A note on the solutions of a second-order evolution inclusion in non separable Banach spaces

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Abstract. We consider a Cauchy problem associated to a second-order evolution inclusion in non separable Banach spaces under Filippov type assumptions and we prove the existence of mild solutions.

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1. Introduction

In this note we study second-order evolution inclusions of the form

$$x''(t) \in A(t)x(t) + F(t, x(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1.1}$$

where $F : [0, T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, y_0 \in X$ and $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t, s)\}_{t,s\in[0,T]}$. The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([9]) and improved by Henriquez ([7]).

The present paper is motivated by several recent papers ([1]–[3], [8], [9]]) where existence results and qualitative properties of mild solutions for problem (1.1) have been obtained by using fixed point techniques. All these approaches are obtained provided that the Banach space X is separable.

De Blasi and Pianigiani ([5]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space X. Even the results in [5] are based on Filippov's ideas ([6]), the approach in [6] has a fundamental difference which consists in the construction of the measurable selections of the multifunction. This construction does not use classical selection theorems as Kuratowsky and Ryll-Nardzewski ([10]) or Bressan and Colombo ([4]).

In the present paper we obtain an existence result for problem (1.1) similar to the one in [5]. We will prove the existence of solutions for problem (1.1) in an arbitrary space X under assumptions on F of Filippov type.

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The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. Preliminaries

Let us denote by I the interval $[0,T]$, $T>0$ and let X be a real Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot,\cdot)$. As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(\cdot):I\to X$ endowed with the norm $|x(\cdot)|_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(\cdot):I\to X$ endowed with the norm $|x(\cdot)|_1=\int_0^T|x(t)|\mathrm{d}t$. By $B(X)$ we denote the Banach space of linear bounded operators on X.

Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of X endowed with the Hausdorff pseudometric

$$\mathrm{d}_H(A,B)=\max\{\mathrm{d}^*(A,B),\mathrm{d}^*(B,A)\},\quad \mathrm{d}^*(A,B)=\sup_{a\in A}\mathrm{d}(a,B),$$

where $\mathrm{d}(x,A)=\inf_{a\in A}|x-a|$, $A\subset X$, $x\in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesgue) measurable subsets of R and, for $A\in\mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of A.

Let X be a Banach space and Y be a metric space. An open (resp. closed) ball in Y with center y and radius r is denoted by $B_Y(y,r)$ (resp. $\overline{B}_Y(y,r)$). In what follows $B=B_X(0,1)$.

A multifunction $F:Y\to\mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H-continuous at $y_0\in Y$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for any $y\in B_Y(y_0,r)$ we have $\mathrm{d}_H(F(y),F(y_0))\leq\varepsilon$. F is called d_H-continuous if it is so at each point $y_0\in Y$.

Let $A\in\mathcal{L}$ with $\mu(A)<\infty$. A multifunction $F:Y\to\mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\varepsilon>0$ there exists a compact set $K_\varepsilon\subset A$h with $\mu(A\backslash K_\varepsilon)<\varepsilon$ such that F restricted to K_ε is d_H-continuous.

It is clear that if $F,G:A\to\mathcal{P}(X)$ and $f:A\to X$ are Lusin measurable then so are F restricted to B ($B\subset A$ measurable), $F+G$ and $t\to\mathrm{d}(f(t),F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is also Lusin measurable.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.1 ([7], [9])**.** A family of bounded linear operators $\mathcal{U}(t,s):X\to X$, $(t,s)\in\Delta:=\{(t,s)\in I\times I;s\leq t\}$ is called an evolution operator of the equation

$$x''(t)=A(t)x(t) \tag{2.1}$$

if the following conditions hold:
(i) for any $x\in X$, the map $(t,s)\to\mathcal{U}(t,s)x$ is continuously differentiable and

(a) $\mathcal{U}(t,t) = 0$, $t \in I$;

(b) if $t \in I$, $x \in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.

(ii) If $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \to \mathcal{U}(t,s)x$ is of class C^2 and

(a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x \equiv A(t)\mathcal{U}(t,s)x$;

(b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x \equiv \mathcal{U}(t,s)A(t)x$;

(c) $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

(iii) If $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and

(a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ and the map $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ is continuous;

(b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [7])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$
$$z(t,0) = z(t,\Pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(\cdot) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R},\mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R},\mathbf{C})$, the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R},\mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n\in\mathbf{Z}} -n^2\langle z, z_n\rangle z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n\in\mathbf{Z}} \cos(nt)\langle z, z_n\rangle z_n$ with the associated sine function $S(t)z = t\langle z, z_0\rangle z_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n}\langle z, z_n\rangle z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R},\mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [7] that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. A continuous mapping $x(\cdot) \in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)\mathrm{d}s,\ t \in I. \tag{2.3}$$

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (1.1) if $f(\cdot)$ verifies (2.2) and $x(\cdot)$ is defined by (2.3).

In what follows X is a real Banach space and we assume the following hypotheses.

Hypothesis 2.3. (i) There exists an evolution operator $\{\mathcal{U}(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

(ii) There exist $M, M_0 \geq 0$ such that $|\mathcal{U}(t,s)|_{B(X)} \leq M$, $|\frac{\partial}{\partial s}\mathcal{U}(t,s)| \leq M_0$, for all $(t,s) \in \Delta$.

(iii) $F(\cdot,\cdot) : I \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and, for any $x \in X$, $F(\cdot, x)$ is Lusin measurable on I.

(iv) There exists $l(\cdot) \in L^1(I, (0,\infty))$ such that, for each $t \in I$,

$$\mathrm{d}_H(F(t,x_1), F(t,x_2)) \leq l(t)|x_1 - x_2|, \quad \forall\, x_1, x_2 \in X.$$

(v) There exists $q(\cdot) \in L^1(I, (0,\infty))$ such that, for each $t \in I$, we have

$$F(t,0) \subset q(t)B.$$

Set $m(t) = \int_0^t l(u)\mathrm{d}u$, $t \in I$. The technical results summarized in the next lemma are essential in the proof of our result. For the proof we refer to [5].

Lemma 2.4. (i) *Let $F_i : I \to \mathcal{P}(X)$, $i = 1,2$ be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, $i = 1,2$ be such that*

$$H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I.$$

Then the multifunction $H : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

(ii) *Assume that Hypothesis 2.1 is satisfied. Then for any $x(\cdot) : I \to X$ continuous, $u(\cdot) : I \to X$ measurable and $\varepsilon > 0$ we have*

(a) *the multifunction $t \to F(t,x(t))$ is Lusin measurable on I;*

(b) *the multifunction $G : I \to \mathcal{P}(X)$ defined by*

$$G(t) := (F(t,x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t,x(t))) + \varepsilon)$$

has a Lusin measurable selection $g : I \to X$.

3. Main result

We are ready now to prove our main result.

Theorem 3.1. *We assume that Hypothesis 2.3 is satisfied. Then, for every $x_0, y_0 \in X$ the Cauchy problem (1.1) has a solution $x(\cdot) : I \to X$.*

Proof: Let us note first that, if $z(\cdot) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t,z(t)) + B$ is Bochner integrable on I. More exactly, for any $t \in I$ we have

$$\begin{aligned}|u(t)| &\leq \mathrm{d}_H(F(t,z(t)) + B, 0) \leq \mathrm{d}_H(F(t,z(t)), F(t,0)) \\ &\quad + \mathrm{d}_H(F(t,0),0) + 1 \leq l(t)|z(t)| + q(t) + 1.\end{aligned}$$

Let $0 < \varepsilon < 1$, $\varepsilon_n = \frac{\varepsilon}{2^{n+2}}$.

Consider $f_0(\cdot) : I \to X$ an arbitrary Lusin measurable function, Bochner integrable and define

$$x_0(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f_0(s)\mathrm{d}s, \quad t \in I.$$

Since $x_0(\cdot)$ is continuous, by Lemma 2.4(ii) there exists a Lusin measurable function $f_1(\cdot) : I \to X$ satisfying, for $t \in I$,

$$f_1(t) \in (F(t, x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), \mathrm{d}(f_0(t), F(t, x_0(t))) + \varepsilon_1)$$

Obviously, $f_1(\cdot)$ is Bochner integrable on I. Define $x_1(\cdot) : I \to X$ by

$$x_1(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f_1(s)\mathrm{d}s, \quad t \in I.$$

By induction, we construct a sequence $x_n : I \to X$, $n \geq 2$ given by

$$x_n(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f_n(s)\mathrm{d}s, \quad t \in I, \tag{3.1}$$

where $f_n(\cdot) : I \to X$ is a Lusin measurable function satisfying, for $t \in I$,

$$f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), \mathrm{d}(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n). \tag{3.2}$$

At the same time, as we saw at the beginning of the proof, $f_n(\cdot)$ is also Bochner integrable.

From (3.2) for $n \geq 2$ and $t \in I$, we obtain

$$\begin{aligned} |f_n(t) - f_{n-1}(t)| &\leq \mathrm{d}(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \\ &\leq \mathrm{d}(f_{n-1}(t), F(t, x_{n-2}(t))) \\ &\quad + \mathrm{d}_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \\ &\leq \varepsilon_{n-1} + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n. \end{aligned}$$

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$ we deduce, for $n \geq 2$, that

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)|x_{n-1}(t) - x_{n-2}(t)|. \tag{3.3}$$

Denote $q_0(t) := \mathrm{d}(f_0(t), F(t, x_0(t)))$, $t \in I$. We prove next, by recurrence, that, for $n \geq 2$ and $t \in I$, we have

$$
\begin{aligned}
|x_n(t) - x_{n-1}(t)| \leq & \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{M^{k+1}(m(t)-m(u))^k}{k!} \mathrm{d}u \\
& + \varepsilon_0 \int_0^t \frac{M^n (m(t)-m(u))^{n-1}}{(n-1)!} \mathrm{d}u \\
& + \int_0^t \frac{M^n (m(t)-m(u))^{n-1}}{(n-1)!} q_0(u) \mathrm{d}u.
\end{aligned} \tag{3.4}
$$

We start with $n = 2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, one has

$$
\begin{aligned}
|x_2(t) - x_1(t)| &\leq \int_0^t |\mathcal{U}(t,s)| \cdot |f_2(s) - f_1(s)| \mathrm{d}s \\
&\leq \int_0^t M[\varepsilon_0 + l(s)|x_1(s) - x_0(s)|] \mathrm{d}s \leq \varepsilon_0 M t \\
&\quad + \int_0^t [M l(s) \int_0^s |\mathcal{U}(s,u)| \cdot |f_1(u) - f_0(u)| \mathrm{d}u] \mathrm{d}s \\
&\leq \varepsilon_0 M t + \int_0^t [M^2 l(s) \int_0^s (q_0(u) + \varepsilon_1) \mathrm{d}u] \mathrm{d}s \\
&\leq \varepsilon_0 M t + \int_0^t [M^2 (q_0(u) + \varepsilon_1) \int_u^t l(s) \mathrm{d}s] \mathrm{d}u \\
&= \varepsilon_0 M t + \int_0^t M^2 (m(t) - m(s))[q_0(s) + \varepsilon_0] \mathrm{d}s,
\end{aligned}
$$

i.e, (3.4) is verified for $n = 2$.

Using again (3.2) and (3.3) we have

$$
\begin{aligned}
|x_{n+1}(t) - x_n(t)| &\leq \int_0^t |\mathcal{U}(t,s)| \cdot |f_{n+1}(s) - f_n(s)| \mathrm{d}s \\
&\leq \int_0^t M[\varepsilon_{n-1} + l(s)|x_n(s) - x_{n-1}(s)|] \mathrm{d}s \\
&\leq \varepsilon_{n-1} M t + \int_0^t l(s) [\sum_{k=0}^{n-2} \int_0^s \varepsilon_{n-2-k} \frac{M^{k+2}(m(s)-m(u))^k}{k!} \mathrm{d}u \\
&\quad + \int_0^s \frac{M^{n+1}(m(s)-m(u))^{n-1}}{(n-1)!} (q_0(u) + \varepsilon_0) \mathrm{d}u] \mathrm{d}s \\
&= \varepsilon_{n-1} M t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t [\int_0^s \frac{M^{k+2}(m(s)-m(u))^k}{k!} l(s) \mathrm{d}u] \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t l(s)(\int_0^s \frac{M^{n+1}(m(s)-m(u))^{n-1}}{(n-1)!} l(s)[q_0(u)+\varepsilon_0]\mathrm{d}u)\mathrm{d}s\\
&=\varepsilon_{n-1}Mt+\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int_0^t(\int_u^t \frac{M^{k+2}(m(s)-m(u))^k}{k!}l(s)\mathrm{d}s)\mathrm{d}u\\
&+\int_0^t(\int_u^t \frac{M^{n+1}(m(s)-m(u))^{n-1}}{(n-1)!}l(s)\mathrm{d}s)[q_0(u)+\varepsilon_0]\mathrm{d}u\\
&=\varepsilon_{n-1}Mt+\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\int_0^t \frac{M^{k+2}(m(s)-m(u))^{k+1}}{(k+1)!}\mathrm{d}u\\
&+\int_0^t \frac{M^{n+1}(m(s)-m(u))^n}{n!}[q_0(u)+\varepsilon_0]\mathrm{d}u\\
&=\sum_{k=0}^{n-1}\varepsilon_{n-1-k}\int_0^t \frac{M^{k+1}(m(s)-m(u))^k}{k!}\mathrm{d}u\\
&+\int_0^t \frac{M^{n+1}(m(s)-m(u))^n}{n!}[q_0(u)+\varepsilon_0]\mathrm{d}u,
\end{aligned}
$$

and the statement (3.4) is true for $n+1$.

From (3.4) it follows that, for $n \geq 2$ and $t \in I$, one has

$$
|x_n(t)-x_{n-1}(t)| \leq a_n, \tag{3.5}
$$

where

$$
a_n=\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\frac{M^{k+1}m(T)^k}{k!}+\frac{M^n m(T)^{n-1}}{(n-1)!}[\int_0^1 q_0(u)\mathrm{d}u+\varepsilon_0].
$$

Obviously, the series whose n-th term is a_n is convergent. So, from (3.5) we have that $x_n(\cdot)$ converges uniformly on I to a continuous function, $x(\cdot): I \to X$.

On the other hand, in view of (3.5) we have

$$
|f_n(t)-f_{n-1}(t)| \leq \varepsilon_{n-2}+l(t)a_{n-1}, \quad t \in I,\ n \geq 3
$$

which implies that the sequence $f_n(\cdot)$ converges to a Lusin measurable function $f(\cdot): I \to X$.

Since $x_n(\cdot)$ is bounded and

$$
|f_n(t)| \leq l(t)|x_{n-1}(t)|+q(t)+1
$$

we infer that $f(\cdot)$ is also Bochner integrable.

Passing with $n \to \infty$ in (3.1) and using Lebesgue dominated convergence theorem we obtain

$$
x(t)=-\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0+\mathcal{U}(t,0)y_0+\int_0^t \mathcal{U}(t,s)f(s)\mathrm{d}s, \quad t \in I.
$$

On the other hand, from (3.2) we get

$$f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, \ n \geq 1$$

and letting $n \to \infty$ we have

$$f(t) \in F(t, x(t)), \quad t \in I.$$

and the proof is complete. □

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