Some results on quasi-Frobenius rings

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Abstract. We give some new characterizations of quasi-Frobenius rings by the generalized injectivity of rings. Some characterizations give affirmative answers to some open questions about quasi-Frobenius rings; and some characterizations improve some results on quasi-Frobenius rings.

Keywords: mininjective ring; YJ-injective ring; 2-injective ring; JGP-injective ring; quasi-Frobenius ring

Classification: 16D50, 16L30, 16L60, 16P60, 16P70

1. Introduction

Throughout this article, R is an associative ring with an identity. For a subset X of R, the right and left annihilators of X are denoted by $\mathbf{r}(X)$ and $\mathbf{l}(X)$, respectively. The Jacobson radical of R is denoted by J or $J(R)$. The right and left socle of R are denoted by S_r and S_l respectively, the right singular ideal of R is denoted by Z_r. Concepts which have not been explained can be found in [7].

Recall that a ring R is quasi-Frobenius if it is right or left self-injective and right or left artinian or, equivalently, if it is right or left self-injective and right or left noetherian. The concept of self-injective rings is generalized by many authors. For example, a ring R is called *right n-injective* [5] if every R-homomorphism from an n-generated right ideal of R to R extends to an endomorphism of R. A right 1-injective ring is also said to be *right P-injective* [5]. A ring R is said to be *right f.g self-injective* [1] if it is right n-injective for each positive integer n. A ring R is called *right YJ-injective* [10], [12] or *right generalized principally injective* (briefly *right GP-injective*) [3], [4] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. A ring R is called *right JGP-injective* [9] if, for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. A ring R is called *right mininjective* [6] if every R-homomorphism from a minimal right ideal of R to R extends to an endomorphism of R. A ring R is called *right AGP-injective* [15] if, for any $0 \neq a \in R$, there exist a positive integer n and a left ideal X_{a^n} such that $a^n \neq 0$ and $\mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$. It is easy to see that the following implications hold:

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right self-injective $\Rightarrow$ right f.g self-injective $\Rightarrow$ right 2-injective $\Rightarrow$ right P-injective $\Rightarrow$ right YJ-injective $\Rightarrow$ right JGP-injective.

By [9, Proposition 3.4], we have JGP-injective $\Rightarrow$ right mininjective. And by [12, Lemma 3], we have right YJ-injective $\Rightarrow$ right AGP-injective.

In this paper, we shall give some new characterizations of quasi-Frobenius rings, some conditions will be given under which a right 2-injective (resp., mininjective, YJ-injective, AGP-injective, JGP-injective) ring is quasi-Frobenius. We shall show that: (1) a two-sided YJ-injective ring with maximum condition on right annihilators is quasi-Frobenius (see Corollary 2.2), which gives an affirmative answer to an open question asked by Roger Yue Chi Ming in [13, Question 4]; (2) a right Johns, right YJ-injective ring is quasi-Frobenius (see Corollary 2.4), which gives an affirmative answer to an open question asked by Roger Yue Chi Ming in [13, Question 3]; (3) a right 2-injective, right perfect, left pseudo-coherent ring is quasi-Frobenius (see Theorem 2.4), which improves a result on f.g self-injective rings obtained by Björk, see [1, Theorem 4.3].

2. Results

The following result is known, see [6, Corollary 4.8] or [11, Theorem 2], here we give a new proof.

Lemma 2.1. *The following statements are equivalent for a ring R:*

(1) *R is a quasi-Frobenius ring;*
(2) *R is a right artinian, two-sided mininjective ring.*

Proof: (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Since R is right artinian, it is a semiprimary ring with maximum condition on right annihilators. Since R is two-sided mininjective, we have $S_r = S_l$ by [6, Corollary 2.6]. Note that a semiprimary ring is semilocal, by [7, Theorem 5.52], S_r is finite dimensional as a left R-module. So, by [2, Lemma 6], R is left artinian. Thus, R is a two-sided artinian, two-sided mininjective ring, by Ikeda's theorem (see [7, Theorem 2.30]), R is a quasi-Frobenius ring. □

Recall that a ring R is called a *right minannihilator ring* [6] if every minimal right ideal of R is a right annihilator.

Theorem 2.2. *The following statements are equivalent for a ring R:*

(1) *R is a quasi-Frobenius ring;*
(2) *R is a right artinian, right mininjective right minannihilator ring.*

Proof: (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) Let $K = Ra$ be a minimal left ideal. Since R is right artinian, aR contains a minimal right ideal $I = bR$. Since $\mathbf{l}(a)$ is a maximal left ideal, $\mathbf{l}(a) = \mathbf{l}(b)$. Now $aR \subseteq \mathbf{rl}(a) = \mathbf{rl}(b) = \mathbf{rl}(bR) = bR$ because R is a right minannihilator ring, so $aR = bR$, which shows that $\mathbf{rl}(a) = aR$. By [6, Lemma 1.1], R is left mininjective. Thus, R is a two-sided mininjective right artinian ring, and so it is quasi-Frobenius by Lemma 2.1. □

Recall that a ring R is called *right GC2* [9], [15] if every right ideal that is isomorphic to R is itself a direct summand; a ring R is called a *right Goldie ring* [7] if it has the maximum condition on right annihilators and R_R is finite dimensional.

Theorem 2.3. *The following statements are equivalent for a ring R:*
(1) *R is a quasi-Frobenius ring;*
(2) *R is a two-sided mininjective, right AGP-injective ring with maximum condition on right annihilators;*
(3) *R is a left mininjective, right JGP-injective, right Goldie, right GC2 ring;*
(4) *R is a semiprimary, two-sided mininjective ring with maximum condition on right annihilators.*

Proof: (1) $\Rightarrow$ (2), and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (4) Since R is a right AGP-injective ring with maximum condition on right annihilators, it is semiprimary by [15, Corollary 1.6], and so (4) follows.

(3) $\Rightarrow$ (4) By the assumptions, R is right GC2 and right finite dimensional, so R is semilocal by [9, Corollary 2.5]. Since R is right JGP-injective, it is right mininjective by [9, Proposition 3.4], and $J \subseteq Z_r$ by [9, Theorem 3.6]. Since R has maximum condition on right annihilators, Z_r is nilpotent by [2, Lemma 1], and so J is nilpotent. Thus, R is semiprimary, and (4) follows.

(4) $\Rightarrow$ (1) Since R is two-sided mininjective, by [6, Theorem 1.14(4)], $S_r = S_l$. Observing that semiprimary ring is semilocal, by [7, Theorem 5.52], S_r is finite dimensional as a left R-module. So, by [2, Lemma 6], R is left artinian. Thus, R is a two-sided mininjective left artinian ring, and hence it is a quasi-Frobenius ring by Lemma 2.1. □

Our next Corollary 2.4 is an inference of Theorem 2.3 or [16, Theorem 2.5], it improves some results in [8, Corollary 1], [13, Theorem 11], [14, Corollary 5, Theorem 7], and gives an affirmative answer to an open question asked by Yue Chi Ming for the case of general rings, see [13, Question 4].

Corollary 2.4. *The following statements are equivalent for a ring R:*
(1) *R is a quasi-Frobenius ring;*
(2) *R is a two-sided YJ-injective ring with maximum condition on right annihilators.*

Recall that a ring R is said to be a right Johns ring if it is right noetherian and every right ideal is a right annihilator, it is easy to see that a right Johns ring is left P-injective with maximum condition on right annihilators. By Theorem 2.3(2), we have the following corollary, which gives an affirmative answer to an open question asked by Yue Chi Ming in [13, Question 3].

Corollary 2.5. *The following statements are equivalent for a ring R:*
(1) *R is a quasi-Frobenius ring;*
(2) *R is a right Johns, right YJ-injective ring.*

Lemma 2.6. *If R is a left Kasch ring, then $J = \mathbf{lr}(J)$.*

Proof: Let T be any maximal left ideal of R. Then $J \subseteq T$, and hence $\mathbf{lr}(J) \subseteq \mathbf{lr}(T)$. But R is a left Kasch ring, by [7, Proposition 1.44], we have $\mathbf{lr}(T) = T$. And so $\mathbf{lr}(J) \subseteq T$. This follows that $\mathbf{lr}(J) \subseteq J$, and therefore $J = \mathbf{lr}(J)$, as required. $\square$

Lemma 2.7. *If R is a ring with the minimum condition on left annihilators of finite subsets of R, then every left annihilator of a subset of R is a left annihilator of a finite subset of R.*

Proof: It is obvious. $\square$

Recall that a ring R is called left pseudo-coherent [1] if the left annihilator of every finite subsets of R is finitely generated; a ring R is *right minfull* [6] if it is semiperfect, right mininjective and $Soc(eR) \neq 0$ for each local idempotent $e \in R$.

Theorem 2.8. *The following statements are equivalent for a ring R:*

(1) *R is a quasi-Frobenius ring;*
(2) *R is a right 2-injective left perfect, left pseudo-coherent ring;*
(3) *R is a right 2-injective, right perfect, left pseudo-coherent ring;*
(4) *R is a right 2-injective left perfect, right pseudo-coherent ring.*

Proof: (1) $\Rightarrow$ (2)–(4) It is clear.

(2) $\Rightarrow$ (1) Since R is left perfect, it is right semiartinian by [7, Theorem B.32], and so $S_r \trianglelefteq R_R$. Thus it is right minfull. By [7, Theorem 3.12(1)], R is left and right Kasch. Since R is left Kasch, we have $J = \mathbf{lr}(J)$ by Lemma 2.6. Since R is right Kasch and right 2-injective, we have that R is left P-injective by [5, Lemma 2.2], and hence R is left mininjective. By [7, Theorem 5.52], $\mathbf{r}(J) = S_l$ is a finitely generated right ideal. But R is left pseudo-coherent, J is a finitely generated left ideal, and so J is nilpotent by [7, Lemma 5.64] since J is left T-nilpotent. Thus, R is semiprimary, and consequently right perfect. Since J/J^2 is a finitely generated left R-module, by Osofsky's Lemma [7, Lemma 6.50], R is left artinian, and therefore R is a quasi-Frobenius ring by Lemma 2.1.

(3) $\Rightarrow$ (1) Since R is right perfect, R has the minimum condition on finitely generated left ideals. Noting that R is left pseudo-coherent, every left annihilator of a finite subset of R is a finitely generated left ideal. So R has the minimum condition on left annihilators of finite subsets of R. By Lemma 2.7, every left annihilator of a subset of R is a left annihilator of a finite subset of R, and thus every left annihilator in R is a finitely generated left ideal. It follows that R has the minimum condition on left annihilators and so R has maximum condition on right annihilators. By [8, Corollary 3], R is a quasi-Frobenius ring.

(4) $\Rightarrow$ (1) Since R is left perfect and right 2-injective, we have that R is two-sided Kasch and two-sided mininjective by the proof of (2) $\Rightarrow$ (1). Since R is right Kasch, we have $J = \mathbf{rl}(J)$ by Lemma 2.6. Since R is semilocal and right mininjective, by [7, Theorem 5.52], $\mathbf{l}(J) = S_r$ is a finitely generated left ideal. But R is right pseudo-coherent, J is a finitely generated right ideal, and then J/J^2 is

a finitely generated right R-module. Now, by Osofsky's Lemma [7, Lemma 6.50] again, we have that R is right artinian, and therefore R is a quasi-Frobenius ring by [8, Corollary 3] again. $\square$

Corollary 2.9 ([1, Theorem 4.3])**.** *The following statements are equivalent for a ring R:*

(1) *R is a quasi-Frobenius ring;*

(2) *R is a right f.g self-injective, right perfect, left pseudo-coherent ring.*

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