Combinatorics of ideals — selectivity versus density

A. Kwela, P. Zakrzewski

Abstract. This note is devoted to combinatorial properties of ideals on the set of natural numbers. By a result of Mathias, two such properties, selectivity and density, in the case of definable ideals, exclude each other. The purpose of this note is to measure the "distance" between them with the help of ultrafilter topologies of Louveau.

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1. Introduction

We are concerned with the following two combinatorial properties of ideals. An ideal $\mathcal{I}$ on ω is:

- *dense* (or *tall*) if every infinite subset of ω contains an infinite subset in $\mathcal{I}$;
- *selective* if for every partition $\{A_n : n \in \omega\}$ of ω such that no finite union of elements of the partition is in the dual filter of $\mathcal{I}$ there is a selector in $\mathcal{I}^+$, i.e., a set S not in $\mathcal{I}$ such that $|S \cap A_n| \leq 1$ for every $n \in \omega$.

While density is a rather common property of ideals, the list of presently known examples of selective ideals is apparently short. It consists of countably generated ideals, ideals generated by AD families of subsets of ω (cf. [6]), ideals of the form

$$I_K(x,(x_n)) = \{M \subseteq \omega : \ x \notin \overline{\{x_n : \ n \in M\}}\}$$

(where K is a topological space with suitable properties, $x \in K$ and (x_n) is a sequence of elements of $K \setminus \{x\}$ accumulating to x, cf. [8, Section 12] and [10]) and the maximal ideals whose duals are Ramsey ultrafilters (cf. [1, Theorem 4.5.2]).

The starting point of this note is the following theorem of Mathias.

Theorem 1.1 (Mathias [6]). *No analytic (or coanalytic) dense ideal is selective.*

Our aim is to show that selectivity of an analytic (or coanalytic) ideal $\mathcal{I}$ is equivalent to $\mathcal{I}$ being nowhere dense in *every* so-called ultrafilter topology on $[\omega]^\omega$ (associated with any ultrafilter extending the dual filter of $\mathcal{I}$) studied earlier by Louveau [5], Todorčević [10] and others, while the property of $\mathcal{I}$ being *not* dense is equivalent to $\mathcal{I}$ being nowhere dense in *at least one* of such topologies.

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1.1 Basic definitions and notation. An *ideal* on ω, the set of natural numbers, is a collection $\mathcal{I}$ of subsets of ω which is closed under taking subsets and finite unions. We always assume that $\mathcal{I} \neq \mathcal{P}(\omega)$ and $\mathcal{I}$ contains all finite subsets of ω.

If $\mathcal{I}$ is an ideal on ω, then $\mathcal{I}^* = \{B \subseteq \omega : \ B^c \in \mathcal{I}\}$ is the *dual filter* and $\mathcal{I}^+ = \mathcal{I}^c = \mathcal{P}(\omega) \setminus \mathcal{I}$ is the *associated coideal* of $\mathcal{I}$ consisting of $\mathcal{I}$*-positive* sets.

We use standard set-theoretic notation. In particular, by $[\omega]^{<\omega}$ ($[\omega]^\omega$, respectively) we denote the collection of all finite (infinite, respectively) subsets of ω. By identifying subsets of ω with their characteristic functions we treat the power set $\mathcal{P}(\omega)$ as the product space 2^ω. Descriptive set theoretic notions concerning ideals on ω always refer to the Cantor set topology on $\mathcal{P}(\omega)$.

1.2 Trees and Grigorieff's characterization of selectivity. Our notation and terminology concerning trees agrees with [7] and is also close to [10].

By $\sqsubseteq$ we denote the initial segment relation on $\mathcal{P}(\omega)$, i.e., for $s \in [\omega]^{<\omega}$ and $A \subseteq \omega$ we have

$$s \sqsubseteq A \quad \Leftrightarrow \quad \big(s = \emptyset \ \vee \ \forall i \leq \max(s) \ (i \in s \Leftrightarrow i \in A)\big).$$

By a *tree* we mean a non-empty set $T \subseteq [\omega]^{<\omega}$ such that $s \in T$ and $t \sqsubseteq s$ imply $t \in T$.

If T is a tree and $s \in T$ then we denote by $succ_T(s)$ the set $\{n > \max(s) : s \cup \{n\} \in T\}$. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we say that T is an $\mathcal{A}$*-tree* if $succ_T(s) \in \mathcal{A}$ for every $s \in T$.

We say that a tree T is a *centered* $\mathcal{I}^+$*-tree* (or a *strong* $\mathcal{I}^+$*-tree*, cf. [2]) if for any finite set $\{s_i : i < n\} \subseteq T$ we have

$$\bigcap\{succ_T(s_i) : \ i < n\} \in \mathcal{I}^+.$$

A set $A \in [\omega]^\omega$ is *a branch* of a tree T if $s \sqsubseteq A$ implies $s \in T$ for every $s \in [\omega]^{<\omega}$. By $[T]$ we denote the set of all branches of T.

The following important characterization of selectivity is due to Grigorieff.

Theorem 1.2 (Grigorieff [2])**.** *An ideal $\mathcal{I}$ on ω is selective if and only if every centered $\mathcal{I}^+$-tree has a branch in $\mathcal{I}^+$.*

1.3 The $\mathscr{U}$-topology on $[\omega]^\omega$. Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω. The following basic facts concerning the so-called $\mathscr{U}$-topology are taken from [10] (cf. [7]).

For $s \in [\omega]^{<\omega}$ and a $\mathscr{U}$-tree T such that $\max(s) < \min(\bigcup T)$ we let

$$[s, T] = \{A \in [\omega]^\omega : \ s \sqsubseteq A \ \wedge \ A \setminus s \in [T]\}.$$

We say that a subset $\mathcal{G}$ of $[\omega]^\omega$ is $\mathscr{U}$*-open* if for every $B \in \mathcal{G}$ there are s and T as above such that $B \in [s, T]$ and $[s, T] \subseteq \mathcal{G}$.

Proposition 1.3 (Louveau [5]; see [10, Lemma 7.36]). *The family of all $\mathscr{U}$-open sets is a topology (called* the $\mathscr{U}$-topology *and denoted by $\tau_{\mathscr{U}}$) on $[\omega]^\omega$ (with the basis $\mathscr{E}_{\mathscr{U}}$ consisting of sets of the form $[s,T]$). The $\mathscr{U}$-topology extends the Polish topology on $[\omega]^\omega$ inherited from $\mathcal{P}(\omega)$ identified with the Cantor set 2^ω.*

Let $\mathcal{X} \subseteq [\omega]^\omega$. We say that $\mathcal{X}$ is

- *completely $\mathscr{U}$-Ramsey* if for every basic set $[s,T] \in \mathscr{E}_{\mathscr{U}}$ there is a $\mathscr{U}$-tree $T' \subseteq T$ such that $[s,T'] \subseteq \mathcal{X}$ or $[s,T'] \subseteq \mathcal{X}^c$;
- *completely $\mathscr{U}$-Ramsey null* if for every basic set $[s,T] \in \mathscr{E}_{\mathscr{U}}$ there is a $\mathscr{U}$-tree $T' \subseteq T$ such that $[s,T'] \subseteq \mathcal{X}^c$.

Theorem 1.4 (Louveau [5]; "Ultra-Ellentuck Theorem", cf. [10]). *A set $\mathcal{X} \subseteq [\omega]^\omega$ is completely $\mathscr{U}$-Ramsey if and only if it has the Baire property in the $\mathscr{U}$-topology. Moreover, $\mathcal{X}$ is completely $\mathscr{U}$-Ramsey null if and only if it is meager in $\tau_{\mathscr{U}}$.*

Corollary 1.5. *If $\mathcal{X} \subseteq [\omega]^\omega$ has the Baire property in the $\mathscr{U}$-topology (in particular, if $\mathcal{I}$ is analytic or coanalytic), then $\mathcal{X}$ is $\tau_{\mathscr{U}}$-dense in $[\omega]^\omega$ if and only if $\mathcal{X}^c$ is $\tau_{\mathscr{U}}$-nowhere dense.*

2. Results

The following results characterize selectivity and density of ideals in terms of ultrafilter topologies.

Theorem 2.1. *Let $\mathcal{I}$ be an ideal on ω. Then:*

(i) *$\mathcal{I}$ is selective if and only if $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$ (equivalently: $\mathcal{I}^* \subseteq \mathscr{U}$);*

(ii) *$\mathcal{I}$ is dense if and only if $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every nonprincipal ultrafilter $\mathscr{U}$;*

(iii) *$\mathcal{I}$ is dense if and only if $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.*

Proof: (i) Assume that $\mathcal{I}$ is selective. Let $[s,T] \in \mathscr{E}_{\mathscr{U}}$ be a basic open set in the $\mathscr{U}$-topology related to an ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.

Then T being a $\mathscr{U}$-tree is also a centered $\mathcal{I}^+$-tree. Hence, by Theorem 1.2 (the Grigorieff's characterization), there is $A \in [T] \cap \mathcal{I}^+$. Then $s \cup A \in [s,T] \cap \mathcal{I}^+$ which shows that $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$.

Now assume that $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Let T be a centered $\mathcal{I}^+$-tree; by Theorem 1.2, to prove that $\mathcal{I}$ is selective it is enough to show that $[T] \cap \mathcal{I}^+ \neq \emptyset$.

Extend the filter $\mathcal{I}^*$ to an ultrafilter $\mathscr{U}$ such that $succ_T(s) \in \mathscr{U}$ for each $s \in T$. Then, T being a $\mathscr{U}$-tree, $[T] \in \tau_{\mathscr{U}}$. Since $\mathcal{I}^+$ is $\tau_{\mathscr{U}}$-dense, $[T] \cap \mathcal{I}^+ \neq \emptyset$.

(ii) and (iii) Assume that $\mathcal{I}$ is dense. Let $[s,T] \in \mathscr{E}_{\mathscr{U}}$ be a basic open set in the $\mathscr{U}$-topology related to a nonprincipal ultrafilter $\mathscr{U}$.

By shrinking T, if necessary, we assume with no loss of generality that $[B]^\omega \subseteq [T]$ for every $B \in [T]$. More precisely, let

$$T^d = \{s \in T :\ \forall t \subseteq s \quad t \in T\}.$$

Then T^d is a $\mathscr{U}$-tree, $T^d \subseteq T$ and $[A]^\omega \subseteq [T^d]$ for any $A \in [T^d]$ (for details see [7, Lemma 1 of 2.2]).

Let $B \in [T]$; since $\mathcal{I}$ is dense, there is $A \in [B]^\omega \cap \mathcal{I}$. Then $A \in [T] \cap \mathcal{I}$ so $s \cup A \in [s, T] \cap \mathcal{I}$ which shows that $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$.

Now assume that $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Let $B \in \mathcal{I}^+$; our aim is to show that $[B]^\omega \cap \mathcal{I} \neq \emptyset$.

Extend the filter $\mathcal{I}^*$ to an ultrafilter $\mathscr{U}$ such that $B \in \mathscr{U}$. Let $T = [B]^{<\omega}$.

Then, T being a $\mathscr{U}$-tree, $[T] \in \tau_{\mathscr{U}}$. Since $\mathcal{I}$ is $\tau_{\mathscr{U}}$-dense, there is $A \in [T] \cap \mathcal{I} \subseteq [B]^\omega \cap \mathcal{I}$ showing that $[B]^\omega \cap \mathcal{I} \neq \emptyset$. □

By the theorem of Mathias recalled in Introduction (cf. Theorem 1.1), selectivity and density in the case of analytic (or coanalytic) ideals exclude each other. The following corollary describes the distance between these properties in the language of $\mathscr{U}$-topologies. We precede it with a general lemma, the second part of which is a zero-one law for the $\mathscr{U}$-topology.

Lemma 2.2. *Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω and assume that a set $\mathcal{X} \subseteq [\omega]^\omega$ is invariant under finite modifications, i.e., $A \in \mathcal{X}$ implies $(A \setminus s_2) \cup s_1 \in \mathcal{X}$ for any $s_1, s_2 \in [\omega]^{<\omega}$ (in particular, $\mathcal{X}$ can be equal to $\mathcal{I} \cap [\omega]^\omega$ for an ideal $\mathcal{I}$ on ω).*

- (i) *$\mathcal{X}$ is not $\tau_{\mathscr{U}}$-dense if and only if $\mathcal{X}$ is $\tau_{\mathscr{U}}$-nowhere dense.*
- (ii) *If, additionally, $\mathcal{X}$ has the Baire property in the $\mathscr{U}$-topology (in particular, if $\mathcal{X}$ is analytic or coanalytic), then either $\mathcal{X}$ or $\mathcal{X}^c$ is $\tau_{\mathscr{U}}$-nowhere dense.*

Proof: To prove the non-obvious part of point (i), assume that $\mathcal{X}$ is not $\tau_{\mathscr{U}}$-dense. This means that there is a basic $\mathscr{U}$-open set $[s_1, T_1] \in \mathscr{E}_{\mathscr{U}}$ disjoint from $\mathcal{X}$. Let $[s_2, T_2] \in \mathscr{E}_{\mathscr{U}}$ be an arbitrary basic $\mathscr{U}$-open set. Then $T = T_1 \cap T_2$ is a $\mathscr{U}$-tree (see [7, Lemma 2 of 3.2]) and $[s_2, T] \subseteq [s_2, T_2]$. Moreover, $[s_2, T] \cap \mathcal{X} = \emptyset$. Indeed, if there was $A \in [s_2, T] \cap \mathcal{X}$, then we would have

$$(A \setminus s_2) \cup s_1 \in [s_1, T] \cap \mathcal{X} \subseteq [s_1, T_1] \cap \mathcal{X},$$

contradicting the assumption that $[s_1, T_1] \cap \mathcal{X} = \emptyset$.

Point (ii) is an immediate consequence of point (i) and Corollary 1.5. □

Theorem 2.3. *If $\mathcal{I}$ is an ideal on ω with the Baire property in the $\mathscr{U}$-topology (in particular, if $\mathcal{I}$ is analytic or coanalytic), then:*

- (i) *$\mathcal{I}$ is selective if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.*
- (ii) *$\mathcal{I}$ is dense if and only if $\mathcal{I}$ contains a $\tau_{\mathscr{U}}$-dense open subset for every nonprincipal ultrafilter $\mathscr{U}$. Moreover, $\mathcal{I}$ is not dense if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for a certain nonprincipal ultrafilter $\mathscr{U}$.*
- (iii) *$\mathcal{I}$ is dense if and only if $\mathcal{I}$ contains a $\tau_{\mathscr{U}}$-dense open subset for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Moreover, $\mathcal{I}$ is not dense if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for a certain ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.*

Proof: Point (i) and the first parts of points (ii) and (iii) follow immediately from the respective points of Theorem 2.1 and Corollary 1.5 of Theorem 1.4 (the ultra-Ellentuck theorem of Louveau). The "moreover" parts of (ii) and (iii) are consequences of Lemma 2.2. □

Remark 2.4. Mathias [6] characterized selective ultrafilters as exactly those ultrafilters which have non-empty intersection with every dense analytic ideal. As a corollary of Theorem 2.3 we may easily generalize this property of selective ultrafilters as follows: if $\mathscr{U}$ is a nonprincipal selective ultrafilter on ω and $\mathcal{I}$ is a dense ideal on ω with the Baire property in the $\mathscr{U}$-topology, then $\mathcal{I} \cap \mathscr{U} \neq \emptyset$.

Indeed, by a result of Todorčević (cf. [9, Theorem 12]), $\mathscr{U}$ being selective is not meager in its own $\mathscr{U}$-topology. Therefore it cannot be disjoint from the ideal $\mathcal{I}$, since by Theorem 2.3(ii), the latter contains a $\tau_{\mathscr{U}}$-dense open set (this argument, simplifying our original reasoning, is due to the referee).

Remark 2.5. The distance between selectivity and density in the case of analytic (or coanalytic) ideals can also be seen with the help of countable diagonalizations of Laflamme [4]. We say that an ideal $\mathcal{I}$ on ω is *ω-diagonalizable* if there is a sequence (A_n) of infinite subsets of ω such that

$$\forall A \in \mathcal{I}\ \exists n\ A \cap A_n = \emptyset.$$

If, moreover, for a certain $\mathcal{A} \subseteq [\omega]^\omega$ all A_n's are members of $\mathcal{A}$, then we say that *$\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{A}$*.

It is easy to see that an ideal $\mathcal{I}$ on ω is *not* dense if and only if it is ω-diagonalizable by elements of a filter contained in $\mathcal{I}^+$. On the other hand, a result of Todorčević [10] says that if $\mathcal{I}$ is analytic (or coanalytic) and selective, then $\mathcal{I}$ is *bisequential*, i.e., ω-diagonalizable by elements of every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$ (cf. [10, Theorem 7.53]). It is also not difficult to prove that for an arbitrary ideal $\mathcal{I}$ the latter condition implies selectivity of $\mathcal{I}$.

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Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, ul. Wita Stwosza 57, 80-308 Gdańsk, Poland

E-mail: Adam.Kwela@ug.edu.pl

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

E-mail: piotrzak@mimuw.edu.pl

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