# Generic extensions of models of ZFC

Lev Bukovský

*Dedicated to the memory of Petr Vopěnka.*

*Abstract.* The paper contains a self-contained alternative proof of my Theorem in *Characterization of generic extensions of models of set theory*, Fund. Math. **83** (1973), 35–46, saying that for models $M \subseteq N$ of **ZFC** with same ordinals, the condition $Apr_{M,N}(\kappa)$ implies that $N$ is a $\kappa$-C.C. generic extension of $M$.

*Keywords:* inner model; extension of an inner model; $\kappa$-generic extension; $\kappa$-C.C. generic extension; $\kappa$-boundedness condition; $\kappa$ approximation condition; Boolean ultrapower; Boolean valued model

*Classification:* Primary 03E45; Secondary 03E40

I present an alternative proof of the main results of my paper [4]. I hope that the proof is interesting in itself. I would like to emphasize that the proof follows the style of reasoning that I have learned in Vopěnka's Seminary in Prague in the sixties of the last century, see e.g. [11] or [13].

Petr Vopěnka died on March 20, 2015.

## 1. Preliminaries

All our considerations are related to the Fraenkel–Zermelo set theory **ZFC** with the axiom of choice. We follow the terminology and notation of T. Jech [7].

A lower case letter always denotes a set.

If $\varphi(x, p)$ is a formula, then

$$C = \{x : \varphi(x, p)\} \tag{1}$$

is a class definable from parameter $p$. We can consider classes definable in an extension of **ZFC**.

We make only one change of Jech's terminology. An **inner model** is a transitive class that is a model of **ZFC** and $On^M = On$. T. Jech does not ask the axiom of choice. It is known that a transitive class $M$ is an inner model if and only if $M$ is almost universal[1], closed under Gödel operations, and **AC** holds true in $(M, \in)$. An inner model $N$ is an **extension** of an inner model $M$ if $M \subseteq N$.

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[1] i.e., for any $x \subseteq M$ there exists a set $y \in M$ such that $x \subseteq y$.

If we work in the Gödel–Bernays set theory then we can omit that a class is defined by a formula and corresponding parameters, compare [7, p. 5].

Let us recall a result of B. Balcar and P. Vopěnka [12].

$$\text{(2)} \qquad \begin{gathered} \textit{If inner models } N_1, N_2 \textit{ are extensions of an inner model } M \\ \textit{and } \mathcal{P}(On) \cap N_1 = \mathcal{P}(On) \cap N_2, \textit{ then } N_1 = N_2. \end{gathered}$$

Thus, investigating the relationship of two extensions of a model, we can restrict our consideration to the sets of ordinals.

Assume that $M$ is an inner model and $a \subseteq M$. Then $M[a]$ is the smallest inner model such that $M \subseteq M[a]$ and $a \in M[a]$. This property cannot be a definition of M, since it contains a metamathematical quantifier "for every inner model". The existence of such an inner model must be proved in a different way, see, e.g., [7, p. 199] or [5, p. 6]. Since $M$ is definable, $M[a]$ is definable as well. Note that for $a, b \subseteq M$ we have $M[a][b] = M[b][a]$.

Let $M \subseteq N$ be inner models, $\kappa$ being an uncountable regular cardinal of $M$. The inner model $N$ is a **$\kappa$-generic extension** of $M$ if there exists a partially ordered set $P \in M$, $|P|^M < \kappa$ and an ultrafilter $G$ on $P$ generic over $M$ such that $N = M[G]$. $N$ is a **$\kappa$-C.C. generic extension** of $M$ if there exists a $\kappa$-C.C. (every antichain has cardinality $< \kappa$) $M$-complete Boolean algebra $B \in M$ and an ultrafilter $G \subseteq B$ generic over $M$ such that $N = M[G]$.

Let $N \supseteq M$ be an extension of the inner model $M$. The **$\kappa$-boundedness condition** $Bd_{M,N}(\kappa)$ says that

$$(\forall x \subseteq On, x \in N)(\exists a \in M)(\exists y \in N)\,(y \subseteq a \wedge |a|^M < \kappa \wedge x = \bigcup y).$$

The **$\kappa$-approximation condition** $Apr_{M,N}(\kappa)$ says[2]

$$\begin{gathered}(\forall f \in N, f \textit{ a function}, \operatorname{dom}(f) \in On, \operatorname{rng}(f) \subseteq On) \\ (\exists g : \operatorname{dom}(f) \longrightarrow M, g \in M)(\forall x \in \operatorname{dom}(f))\,(f(x) \in g(x) \wedge |g(x)|^M < \kappa).\end{gathered}$$

$Bd_{M,N}(\kappa)$ implies $Apr_{M,N}(\kappa)$. Indeed, let $f : \alpha \longrightarrow On$, $f \in N$, $\alpha \in On$. Then there exists a set $F \in M$, $|F|^M < \kappa$, and a set $Y \subseteq F$ such that $f = \bigcup Y$. We may assume that every element of $F$ is a partial function from ordinals into ordinals. For $\xi \in \alpha$ we set

$$h(\xi) = \{\eta : (\exists g \in F)\, g(\xi) = \eta\}.$$

Evidently $f(\xi) \in h(\xi)$ and $|h(\xi)|^M < \kappa$ for each $\xi \in \alpha$.

## 2. Main results

Let $M \subseteq N$ be inner models. Our main results read as follows:

[2]In [5] the authors say that $M$ $\kappa$-globally covers $N$.

**Theorem 1** (essentially P. Vopěnka). *$N$ is a $\kappa$-generic extension of $M$ if and only if $Bd_{M,N}(\kappa)$ holds true.*

**Theorem 2** (L. Bukovský). *$N$ is a $\kappa$-C.C. generic extension of $M$ if and only if $Apr_{M,N}(\kappa)$ holds true.*

A weaker form of Theorem 1 was proved in [13], p. 207. Both Theorems 1 and 2 were proved by the author in [4].

The implications from left to right in both theorems are trivial.

Indeed, if $N = M[G]$, where $G$ is a generic ultrafilter on a partially ordered set $P \in M$, $|P|^M < \kappa$, then for every $x \subseteq M$, $x \in N$, there exists a relation $r \in M$ such that[3] $x = r''G$. We may assume that $r \subseteq P \times M$. Set

$$a = \{\{s : \langle t, s\rangle \in r\} : t \in P\}, \quad y = \{\{s : \langle t, s\rangle \in r\} : t \in G\}.$$

Then $a \in M$, $|a|^M < \kappa$, $y \subseteq a$ and $x = \bigcup y$.

Similarly, if $N = M[G]$, where $G$ is a filter on an $M$-complete $\kappa$-C.C. Boolean algebra $B \in M$ generic over $M$, then for every function $f : \alpha \longrightarrow M$, $\alpha \in On$, $f \in N$, there exists a function $h : \alpha \times \mathrm{rng}(f) \longrightarrow B$, $h \in M$ such that $f = h^{-1}(G)$. We can assume that $h(\xi, y_1) \wedge h(\xi, y_2) = 0$ for $y_1 \neq y_2$. We set

$$g(\xi) = \{y : h(\xi, y) \neq 0\}.$$

Since $B$ is $\kappa$-C.C. we obtain that $|g(\xi)|^M < \kappa$ for each $\xi \in \alpha$. Evidently $f(\xi) \in g(\xi)$ for every $\xi \in \alpha$.

Later we show that Theorem 1 follows from Theorem 2.

Recently, S.D. Friedman, S. Fuchino and H. Sakai [5] have found a proof of Theorem 2 different than that of [4]. We present a proof that is different than those of [4] and [5]. Independently J.L. Krivine has found similar proof of a weaker result using essentially the results of [3].

## 3. Support

A set $\sigma \subseteq M$ is a support over $M$ if for any relations $r_1, r_2 \in M$ there exists a relation $r \in M$ such that

$$r''\sigma = r_1''\sigma \setminus r_2''\sigma.$$

If $x = r''\sigma$, $r \in M$ then $x \in M[\sigma]$.

If $N = M[G]$, where $G$ is an ultrafilter on a partially ordered set generic over $M$, then $G$ is a support over $M$. Actually, for every $x \subseteq M$, $x \in M[G]$, there exists a relation $r \in M$, such that $x = r''G$. If $G$ is an ultrafilter on a complete Boolean algebra, then for any such $x$ even $x = f^{-1}(G)$ for some function $f \in M$.

A first form of the next theorem presented in the language of the theory of semisets was proved in [13] as Theorem 4233.

[3] Recall that $r''a = \{y \in \mathrm{rng}(r) : (\exists x \in a)\, \langle x, y\rangle \in r\}$.

**Theorem 3** (P. Vopěnka and B. Balcar)**.** *If $\sigma \subseteq M$ is a support, then $M[\sigma]$ is a generic extension of $M$. Moreover, if $\sigma \subseteq P$ for some $P \in M$, $|P|^M < \kappa$, then $M[\sigma]$ is a $\kappa$-generic extension.*

B. Balcar [1] gave a nice simple proof of the result as stated above. The proof was presented in the language of semiset theory. A proof in the language of set theory is presented in B. Balcar and P. Štěpánek [2] in Czech. Since I do not know about any published proof of the theorem in the language of set theory in English, for the convenience of the reader, I sketch the idea of Balcar's proof. Actually I follow [2].

We begin with a motivation for Balcar's proof.

If $P$ is a partially ordered set in $M$ and $G \subseteq P$ is an ultrafilter generic over $M$, we let

$$r = \{\langle x, y\rangle : x, y \in P \text{ and } x \wedge y = 0\}.$$

Then $r \in M$ and we have:

(i) $r$ is a symmetric antireflexive relation;
(ii) $r''\{x\} \subseteq P \setminus G$ for any $x \in G$;
(iii) for any $u \subseteq P \setminus G$, $u \in M$, there exists an $x \in G$ such that $u \subseteq r''\{x\}$;
(iv) $x \leq y \equiv r''\{x\} \supseteq r''\{y\}$ for any $x, y \in P$.

Let us set

$$R = \{\langle x, a\rangle : x \in P \wedge a \subseteq P \wedge a \in M \wedge (\forall y \in a)\, x \wedge y = 0\}.$$

Then

$$R''G = \mathcal{P}(P \setminus G) \cap M. \tag{3}$$

Note that

$$r = \{\langle x, y\rangle : (\exists a)\,(y \in a \wedge \langle x, a\rangle \in R)\}. \tag{4}$$

Proof of Theorem 3: Assume that $\sigma \subseteq P \in M$ is a support. If we set

$$R_1 = \{x\} \times (\mathcal{P}(P) \cap M) \text{ for fixed } x \in \sigma,$$
$$R_2 = \{\langle y, u\rangle : y \in u \wedge u \subseteq P\} \cap M,$$

then $R_1''\sigma = \mathcal{P}(P) \cap M$ and $R_2''\sigma = (\mathcal{P}(P) \setminus \mathcal{P}(P \setminus \sigma)) \cap M$. Since $\sigma$ is a support, there exists a relation $R \in M$ such that

$$R''\sigma = R_1''\sigma \setminus R_2''\sigma = \mathcal{P}(P \setminus \sigma) \cap M. \tag{5}$$

Following (4) we set

$$r_0 = \{\langle x, y\rangle : (\exists u)\,(y \in u \wedge \langle x, u\rangle \in R)\},$$
$$r = (r_0 \cup r_0^{-1}) \setminus \{\langle x, x\rangle : x \in P\}.$$

Then $r \in M$ and we show that (i) – (iii) hold true with $G = \sigma$.

(i) is evident.

Assume that $x \in \sigma$ and $y \in r''\{x\}$. Then either there exists $u \in M$ such that $\langle x, u\rangle \in R$ and $y \in u$ or there exists $u \in M$ such that $\langle y, u\rangle \in R$ and $x \in u$. In the former case by (5) we obtain $u \subseteq P \setminus \sigma$, therefore $y \notin \sigma$. In the latter case $u \nsubseteq P \setminus \sigma$, so by (5) we obtain $y \notin \sigma$. Thus (ii) holds true.

Now assume that $u \subseteq P \setminus \sigma$, $u \in M$. Then by (5) there exists an $x \in \sigma$ such that $\langle x, u\rangle \in R$. Thus we have $u \subseteq r_0''\{x\} \subseteq r''\{x\}$ and we obtain (iii).

Considering $r$ as the relation of incompability on $P$, we define a preorder $\leq$ on $P$ by (iv):

$$x \leq y \equiv r''\{x\} \supseteq r''\{y\}.$$

We show that $\sigma$ is basis of a generic filter over $M$. More precisely, we let

$$\sigma^* = \{p \in P : (\exists q \in \sigma)\, q \leq p\}.$$

By (ii) and (iii), $\sigma^*$ is a filter on $P$. We show that $\sigma^*$ is generic over $M$.

So, let $D \subset P$, $D \in M$ be a dense set. We want to show that $D \cap \sigma^* \neq \emptyset$. Let us suppose, to get a contradiction, that $D \subset P \setminus \sigma^* \subset P \setminus \sigma$. Then by (iii) there exists $x \in \sigma$ such that $D \subseteq r''\{x\}$. We show that $x \wedge y = 0$ for each $y \in D$, i.e. $D$ is not dense. Indeed, suppose that there exist $y \in D$ and $z$ such that $z \leq x$ and $z \leq y$. Since $r''\{x\} \subseteq r''\{z\}$, $r''\{y\} \subseteq r''\{z\}$ and the relation $r$ is symmetric we obtain

$$y \in D \to y \in r''\{x\} \to x \in r''\{y\} \to x \in r''\{z\} \to z \in r''\{x\} \to z \in r''\{z\},$$

i.e. $\langle z, z\rangle \in r$, what is a contradiction. Hence $D \cap \sigma \neq \emptyset$.

Let $\sim$ be the equivalence relation on $P$ defined as

$$x \sim y \equiv r''\{x\} = r''\{y\}.$$

Note that if $x \in \sigma^*$ and $x \sim y$, then $y \in \sigma^*$. Thus $\sigma^*/\sim$ is a filter on the partially ordered set $P/\sim$ generic over $M$. If $x \subseteq M$, $x = r''\sigma$, $r \in M$, then also $x = s''(\sigma^*/\sim)$ for suitable $s \in M$. Therefore, by Balcar–Vopěnka Theorem 2 we obtain $M[\sigma^*/\sim] = M[\sigma]$.

Thus $M[\sigma] = M[\sigma^*/\sim]$ is a generic extension of $M$. □

Note that we have actually showed that

(6) $\qquad \sigma \subseteq P$ *is a support* $\equiv (\exists R \in M)\, R''\sigma = \mathcal{P}(P \setminus \sigma) \cap M.$

## 4. Set of integers and $Apr_{M,N}(\aleph_1)$

For our proof of the Basic Lemma 5 we shall need the following

**Theorem 4.** *Let $N \supseteq M$ be an extension of an inner model. If $a \subseteq \omega_0$, $a \in N$ and $Apr_{M,N}(\aleph_1)$ holds true, then $M[a]$ is a generic extension of $M$.*

The proof follows that of the main result of [3].

Proof: Let $\mathcal{B}$ denote the family of Borel subsets of the Cantor space ${}^{\omega_0}2$. There exist a mapping $\# : \mathcal{B}^M \longrightarrow \mathcal{B}$ preserving complement and unions of countable families belonging to $M$ – for a proof see R.M. Solovay [10] or Lemma 25.46 of [7]. We can consider the set $a$ as an element of ${}^{\omega_0}2$ and we set

$$j = \{A \in \mathcal{B}^M : a \in \#(A)\}.$$

$j$ is an ultrafilter on $\mathcal{B}^M$ closed under intersections of countable families from $M$ and $M[a] = M[j]$. We show that $j$ is a support.

We begin with showing that for any relation $r \in M$ there exists a function $h \in M$ such that $r''j = h^{-1}(j)$.

Since $r''j \subseteq M$ and $M$ is an almost universal class, there exists a set $A \in M$ such that $r''j \subseteq A$. We can assume that $r \subseteq \mathcal{B}^M \times A$.

Let $\mathfrak{S} = \mathcal{P}(\mathcal{B}^M) \cap M$. For $u \in \mathfrak{S}$ we set

$$A_u = \{x \in A : \{B \in \mathcal{B}^M : \langle B, x\rangle \in r\} = u\}.$$

Then $\{A_u; u \in \mathfrak{S}\} \in M$ is a family of pairwise disjoint sets. Some elements $A_u$ may be empty. For every $x \in A$ there exists unique $u \in \mathfrak{S}$ such that $x \in A_u$. We set $U(x) = u$. The function $U : A \longrightarrow \mathfrak{S}$ is defined in $M$, hence $U \in M$. Evidently

$$r = \bigcup_{u \in \mathfrak{S}} u \times A_u.$$

By the axiom of choice, there exists a function $f : A \longrightarrow \mathcal{B}^M$, $f \in M[a]$ such that $f(x) \in j \cap U(x)$ if $j \cap U(x) \neq \emptyset$ and $f(x) = \emptyset$ otherwise. By $Apr_{M,N}(\aleph_1)$ there exists a function $g : A \longrightarrow [\mathcal{B}^M]^{\leq \aleph_0}$, $g \in M$, such that $f(x) \in g(x)$ for each $x \in A$. We set

$$h(x) = \bigcup (g(x) \cap U(x)) \in \mathcal{B}^M.$$

Then $h \in M$. Since $g(x) \cap U(x) \in M$ is countable, by the completeness of $j$ we obtain

$$j \cap U(x) = \emptyset \to h(x) = \bigcup (g(x) \cap U(x)) \notin j.$$

Vice versa, if $j \cap U(x) \neq \emptyset$, then $f(x) \in j \cap U(x) \cap g(x)$. Thus $h(x) \in j$. Therefore

$$h(x) \in j \equiv j \cap U(x) \neq \emptyset.$$

Consequently we have $h^{-1}(j) = r''j$.

Now, if $y_i = h_i^{-1}(j)$, $h_i \in M$ are functions with values in $\mathcal{B}_M$ for $i = 1, 2$, we set

$$h(x) = \begin{cases} h_1(x) \setminus h_2(x) & \text{if } x \in \operatorname{dom}(h_1) \cap \operatorname{dom}(h_2), \\ h_1(x) & \text{if } x \in \operatorname{dom}(h_1) \setminus \operatorname{dom}(h_2). \end{cases}$$

Then $h \in M$ and $y_1 \setminus y_2 = h^{-1}(j)$.

The theorem follows by Theorem 3. □

Note the following. For the proof we needed actually only that there exists a relation $r \in M$ such that $r''j = \mathcal{P}(\mathcal{B}^M \setminus j) \cap M$. Thus we have dealt with a relation $r \subseteq \mathcal{B}^M \times \mathfrak{S}$ only. Therefore, instead of $Apr_{M,N}(\aleph_1)$ we can use the seemingly weaker condition

*for every* $f : (2^\kappa)^M \longrightarrow \kappa, f \in N,$ *there exists a function* $h : (2^\kappa)^M \longrightarrow [\kappa]^{\leq\aleph_0}$, $h \in M$, *such that* $f(\xi) \in h(\xi)$ *for each* $\xi \in (2^\kappa)^M$, *where* $\kappa = |\mathcal{P}(\omega) \cap M|^M$.

## 5. Basic lemma

**Lemma 5** (Basic lemma)**.** *If* $Apr_{M,N}(\lambda)$ *and* $a \subseteq \lambda$, $a \in N$, *then the inner model* $M[a]$ *is a generic extension of* $M$.

The proof of Lemma 5 in [4] is based on an embedding of the free $\lambda$-complete Boolean algebra with $\lambda$ generators constructed in $M$ into the similar Boolean algebra constructed in the universe $V$ that preserves unions of sets from $M$ of cardinality $<\lambda$. The presented proof reduced this problem to the $\aleph_1$-free Boolean algebra $\mathcal{B}$ with $\aleph_0$ generators and Theorem 4.

We begin with a weaker result. We recall that $({}^{<\omega}\lambda, \supseteq)$ is a partially ordered set "making" the regular cardinal $\lambda$ countable in the corresponding Boolean valued model. Let us consider a theory $\mathbf{T}$ that is stronger than

$$\mathbf{ZFC} + M, N \text{ are inner models} + Apr_{M,N}(\lambda) +$$
$$\lambda \text{ is regular cardinal in } M + a \subseteq \lambda + a \in N.$$

The main result is contained in

**Lemma 6** (Reduction)**.** *In the theory* $\mathbf{T}$ + *"there exists a filter* $G \subseteq {}^{<\omega}\lambda$ *generic over* $M[a]$*" it is provable that the model* $M[a]$ *is a generic extension of* $M$.

Proof: Let $a \subseteq \lambda$, $\lambda$ being a regular cardinal, $a \in N$ and $Apr_{M,N}(\lambda)$ hold true.

Let $G \subseteq {}^{<\omega_0}\lambda$ be an ultrafilter generic over $M[a]$. Note that $G$ is generic over $M$ as well. Since $\lambda$ is countable in $M[a][G]$, one can find a set $b \subseteq \omega_0$ such that $M[a][G] = M[b]$. We show that $Apr_{M[G],M[b]}(\aleph_1)$ holds true.

The partially ordered set $({}^{<\omega}\lambda, \supseteq)$ is $\lambda^+$-C.C., therefore $Apr_{M[a],M[b]}(\lambda^+)$ holds true. Let $f : \alpha \longrightarrow \beta$, $f \in M[b]$. Then there exists a function $g \in M[a]$, $g : \alpha \longrightarrow ([\beta]^{\leq\lambda})^{M[a]}$, such that $f(\xi) \in g(\xi)$ for each $\xi \in \alpha$. Since $Apr_{M,M[a]}(\lambda)$, every set from $([\beta]^{\leq\lambda}) \cap M[a]$ is a subset of a set from $([\beta]^{\leq\lambda}) \cap M$. So, we may assume that all values of $g$ are in $([\beta]^{\leq\lambda}) \cap M$. Now, by $Apr_{M,M[a]}(\lambda)$ there exists a function $h : \alpha \longrightarrow [([\beta]^{\leq\lambda})]^{<\lambda} \cap M$ such that $g(\xi) \in h(\xi)$ for each $\xi \in \alpha$. Set $d(\xi) = \bigcup h(\xi)$. Then $d \in M$ and $f(\xi) \in d(\xi)$ for each $\xi \in \alpha$. Since $|d(\xi)|^M \leq \lambda$ we have $|d(\xi)|^{M[G]} \leq \aleph_0$.

Thus, by Theorem 4, $M[b]$ is a generic extension of $M[G]$, hence a generic extension of $M$ as well. Since $M[a] \subseteq M[b]$, we obtain that $M[a]$ is a generic extension of $M$ as well (folklore, see e.g. T. Jech [7, Lemma 15.43]). $\square$

## 6. Proof of the basic lemma

Actually, the Basic lemma follows from Lemma 6 by standard argument as presented e.g. by K. Kunen [8, p. 280]. I present a proof by the methods I have learned in Vopěnka's Seminary.

We follow the terminology and notations of T. Jech [7], Sections 12–15. Assume that the language $\{\in\}$ of the set theory is enlarged by some other predicates to the language $\mathcal{L}$. If $M$ is a class, $E$ is a binary relation on $M$, and for every predicate of $\mathcal{L}$ we have corresponding relation on $M$, then $(M, E, \dots)$ is an interpretation of the language $\mathcal{L}$. Let $\varphi(x_1, \dots, x_k)$ be a formula in the language $\mathcal{L}$. The relativization of $\varphi$ to $(M, E, \dots)$ is the formula

$$\varphi^{(M,E,\dots)}(x_1, \dots, x_k) \tag{7}$$

defined similarly as $\varphi^{M,E}$ in [7, p. 161], i.e., replacing each predicate of $\mathcal{L}$, including $\in$, by its interpretation in $(M, E, \dots)$ and relativizing all quantifier to $M$. Instead of (7) we shall write

$$(M, E, \dots) \models \varphi(x_1, \dots, x_k).$$

If $B$ is a complete Boolean algebra, $M$ is an inner model, then ${}^B M$ is the class of all functions $f : P \longrightarrow M$ defined on a partition $P$ of $B$. We shall assume that each $f$ is an injection. For sake of simplicity, if $b \in B$, $b \leq a \in P$, we set $\bar{f}(b) = f(a)$.

Assume that $\mathbf{S}$ is a theory stronger than $\mathbf{ZFC}$ in the language $\{\in, R. \dots\}$, where $R$ is a $k$-ary predicate. If $M$ is an inner model of $\mathbf{S}$, $j \subseteq B$ is an ultrafilter, we define $=_j$, $\in_j$ and $R_j$ on ${}^B M$ as

$$\begin{aligned} f =_j g &\equiv \bigvee\{a \in B : \bar{f}(a) = \bar{g}(a)\} \in j, \\ f \in_j g &\equiv \bigvee\{a \in B : \bar{f}(a) \in \bar{g}(a)\} \in j, \\ R_j(f_1, \dots, f_k) &\equiv \bigvee\{a \in B : R(\bar{f}_1(a), \dots, \bar{f}_k(a)\} \in j. \end{aligned}$$

The quotient of ${}^B M$ by the equivalence relation $=_j$ will be denoted by ${}^B M/j$. The interpretation

$$({}^B M/j) = ({}^B M/j, =_j, \in_j, R_J, ...)$$

is the **Boolean ultrapower** of $M$.

One can easily extend the classical result as

**Theorem 7** (J. Łoś)**.** *If $\varphi$ is a formula in the language of $\mathbf{S}$, $M$ is an inner model and $f_1, \dots, f_n \in {}^B M$, then*

$$\begin{gathered} ({}^B M/j) \models \varphi(f_1, \dots, f_n) \equiv \\ \bigvee\{a \in B : (M, \in, R, \dots) \models \varphi(\bar{f}_1(a), \dots, \bar{f}_n(a))\} \in j. \end{gathered}$$

Therefore, the Boolean ultrapower $({}^B M/j)$ is also a model of $\mathbf{S}$.

We set $\Xi(x) = \tilde{x}$, where $\tilde{x}(1) = x$ for any $x \in M$. Then $\Xi : M \longrightarrow {}^B M/j$ is an elementary embedding.

If $B$ is a complete Boolean algebra then the Boolean valued model $V^B$ is defined in [7, pp. 209–214]. We define $=_j$ and $\in_j$ similarly as above:

$$f =_j g \equiv \|f = g\| \in j, \quad f \in_j g \equiv \|f \in g\| \in j,$$

and we denote by $V^B/j$ the quotient of $V^B$ by the equivalence relation $=_j$. Then $(V^B/j, \in_j)$, denoted as $(V^B/j)$, is a model of **ZFC**. We have similar equivalence to the Łoś Theorem

$$(V^B/j) \models \varphi(f_1, \ldots, f_n) \equiv \|\varphi(f_1, \ldots, f_n)\| \in j.$$

Let $\Phi : {}^BV \longrightarrow V^B$ be defined as $\Phi(f) = g$, where $g \in V^B$ is such that $\|g = \check{x}\| \geq a$ for every $a \in \mathrm{dom}(f)$ and $x = f(a)$. Then $\Phi$ induces an embedding of ${}^BV/j$ into $V^B/j$ such that $(V^B/j)$ is a generic extension of $\Phi({}^BV/j)$ by the ultrafilter $G$ on $\Phi(\tilde{B})$ with the canonical name $\dot{G}$ generic over $\Phi({}^BV/j)$.

In the next we shall identify $f \in {}^BV$ with $\Phi(f)$.

If the inner models $M$, $N$ are definable in $V$ by formulas $\varphi$, $\psi$ and parameters $p$, $q$, respectively, then $({}^BM/j)$, $({}^BN/j)$ are definable in $({}^BV/j)$ by same formulas and parameters $\tilde{p}$, $\tilde{q}$, respectively. Since by R. Laver [9], the inner model $\Phi({}^BV/j)$ is definable in $(V^B/j)$, both inner models $({}^BM/j)$ and $({}^BN/j)$ are definable in $(V^B/j)$.

Assume that $M$ is an inner model. Let $\psi(Z, x)$ denote the formula

$$(\exists P \in M)\,(P \text{ is a partially ordered set,}$$
$$Z \subseteq P \text{ is a filter generic over } M \text{ and } (\exists r \in M)\, x = r''Z).$$

We have

(8) $$(\forall x \subseteq M)((\exists Z)\,\psi(Z, x) \equiv M[x] \text{ is a generic extension of } M).$$

Moreover, we have the following implications

(9) $$(\exists Z)\,\psi(Z, x) \rightarrow (\exists Z \in M[x])\,\psi(Z, x) \rightarrow M[x] \models (\exists Z)\,\psi(Z, x).$$

Proof of Lemma 5: Let $B = B({}^{<\omega}\lambda)$, $j$ being an ultrafilter on $B$. Then $V^B/j$ is a model of the theory $\mathbf{T}+$"there exists a filter $G \subseteq {}^{<\omega}\lambda$ generic over $({}^BM/j)[\tilde{a}]$" of Lemma 6. Hence, by Lemma 6, $({}^BM/j)[\tilde{a}]$ is a generic extension of ${}^BM/j$. Since ${}^BM/j[\tilde{a}] \subseteq {}^BV/j$, by (8) and (9) we obtain

$$({}^BV/j) \models (\exists Z)\Psi(Z, \tilde{a}).$$

Since the models $({}^BV/j)$ and $V$ are elementary equivalent, we obtain

$$V \models (\exists Z)\Psi(Z, a).$$

By (8), $M[a]$ is a generic extension of the inner model $M$. $\square$

## 7. Auxiliary results

**Lemma 8.** *If $N$ is a generic extension of $M$ and $Apr_{M,N}(\kappa)$ holds true, then $N$ is a $\kappa$-C.C. generic extension of $M$.*

Proof: The proof is the same as the argumentation in [4] on p. 42, lines 14–28.

Assume that $N = M[G]$, where $G$ is an ultrafilter on an $M$-complete Boolean algebra $B$ generic over $M$. Let $\mathcal{P} = \{P \subseteq B : P$ is a partition of B $\wedge P \in M\}$. We set $f(P) = a \in G \cap P$ for $P \in \mathcal{P}$. By $Apr_{M,N}(\kappa)$ there exists $g : \mathcal{P} \longrightarrow [B]^{<\kappa}$, such that $g \in M$ and $f(P) \in g(P)$ for each $P \in \mathcal{P}$. Then $a = \bigwedge_{P\in\mathcal{P}} \bigvee g(P) \in G$ and the Boolean algebra $B|a$ is $\kappa$-C.C. $\square$

For the sake of completeness we repeat Theorem 2.1 of [4] as

**Lemma 9.** *If $B$ is a complete atomless $\kappa$-C.C. Boolean algebra, then the first cardinal $\lambda$ such that $B$ is not $(\lambda, 2)$-distributive is $\lambda \leq \kappa$. Thus if $M \subseteq N$ are inner models, $Apr_{M,N}(\kappa)$ holds true, then $N = M[A]$, where $\lambda = |\mathcal{P}(\kappa) \cap N|^N$ and $A \subset \lambda \times \kappa$ is such that*

$$\mathcal{P}(\kappa) \cap N = \{\{\xi \in \kappa : (\eta, \xi) \in A\} : \eta \in \lambda\}.$$

Note that $2^{<\kappa}$ may be greater than $\kappa$, therefore Lemma 8 is stronger than Lemma 2.2 of [5].

We know that a complete $\aleph_1$-C.C., $(\aleph_0, 2)$-distributive and $(\aleph_1, 2)$-non-distributive Boolean algebra produces a Suslin tree (that was essentially proved by H. Gaifman [6]). Thus, we obtain

**Corollary 10.** *If $V$ is a generic extension of an inner model $M$, $\mathcal{P}(\omega_0) \subseteq M$, $\mathcal{P}(\omega_1) \nsubseteq M$ and $Apr_{M,N}(\aleph_1)$ holds true, then in $M$ there exists a Suslin tree.*

Proof of Lemma 9: Assume that $B$ is a complete atomless $\kappa$-C.C., $(\kappa, 2)$-distributive Boolean algebra. Then $B$ is $(\kappa, \kappa)$-distributive as well.

If $P$ and $R$ are partitions of the unit element, we say that $R$ strongly refines $P$, if for any $a \in R$ there exists a $b \in P$ such that $a < b$. Since $B$ is atomless, for every partition $P$ there exists a partition strongly refining $P$. We construct a sequence of partitions $\{P_\xi : \xi < \kappa\}$ as follows. If $P_\xi$ is constructed we take for $P_{\xi+1}$ any partition strongly refining $P_\xi$. Since the algebra $B$ is $(\kappa, \kappa)$-distributive, for a limit ordinal $\xi < \kappa$, there exists a common refinement $P_\xi$ of all partitions $P_\eta$, $\eta < \xi$. Again, since the algebra $B$ is $(\kappa, \kappa)$-distributive, there exists a common refinement $P$ of all partitions $P_\xi$, $\xi < \kappa$. Let $a \in P$, $a \neq 0$. Then for each $\xi < \kappa$ there exists an $a_\xi \in P_\xi$ such that $a < a_\xi$. One can easily see that $\{a_\xi : \xi \in \kappa\}$ is a strictly decreasing sequence, what contradicts $\kappa$-C.C. condition.

Let $M \subseteq N$ and $A$ be as in the Lemma and $M[A] \neq N$. Thus for some $\mu > \kappa$ there exists a set of ordinals $a \subseteq \mu$, $a \in N$ such that $a \notin M[A]$. Since $Apr_{M[A],N}(\kappa)$ holds true, by Lemma 5, $M[A][a]$ is a generic extension of $M[A]$. Therefore there exists a $\kappa$-C.C. Boolean algebra $B$ and an ultrafilter $G \subseteq B$ generic over $M[A]$ such that $M[A][a] = M[A][G]$. Since $\mathcal{P}(\kappa) \cap N \subseteq M[A][a]$, we can assume that the Boolean algebra $B$ is $(\kappa,\kappa)$-distributive. Since $a \notin M[A]$, the Boolean algebra $B$ is not $(\mu,2)$-distributive – a contradiction. $\square$

## 8. Proofs of the main results

Proof of Theorem 2: The implication from left to right was already proved. Assume that $Apr_{M,N}(\kappa)$ holds true and $A$ is as in Lemma 9. By Lemma 5, $M[A]$ is a generic extension of $M$. Then by Lemma 8, $M[A]$ is a $\kappa$-C.C. generic extension of $M$. By Lemma 9 we obtain $N = M[A]$. $\square$

Proof of Theorem 1: The implication from left to right was already proved.

Let $Bd_{M,N}(\kappa)$ hold true. Since $Bd_{M,N}(\kappa)$ implies $Apr_{M,N}(\kappa)$, $N$ is a generic extension of $M$. Let $B$ be an $M$-complete Boolean algebra, $G \subseteq B$ being an ultrafilter generic over $M$ such that $N = M[G]$. By $Bd_{M,N}(\kappa)$ there exists a set $A \in M$, $|A|^M < \kappa$, and a set $Y \subseteq A$, $Y \in N$, such that $G = \bigcup Y$. We set

$$r = \{\langle x,y\rangle : x \in A \wedge y \in x\}.$$

Then $G = r''Y$. For every set $x \subseteq M$, $x \in M[G]$, there exists a function $f \in M$ such that $x = f^{-1}(G)$. Then $x = f^{-1}(r''Y)$. Hence $Y$ is a support over $M$. Since $|A|^M < \kappa$, by Theorem 3, $M[Y]$ is a $\kappa$-generic extension of $M$. Since $G = r''Y$, we obtain $N = M[Y]$. $\square$

**Remarks.** If $M$, $N$ are sets and models of **ZFC** such that $On^M = On^N$, then Theorems 1 and 2 are true as well and the proofs work equally as above.

If $M$, $N$ are countable models of **ZFC** with $On^M = On^N$, then there exists an ultrafilter $G \subset {}^{<\omega}\lambda$ generic over $M$. Hence the proof of Lemma 6 is actually a proof of the Basic Lemma 5. Thus, the considerations of Section 6 may be omitted.

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## References

[1] Balcar B., *A theorem on supports in the theory of semisets*, Comment. Math. Univ. Carolin. **14** (1973), 1–6.
[2] Balcar B., Štěpánek P., *Teorie množin* (Set Theory, Czech), Academia, Prague, 1986, second edition 2003.
[3] Bukovský L., *Ensembles génériques d'entiers*, C.R. Acad. Sci. Paris **273** (1971), 753–755.
[4] Bukovský L., *Characterization of generic extensions of models of set theory*, Fund. Math. **83** (1973), 35–46.
[5] Friedman S.D., Fuchino S., Sakai H., *On the set-generic multiverse*, preprint.
[6] Gaifman H., *Concerning measures on Boolean algebras*, Pacific J. Math. **14** (1964), 61–73.
[7] Jech T., *Set Theory*, the third millenium edition, revised and expanded, Springer, Berlin, 2003.
[8] Kunen K., *Set Theory*, Studies in Logic 34, College Publications, London, 2013.
[9] Laver R., *Certain very large cardinals are not created in small forcing extensions*, Ann. Pure Appl. Logic **149** (2007), 1–6.

[10] Solovay R., *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. **92** (1970), 1–56.
[11] Vopěnka P., *General theory of $\nabla$-models*, Comment. Math. Univ. Carolin. **8** (1967), 145–170.
[12] Vopěnka P., Balcar B., *On complete models of the set theory*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **15** (1967), 839–841.
[13] Vopěnka P., Hájek P., *The Theory of Semisets*, Academia, Prague, 1972.

Institute of Mathematics, Faculty of Sciences, P.J. Šafárik University, Košice, Slovakia

*E-mail:* lev.bukovsky@upjs.sk

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