mathfrak{p}$ of R such that $a, b \in \mathfrak{p}$.

2. Some properties of the prime ideals intersection graph

In this section, all rings R whose prime ideals intersection graph is not connected will be characterized. We prove that if $G_p(R)$ is a connected graph for a ring R, then its diameter is at most 2. We also prove that if the prime ideals intersection graph contains a cycle, then its girth is 3. Furthermore, we characterize all rings whose prime ideals intersection graph is a star, path, or cycle. At the end of this paper, we investigate the planarity and outerplanarity of $G_p(R)$.

In the beginning, we state necessary and sufficient conditions on a ring R such that $G_p(R)$ is disconnected.

Theorem 1. *Let R be a ring. Then $G_p(R)$ is disconnected if and only if $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

PROOF: Let $G_p(R)$ be a disconnected graph. Suppose that $C_1, C_2, \dots, C_k$ are the connected components of $G_p(R)$. Because $G_p(R)$ is disconnected, $k \geqslant 2$. Let a and b be two arbitrary vertices of $G_p(R)$. Without loss of generality, we can consider $a \in C_i$ and $b \in C_j$ for $i \neq j$. In the sequel, we show that $G_p(R)$ is disconnected only in the case that $R \cong F_1 \times F_2$. For this purpose, we consider the following cases.

(1) If $ab \neq 0$, then there exist two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R such that $a \in \mathfrak{p}_1$ and $b \in \mathfrak{p}_2$. If $ab = a$, then $a(1-b) = 0 \in \mathfrak{p}_2$, and we have $a \in \mathfrak{p}_2$ or $(1-b) \in \mathfrak{p}_2$, which is a contradiction. Hence $ab \neq a$. Similarly, $ab \neq b$. Therefore ab is a vertex of $G_p(R)$ different from a, b. Now it is clear that $a — ab — b$ is a path between a and b. This contradicts our assumption that a and b are in different components.

(2) Let $ab = 0$ and R be a non-reduced ring. Then there exists $0 \neq c \in Nil(R)$, and so we have the path $a — c — b$ between a and b, which is a contradiction to the disconnectivity of $G_p(R)$.

(3) Let $ab = 0$ and R be a reduced ring. We consider two cases, $|\operatorname{Min}(R)| \geq 3$ and $|\operatorname{Min}(R)| < 3$.

   (a) Let $|\operatorname{Min}(R)| \geq 3$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ be distinct minimal prime ideals of R. Suppose that $a \in \mathfrak{p}_1$ and $b \in \mathfrak{p}_2$. We have $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. Because if $0 \neq c \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, then we have the path $a — c — b$ between a and b, which is a contradiction. Therefore, $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$, but $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subseteq \mathfrak{p}_3$. So $\mathfrak{p}_1 \subseteq \mathfrak{p}_3$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}_3$, which is a contradiction.

   (b) Let $|\operatorname{Min}(R)| < 3$. If $|\operatorname{Min}(R)| = 1$, then R is an integral domain, which is a contradiction to $ab = 0$. Now let $|\operatorname{Min}(R)| = 2$. So there exist two minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\operatorname{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and $a \in \mathfrak{p}_1$, $b \in \mathfrak{p}_2$. Now we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. We claim that R does not contain any regular elements. Because if c is a regular element of R, then we have the path $a — ac — c — bc — b$, which is a contradiction. So in this case, R has no regular elements and every element of R is a unit or a zero-divisor. Now for every maximal ideal $\mathfrak{m} \in \operatorname{Max}(R)$, we have $\mathfrak{m} \subseteq Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$, and by the Prime Avoidance Theorem [6, Proposition 1.11], we have $\mathfrak{m} \subseteq \mathfrak{p}_1$ or $\mathfrak{m} \subseteq \mathfrak{p}_2$. So, for every maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{m} = \mathfrak{p}_1$ or $\mathfrak{m} = \mathfrak{p}_2$. If the ring R is local, then the graph $G_p(R)$ is connected. Hence R is not local. Let $\mathfrak{m}_1$, $\mathfrak{m}_2$ be two distinct maximal ideals of R. So $\mathfrak{m}_1 = \mathfrak{p}_1$, $\mathfrak{m}_2 = \mathfrak{p}_2$ or $\mathfrak{m}_1 = \mathfrak{p}_2$,

$\mathfrak{m}_2 = \mathfrak{p}_1$. Now the Chinese Remainder Theorem [6, Proposition 1.10] implies that $R \cong \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \cong F_1 \times F_2$.

Conversely, let $R \cong F_1 \times F_2$. Then R is an Artinian ring and hence every element of R is a unit or a zero-divisor. Therefore each vertex of $G_p(R)$ is in $S_1 = \{(a, 0) : 0 \neq a \in F_1\}$ or $S_2 = \{(0, b) : 0 \neq b \in F_2\}$. It is clear that there exists no edge between any vertex of S_1 and any vertex of S_2. Therefore, $G_p(R)$ is disconnected. □

Remark 2. Let R be a commutative ring. It is clear that $G_p(R)$ is the null graph if and only if R is a field.

Remark 3. Let R be a commutative ring. It is obvious that the non-zero elements of every ideal of R form a clique in the graph $G_p(R)$. So the non-zero elements of every maximal ideal of R form a maximal clique in the graph $G_p(R)$.

Now, we characterize the diameter of $G_p(R)$.

Theorem 4. *Let R be a ring. Then* $\operatorname{diam}(G_p(R)) \in \{0, 1, 2, \infty\}$ *and we have:*

(1) *if $G_p(R)$ is a singleton, then* $\operatorname{diam}(G_p(R)) = 0$;
(2) *if R is a local ring, then* $\operatorname{diam}(G_p(R)) = 1$;
(3) *if R is the direct product of two fields, then* $\operatorname{diam}(G_p(R)) = \infty$;
(4) *in all other cases,* $\operatorname{diam}(G_p(R)) = 2$.

Proof: (1) This is clear.

(2) This is clear by Remark 3.

(3)–(4) This is clear by the proof of Theorem 1. □

In the following, we show when $G_p(R)$ has isolated vertices.

Proposition 5. *Let R be a ring.*

(1) *$G_p(R)$ has an isolated vertex if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a field.*
(2) *$G_p(R)$ has two isolated vertices if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof: (1) If $R \cong \mathbb{Z}_2 \times F$, where F is a field, then it is clear that $G_p(R)$ has an isolated vertex. Now suppose that $G_p(R)$ has an isolated vertex. So $G_p(R)$ is a disconnected graph, and hence by Theorem 1, we have $R \cong F_1 \times F_2$, where F_1 and F_2 are two fields. We know that the maximal ideals of R are $F_1 \times \{0\}$ and $\{0\} \times F_2$. But $G_p(R)$ has an isolated vertex. So $F_1 \times \{0\}$ has only one non-zero element. Therefore we have $R \cong \mathbb{Z}_2 \times F$.

(2) If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then it is clear that $G_p(R)$ has two isolated vertices. Now suppose that $G_p(R)$ has two isolated vertices. So $G_p(R)$ is a disconnected graph, and hence by Theorem 1, we have $R \cong F_1 \times F_2$, where F_1 and F_2 are two fields. We know that the maximal ideals of R are $F_1 \times \{0\}$ and $\{0\} \times F_2$. But the graph $G_p(R)$ has two isolated vertices. So $F_1 \times \{0\}$ and $\{0\} \times F_2$ have only one non-zero element. Therefore we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Next, we obtain the girth of $G_p(R)$ for a ring R.

Theorem 6. *Let R be a ring. Then* $\operatorname{gr}(G_p(R)) \in \{3, \infty\}$.

PROOF: Let $G_p(R)$ have a cycle and $\mathrm{gr}(G_p(R)) = n > 3$. Let $x_1, x_2, \dots, x_n$ be distinct vertices of $G_p(R)$ and $x_1 — x_2 — x_3 — \cdots — x_n — x_1$ be the shortest cycle in $G_p(R)$. So there exist distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{n-1}, \mathfrak{p}_n$ such that $x_i, x_{i+1} \in \mathfrak{p}_i$ for $1 \le i < n$ and $x_n, x_1 \in \mathfrak{p}_n$. If R is non-reduced, then there exists $0 \ne c \in \mathrm{Nil}(R)$. Therefore we have the cycle $x_1 — c — x_2 — x_1$, which is a contradiction. Now let R be reduced. We consider two cases: $x_1x_2 = 0$ and $x_1x_2 \ne 0$.

(a) If $x_1x_2 = 0$, then $x_1 \in \mathfrak{p}_3$ or $x_2 \in \mathfrak{p}_3$. Therefore we have the cycle $x_1 — x_2 — x_3 — x_1$ or the cycle $x_2 — x_3 — x_4 — x_2$, which is a contradiction in either case.

(b) If $x_1x_2 \ne 0$, then we consider three cases.
 (1) If $x_1x_2 = x_i$ for $2 < i < n$, then we have the cycle $x_1 — x_2 — x_i — x_1$, which is a contradiction.
 (2) Let $x_1x_2 = x_1$ or $x_1x_2 = x_2$. If $x_1x_2 = x_1$, then $x_1(x_2 - 1) = 0$. So we have $x_1 \in \mathfrak{p}_2$ or $x_2 - 1 \in \mathfrak{p}_2$. If $x_1 \in \mathfrak{p}_2$, then we have the cycle $x_1 — x_2 — x_3 — x_1$, which is a contradiction. If $x_2 - 1 \in \mathfrak{p}_2$, then we have $1 \in \mathfrak{p}_2$, which is a contradiction. By a similar argument, if $x_1x_2 = x_2$, then we have a contradiction.
 (3) Let $x_1x_2 \ne x_1$ and $x_1x_2 \ne x_2$. Thus we have the cycle $x_1 — x_1x_2 — x_2 — x_1$, which is a contradiction.

□

Corollary 7. *Let R be a ring such that $G_p(R)$ contains no cycles. Then R is an Artinian ring.*

PROOF: Since $G_p(R)$ contains no cycles, for every maximal ideal $\mathfrak{m}$ of R, we have $|\mathfrak{m}| < 4$. Because if there exists a maximal ideal $\mathfrak{m}$ of R such that $|\mathfrak{m}| \ge 4$, then the non-zero elements of $\mathfrak{m}$ form a cycle. Since all maximal ideals of R are finite, R is an Artinian ring. □

Example 8. Let R be a ring. In Theorem 6, we showed that $\mathrm{gr}(G_p(R)) \in \{3, \infty\}$. For example, if $R = \mathbb{Z}_8$, then $\mathbb{Z}_8$ is a local ring with maximal ideal $Z(R) = \mathfrak{m} = \{0, 2, 4, 6\}$. So $G_p(\mathbb{Z}_8) \approx \mathrm{C}_3$ and $\mathrm{gr}(G_p(\mathbb{Z}_8)) = 3$.

The next theorem shows when the prime ideals intersection graph is a bipartite graph.

Theorem 9. *Let R be a ring. Then the following statements are equivalent:*

(1) *$G_p(R)$ is a bipartite graph;*
(2) *$G_p(R)$ is a tree;*
(3) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|\mathfrak{m}| = 3$, or R has two maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2| = 3$ and $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = 2$.*

PROOF: $(1) \Longrightarrow (2)$ Since $G_p(R)$ is a bipartite graph, by [11, Theorem 1.2.18], $G_p(R)$ contains no odd cycles. By Theorem 6, if $G_p(R)$ contains a cycle, then $\mathrm{gr}(G_p(R)) = 3$. Therefore, $G_p(R)$ contains no cycles and hence it is a tree.

(2) $\Longrightarrow$ (3) Assume that $G_p(R)$ is a tree. So $G_p(R)$ has no cycles, and hence we have $|\mathfrak{m}| \leqslant 3$ for every maximal ideal $\mathfrak{m}$ of R. Because if there exists a maximal ideal $\mathfrak{m}$ such that $|\mathfrak{m}| > 3$, then we have a cycle, which is a contradiction. Therefore, R is a local ring with maximal ideal $\mathfrak{m}$ such that $|\mathfrak{m}| = 3$, or R has two maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2| = 3$ and $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = 2$.

(3) $\Longrightarrow$ (1) This is clear. □

In the next theorem, we show that if $G_p(R)$ is a star graph, then $G_p(R) \approx K_{1,1}$ or $G_p(R) \approx K_{1,2}$.

Theorem 10. *Let R be a ring and $G_p(R) \approx K_{1,n}$. Then $n \leq 2$.*

Proof: Suppose to the contrary that $n > 2$. Let $a, a_1, a_2, a_3, \ldots, a_n$ be distinct vertices of $G_p(R)$ and a be the center of the star. Therefore, there exist prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_n$ such that $a, a_i \in \mathfrak{p}_i$ for $1 \leq i \leq n$. It is clear that $a_1a_2 = 0$ or a_1a_2 is a vertex of $G_p(R)$. We consider three cases.

(1) If $a_1a_2 = 0$, then we have $a_1 \in \mathfrak{p}_3$ or $a_2 \in \mathfrak{p}_3$, and we have edges $a_1 — a_3$ or $a_2 — a_3$, which is a contradiction.
(2) If $a_1a_2 = a$ or $a_1a_2 = a_i$ and $i > 2$, then we have $a_1 \in \mathfrak{p}_i$ or $a_2 \in \mathfrak{p}_i$, and we have edges $a_1 — a_i$ or $a_2 — a_i$, which is a contradiction.
(3) If $a_1a_2 = a_1$, then $a_1(a_2 - 1) = 0$. So we have $a_1 \in \mathfrak{p}_2$ or $a_2 - 1 \in \mathfrak{p}_2$, which is a contradiction. If $a_1a_2 = a_2$, then $a_2(a_1 - 1) = 0$. So we have $a_2 \in \mathfrak{p}_1$ or $a_1 - 1 \in \mathfrak{p}_1$, which is a contradiction.

□

Example 11. Let R be a ring. In Theorem 10, we showed that if $G_p(R)$ is a star graph, then $G_p(R) \approx K_{1,1}$ or $G_p(R) \approx K_{1,2}$. For example, if $R = \mathbb{Z}_9$, then $\mathbb{Z}_9$ is a local ring with maximal ideal $M = \{0, 3, 6\}$. So $G_p(\mathbb{Z}_9) \approx K_{1,1}$.

Remark 12. It is well known that if p be a prime number and R is a ring with $|Z(R)| = p$, then R is isomorphic to one of the rings $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p[x]/(x^2)$, or $F_{q_1} \times F_{q_2} \times \cdots \times F_{q_n}$, where $F_{q_1}, F_{q_2}, \ldots, F_{q_n}$ are fields and $p = q_1q_2 \cdots q_n - (q_1 - 1)(q_2 - 1) \cdots (q_n - 1)$.

Using Remark 12, we show that $G_p(R)$ is a star graph only for the rings $R = \mathbb{Z}_9$ and $R = \mathbb{Z}_3[x]/(x^2)$.

Corollary 13. *Let R be a ring and $G_p(R)$ be a star graph. Then $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$, and $G_p(R) \approx K_{1,1}$.*

Proof: Since $G_p(R)$ is a star graph, then by Theorem 10, we have $G_p(R) \approx K_{1,1}$ or $G_p(R) \approx K_{1,2}$. If $G_p(R) \approx K_{1,1}$, then $|Z(R)| = 3$ and by Remark 12, we have $R \cong \mathbb{Z}_9$, $R \cong \mathbb{Z}_3[x]/(x^2)$, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. But only for the rings $R = \mathbb{Z}_9$ and $R \cong \mathbb{Z}_3[x]/(x^2)$, $G_p(R)$ is a star graph.

Now let $G_p(R) \approx K_{1,2}$. So R is a local ring with maximal ideal $\mathfrak{m}$ such that $|\mathfrak{m}| = 4$, or has two maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2| = 3$ and $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = 2$. In the first case, $G_p(R)$ has a cycle, which is a contradiction. In the second case, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, so $G_p(R)$ is disconnected. □

In the following, we determine all rings R such that $G_p(R)$ is a path or cycle.

Theorem 14. *Let R be a ring and $G_p(R) \approx \mathrm{P}_n$. Then $n \leq 3$.*

Proof: Let R be a ring such that $G_p(R) \approx \mathrm{P}_n$ and $n > 3$. Let $a_1, a_2, a_3, \ldots, a_n$ be distinct vertices of $G_p(R) \approx \mathrm{P}_n$. So we have the path graph $a_1 — a_2 — a_3 — \cdots — a_n$. Therefore there exist prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_n$ such that $a_i, a_{i+1} \in \mathfrak{p}_i$, for $1 \leq i \leq n$. It is clear that $a_1 a_2 = 0$ or $a_1 a_2$ is a vertex of $G_p(R)$. We consider two cases.

(1) If $a_1 a_2 = 0$, then we have $a_1 \in \mathfrak{p}_3$ or $a_2 \in \mathfrak{p}_3$, and we have edges $a_1 — a_3$ or $a_2 — a_4$, which is a contradiction.
(2) If $a_1 a_2 = a_i$ and $i > 2$, then we have $a_1 \in \mathfrak{p}_i$ or $a_2 \in \mathfrak{p}_i$, and we have edges $a_1 — a_i$ or $a_2 — a_i$, which is a contradiction.

□

Corollary 15. *Let R be a ring such that $G_p(R) \approx \mathrm{P}_n$. Then R is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3[x]/(x^2)$, or $\mathbb{Z}_9$, and $G_p(R) \approx \mathrm{P}_1$ or $G_p(R) \approx \mathrm{P}_2$.*

Proof: By Theorem 14, we have $G_p(R) \approx \mathrm{P}_1$, $G_p(R) \approx \mathrm{P}_2$, or $G_p(R) \approx \mathrm{P}_3$. If $G_p(R) \approx \mathrm{P}_1$, then $|Z(R)| = 2$, and hence by Remark 13, $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$. If $G_p(R) \approx \mathrm{P}_2$, then $|Z(R)| = 3$ and hence by Remark 12, $R \cong \mathbb{Z}_9$, $R \cong \mathbb{Z}_3[x]/(x^2)$, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. But $G_p(\mathbb{Z}_9)$ and $G_p(\mathbb{Z}_3[x]/(x^2))$ are isomorphic to P_2. If $G_p(R) \approx \mathrm{P}_3$, then R is a local ring with maximal ideal $\mathfrak{m}$ such that $|\mathfrak{m}| = 4$, or R has two maximal ideals $\mathfrak{m}_1$ $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2| = 3$ and $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = 2$. In the first case, $G_p(R)$ has a cycle, which is a contradiction. In the second case, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and $G_p(R)$ is disconnected. □

Similarly, we have the next theorem.

Theorem 16. *Let R be a ring and $G_p(R) \approx \mathrm{C}_n$. Then $n = 3$.*

Corollary 17. *Let R be a ring and $G_p(R) \approx \mathrm{C}_n$. Then R is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/[x^3]$, $F_4[x]/[x^2]$, or $GR(16,4)$, where $GR(16,4)$ is the Galois ring of order 2.*

Proof: Since $G_p(R) \approx \mathrm{C}_n$ by Theorem 16, we have $n = 3$. Hence R is a local ring with maximal ideal $Z(R) = \mathfrak{m}$ such that $|\mathfrak{m}| = 4$. Therefore, by [8, p. 687] and [9, Theorem 12], R is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/[x^3]$, $F_4[x]/[x^2]$, or $GR(16,4)$, where $GR(16,4)$ is the Galois ring of order 2. □

In the next theorem, we obtain the clique number for the prime ideals intersection graph.

Theorem 18. *Let R be a ring. Then $\omega(G_p(R)) = \sup\{|\mathfrak{m}| - 1 : \mathfrak{m} \in \mathrm{Max}(R)\}$.*

Proof: It is clear that the non-zero elements of every ideal of R form a clique. Therefore, the non-zero elements of every maximal ideal of R form a maximal clique because every ideal is contained in a maximal ideal. So we have $\omega(G_p(R)) = \sup\{|\mathfrak{m}| - 1 : \mathfrak{m} \in \mathrm{Max}(R)\}$. □

Corollary 19. *Let (R, m) be a ring. Then $G_p(R)$ is a complete graph.*

Proof: Let R be a local ring with maximal ideal $\mathfrak{m}$. Therefore, $\mathfrak{m}$ contains all non-unit elements of R. So $\omega(G_p(R)) = |\mathfrak{m}| - 1$, and hence $G_p(R)$ is a complete graph. $\square$

Corollary 20. *Let R be a ring such that $\omega(G_p(R)) < \infty$. Then R is an Artinian ring.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of R with the greatest cardinal of any maximal ideal of R. Since $\omega(G_p(R)) < \infty$, therefore $|\mathfrak{m}| < \infty$. So R is an Artinian ring. $\square$

Let R be a ring and let the zero-divisor graph of R be denoted by $\Gamma(R)$. The set of vertices of $\Gamma(R)$ is $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. Since $Z(R)$ is a union of prime ideals of R, there exist some relationships between the two graphs $\Gamma(R)$ and $G_p(R)$. In the following, we state two examples such that $\Gamma(R)$ and $G_p(R)$ are not a subgraph of each other.

Example 21. Let $R = \mathbb{Z}_6$. Then $2 — 4$ is an edge of $G_p(R)$ that is not an edge of $\Gamma(R)$.

Example 22. Let $R = \mathbb{Z}_{12}$. Then $3 — 8$ is an edge of $\Gamma(R)$ that is not an edge of $G_p(R)$.

In the next theorem, we state some conditions on the ring R so $\Gamma(R) = G_p(R)$.

Theorem 23. *Let (R, m) be a zero-dimensional ring. If $m^2 = 0$, then $\Gamma(R) = G_p(R)$.*

Proof: Since R is a zero-dimensional ring, every non-unit element of R is a zero-divisor. Therefore, the vertices of $\Gamma(R)$ and $G_p(R)$ are equal. Also, R is a local ring. So we have $Z(R) = \mathfrak{m}$. Now suppose that a and b are two vertices of $\Gamma(R)$ (or $G_p(R)$). Thus there exists an edge between a and b in $G_p(R)$ since $Z(R) = \mathfrak{m}$. Also, by [5, Theorem 2.8], there exists an edge between a and b in $\Gamma(R)$. Therefore, the proof is complete. $\square$

In the sequel, we state some results about the planarity and outerplanarity of the prime ideals intersection graph.

Theorem 24. *Let R be a ring. If there exists a maximal ideal $\mathfrak{m}$ of R with $|\mathfrak{m}| \geq 6$, then $G_p(R)$ is not planar.*

Proof: Since $|\mathfrak{m}| \geq 6$ for a maximal ideal $\mathfrak{m}$ of R, the non-zero elements of $\mathfrak{m}$ form the graph K_5 as an induced subgraph of $G_p(R)$. Therefore, by [11, Theorem 6.2.2], $G_p(R)$ is not a planar graph. $\square$

Corollary 25. *Let R be a ring. If $G_p(R)$ is a planar graph, then R is an Artinian ring.*

Proof: Let $G_p(R)$ be planar. Then by Theorem 24, for every maximal ideal $\mathfrak{m}$ of R we have $|\mathfrak{m}| \leq 5$. So R is an Artinian ring. $\square$

Theorem 26. *Let R be a ring. If there exists a maximal ideal $\mathfrak{m}$ of R with $|\mathfrak{m}| \geq 5$, then $G_p(R)$ is not an outerplanar.*

PROOF: Since $|\mathfrak{m}| \geq 5$ for a maximal ideal $\mathfrak{m}$ of R, the non-zero elements of $\mathfrak{m}$ form the graph K_4 as an induced subgraph of $G_p(R)$. Therefore, by [10, Theorem 1] the graph $G_p(R)$ is not an outerplanar graph. □

Corollary 27. *Let R be a ring. If $G_p(R)$ is outerplanar, then R is an Artinian ring.*

PROOF: Let $G_p(R)$ be outerplanar. Then by Theorem 26, for every maximal ideal $\mathfrak{m}$ of R, we have $|\mathfrak{m}| \leq 4$. So R is an Artinian ring. □

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FACULTY OF MATHEMATICS, K. N. TOOSI UNIVERSITY OF TECHNOLOGY, P.O. BOX 16315-1618, TEHRAN, IRAN

E-mail: nikmehr@kntu.ac.ir
b.soleymanzadeh@mail.kntu.ac.ir

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