On the structure of universal differentiability sets

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Abstract. A subset of $\mathbb{R}^d$ is called a universal differentiability set if it contains a point of differentiability of every Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$. We show that any universal differentiability set contains a 'kernel' in which the points of differentiability of each Lipschitz function are dense. We further prove that no universal differentiability set may be decomposed as a countable union of relatively closed, non-universal differentiability sets.

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1. Introduction

Subsets of $\mathbb{R}^d$ containing a point of differentiability of every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ form a complex and still somewhat mysterious class of sets, despite significant modern progress. Such sets are called *universal differentiability sets* (or UDSs), a term introduced in [6]. The classical Rademacher's Theorem states that Lipschitz functions on Euclidean spaces are differentiable almost everywhere with respect to the Lebesgue measure. Thus, every set of positive measure is a universal differentiability set. Whilst one may characterise universal differentiability sets in $\mathbb{R}$ as sets of positive (outer) Lebesgue measure (see [17] or [9, Theorem 1]), this description fails in all Euclidean spaces of higher dimension. Preiss proves, in [14], that $\mathbb{R}^2$ contains a dense, G_δ universal differentiability set of Lebesgue measure zero. In [4], Doré and Maleva verify the existence of compact universal differentiability sets of Lebesgue measure zero in all Euclidean spaces $\mathbb{R}^d$ with $d \geq 2$. This result is strengthened in [6] and [8]: [6] establishes the existence of a compact universal differentiability set in $\mathbb{R}^d$ for $d \geq 2$ with Hausdorff dimension one, whilst [8] further proves that such a set can be found with upper Minkowski dimension one. Moreover, it is shown that for both the Hausdorff and Minkowski dimensions, dimension one is the smallest possible for a universal differentiability set in $\mathbb{R}^d$.

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In this paper we examine the structural properties of universal differentiability sets in Euclidean spaces. Our approach is motivated by the work [19], of Zelený and Pelant on the structure of non-σ-porous sets; we refer the reader to [18] for a survey on the notions of porosity and σ-porosity. In particular, we prove that, like non-σ-porous sets, universal differentiability sets contain a 'kernel', which in some sense captures the core or essence of the set. In the papers [4], [6] and [5], Doré and Maleva observe that the universal differentiability sets constructed possess the property that the differentiability points of each Lipschitz function form a dense subset. We verify that every universal differentiability set contains a universal differentiability set with this property. We go on to establish that no universal differentiability set can be expressed as a countable union of relatively closed, non-universal differentiability sets. Our main results are stated in Section 2 and proved in Section 3. Finally, in Section 4 we give an application to a question of Godefroy relating to the existence of exceptional universal differentiability sets.

2. Main results

In this section we present our main results and discuss their connections to the existing theory. Our two main theorems are based on the following lemma:

Lemma 2.1. *Let $F \subseteq \mathbb{R}^d$ be a universal differentiability set and suppose that A is a relatively closed subset of F. Then either A or $F \setminus A$ is a universal differentiability set.*

In general, a universal differentiability set in $\mathbb{R}^d$ may be decomposed as a union of two non-universal differentiability sets. In [14, Corollary 6.5], Preiss proves that any G_δ subset of $\mathbb{R}^2$ containing all lines passing through two different points of $\mathbb{Q}^2$ is a universal differentiability set. Csörnyei, Preiss and Tišer construct, in [3], a universal differentiability set $S \subseteq \mathbb{R}^2$ of this form and a pair of Lipschitz mappings $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ such that f and g have no common points of differentiability inside S; see [3, Theorem 1.2(i)]. Writing D_f for the set of points of differentiability of f, we get that

$$S = (S \setminus D_f) \cup (S \cap D_f)$$

is a decomposition of S as a union of two non-universal differentiability sets.

G_δ sets arise naturally in the theory of universal differentiability sets because they admit an equivalent metric with respect to which they are complete, see [14] and [15]. We therefore considered the question of whether it is possible to weaken the assumption on A in Lemma 2.1 to be G_δ rather than closed. It turns out that, based on announcements in [3] and [1], it is not possible to improve Lemma 2.1 in this manner: Csörnyei, Preiss and Tišer construct, in [3], a universal differentiability set $S \subseteq \mathbb{R}^2$ which admits a Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose points of differentiability D_f intersect S in a uniformly purely unrectifiable set; see [3, Theorem 1.2(ii)]. For a definition of uniformly purely unrectifiable sets we refer to [3, p. 363].

The paragraph that follows is based on the claim, appearing in [3, p. 363] and [1, Theorem 1.15], that any uniformly purely unrectifiable set is a non-universal differentiability set. In fact [3] and [1] claim something stronger, namely that any uniformly purely unrectifiable set $U \subseteq \mathbb{R}^d$ admits a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ such that f has no directional derivatives at any point in U. This statement does not have a published proof; a proof will appear in the paper [13].

Let us return to the set S given by [3] discussed above. Using that any uniformly purely unrectifiable set in $\mathbb{R}^d$ is both a non-universal differentiability set and contained in a G_δ uniformly purely unrectifiable set (see [3, p. 364]), we may find a G_δ non-universal differentiability set U such that $S \cap D_f \subseteq U$. Then,

$$S = (S \cap U) \cup (S \setminus U)$$

is a decomposition of S as a union of two non-universal differentiability sets, the first of which is a relatively G_δ subset of S.

We define the kernel, $\ker(S)$, of a set $S \subseteq \mathbb{R}^d$ with respect to universal differentiability similarly to the kernel of S with respect to non-σ-porosity, see [19, Definition 3.2].

Definition 2.2. Given a set $S \subseteq \mathbb{R}^d$, we let

$$\ker(S) = S \setminus \{x \in S : \exists \varepsilon > 0 \text{ such that } B(x,\varepsilon) \cap S \text{ is a non-UDS}\}.$$

Note that $\ker(S)$ is closed as a subset of S. Through an application of Lemma 2.1 we obtain the following theorem, which shows that the kernel of a universal differentiability set can be thought of as the core of the set. We remark that universal differentiability sets behave similarly to non-σ-porous sets in this respect — see [19, Lemma 3.4].

Theorem 2.3. *Suppose $F \subseteq \mathbb{R}^d$ is a universal differentiability set. Then,*

(i) $\ker(F) \subseteq F$ *is a universal differentiability set;*
(ii) $\ker(\ker(F)) = \ker(F)$ *and* $F \setminus \ker(F)$ *is a non-universal differentiability set. In particular, for each Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, the differentiability points of f in $\ker(F)$ form a dense subset of $\ker(F)$.*

An iterative construction based on the proof of Lemma 2.1 leads to our second main theorem:

Theorem 2.4. *Suppose that $E \subseteq \mathbb{R}^d$ is a universal differentiability set and that $(A_i)_{i=1}^{\infty}$ is a collection of relatively closed subsets of E satisfying $E = \bigcup_{i=1}^{\infty} A_i$. Then at least one of the sets A_i is a universal differentiability set.*

3. Construction

In this section we will prove the main results stated in Section 2. We begin with a summary of the notation that we will use: we fix an integer $d \geq 2$ and let $e_1, e_2, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$. For a point $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we let $B(x,\varepsilon)$ (respectively $\overline{B}(x,\varepsilon)$) denote the open (respectively closed) ball with centre x and radius ε. Given a set $S \subseteq \mathbb{R}^d$, we let $\mathrm{Int}(S)$ denote the interior, Cl(S) denote the closure and ∂S denote the boundary of S. For non-empty subsets A and B of $\mathbb{R}^d$ we let

$$\operatorname{diam}(A) = \sup\{\|a' - a\| : a, a' \in A\} \text{ and}$$
$$\operatorname{dist}(A, B) = \inf\{\|b - a\| : a \in A, b \in B\}.$$

When $A = \{a\}$ is a singleton, we will just write $\operatorname{dist}(a, B)$ rather than $\operatorname{dist}(\{a\}, B)$. We also adopt the convention $\operatorname{dist}(A, \emptyset) = 1$ for all $A \subseteq \mathbb{R}^d$. We write $\operatorname{Lip}(f)$ for the Lipschitz constant of a Lipschitz function f. Moreover, given $e \in S^{d-1}$ and function $f \colon \mathbb{R}^d \to \mathbb{R}$ we let $f'(x, e) := \lim_{t \to 0+} \frac{f(x+te)-f(x)}{t}$ denote the one-sided directional derivative of f, provided that the limit exists. The restriction of f to a set S is denoted by $f|_S$ and the support of f by $\operatorname{supp}(f)$.

A subset U of $\mathbb{R}^d$ is called a box if $U = I_1 \times I_2 \times \ldots \times I_d$ for some sequence of closed, bounded intervals $I_1, \ldots, I_d \subseteq \mathbb{R}$. Note that this definition allows a box to be empty. Writing $I_k = [a_k, b_k]$ for each k, we call a set $Y \subseteq \partial U$ a face of U if there exist $m \in \{1, \ldots, d\}$ and $y \in \{a_m, b_m\}$ such that $Y = I_1 \times \ldots \times I_{m-1} \times \{y\} \times I_{m+1} \times \ldots \times I_d$.

Lemma 3.1. *Let A be a relatively closed subset of $F \subseteq \mathbb{R}^d$.*

(i) *There exists a collection $\{U_i\}_{i=1}^\infty$ of boxes with pairwise disjoint interiors such that*

$$F \cap \bigcup_{i=1}^\infty U_i = F \setminus A, \quad \operatorname{dist}(U_i, A) > 0,\ i \in \mathbb{N} \quad \text{and} \quad \frac{\operatorname{diam}(U_i)}{\operatorname{dist}(U_i, A)} \to 0 \text{ as } i \to \infty.$$

(ii) *Furthermore, if $\{V_i\}_{i=1}^\infty$ is a collection of boxes with pairwise disjoint interiors satisfying $F \setminus A \subseteq F \cap \bigcup_{i=1}^\infty V_i$, then a collection $\{U_i\}_{i=1}^\infty$ satisfying the conclusion of* (i) *can be additionally chosen so that for each $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $U_i \subseteq V_j$.*

Proof: We prove (i) and (ii) simultaneously. Let $\Omega \subseteq \mathbb{R}^d$ be an open set satisfying $F \setminus \Omega = A$. By applying the Whitney Covering Lemma [10, Proposition 3.3.4, p. 175] to Ω, we may find a collection $\{S_i\}_{i=1}^\infty$ of boxes with pairwise disjoint interiors such that $F \cap \bigcup_{i=1}^\infty S_i = F \setminus A$ and $\operatorname{dist}(S_i, A) > 0$ for all $i \in \mathbb{N}$. For the proof of (i) we set $W_i = S_i$ for all $i \in \mathbb{N}$. For the proof of (ii) we let $\{W_i\}_{i=1}^\infty$ be an enumeration of the collection $\{S_i \cap V_k : i, k \in \mathbb{N}\}$. In both cases the boxes $\{W_i\}_{i=1}^\infty$ have pairwise disjoint interiors and satisfy $F \cap \bigcup_{i=1}^\infty W_i = F \setminus A$ and $\operatorname{dist}(W_i, A) > 0$.

Set $p_0 = 0$. For each $i \geq 1$, partition the box W_i into a finite number of boxes $U_{p_{i-1}+1}, \ldots, U_{p_i}$ with pairwise disjoint interiors such that

$$\frac{\operatorname{diam}(U_j)}{\operatorname{dist}(U_j, A)} \leq 2^{-i} \text{ for } p_{i-1} + 1 \leq j \leq p_i.$$

The assertions of the lemma are now readily verified. $\square$

Using the next lemma, we will later be able to ignore points lying in the boundaries of boxes.

Lemma 3.2. *Suppose E is a subset of $\mathbb{R}^d$ and $\{U_i\}_{i=1}^{\infty}$ is a collection of boxes in $\mathbb{R}^d$ such that $E \setminus \bigcup_{i=1}^{\infty} \partial U_i$ is a non-universal differentiability set. Then E is a non-universal differentiability set.*

Proof: For $j = 1, \dots, d$, let H_j denote the union of all faces of the boxes U_i which are orthogonal to e_j. Then $\bigcup_{i=1}^{\infty} \partial U_i = \bigcup_{j=1}^{d} H_j$. Moreover, writing p_j for the j-th co-ordinate projection map on $\mathbb{R}^d$, we have that $p_j(H_j)$ is a subset of $\mathbb{R}$ with one-dimensional Lebesgue measure zero, (in fact it is a countable set) for each $j = 1, \dots, d$. The result now follows from [8, Lemma 2.1]. $\square$

Lemma 3.3. *Let $E \subseteq \mathbb{R}^d$, $\eta > 0$ and $\{U_i\}_{i=1}^{\infty}$ be a collection of boxes with pairwise disjoint interiors such that $E \cap \mathrm{Int}(U_i)$ is a non-universal differentiability set for each i. Then there exists a Lipschitz function $g\colon \mathbb{R}^d \to \mathbb{R}$ such that*

$$\|g\|_\infty \le \eta \text{ and } \mathrm{Lip}(g) \le \eta, \tag{3.1}$$

$$g \text{ is nowhere differentiable in } \bigcup_{i=1}^{\infty} E \cap \mathit{Int}(U_i), \text{ and} \tag{3.2}$$

$$g(x) = 0 \quad \text{whenever } x \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \mathit{Int}(U_i) \right). \tag{3.3}$$

Proof: For $x \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \mathrm{Int}(U_i) \right)$ we define $g(x)$ according to (3.3). Given $i \in \mathbb{N}$ we define g on $\mathrm{Int}(U_i)$ as follows: let $\varphi = \varphi_i \in C^\infty(\mathbb{R}^d)$ be a smooth function satisfying $\varphi(x) > 0$ for all $x \in \mathrm{Int}(U_i)$, $\|\varphi\|_\infty \le 1$, $\mathrm{Lip}(\varphi) \le 1$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \setminus \mathrm{Int}(U_i)$. To verify that such a function exists, we may assume that $U_i = [-1,1]^d$ and then define $\varphi(x) = \delta \prod_{k=1}^{d} \alpha_k(x)$, where $\alpha_k \colon \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\alpha_k(x) = \begin{cases} \exp\left(\frac{-1}{1-x_k^2}\right) & \text{if } x_k := \langle x, e_k \rangle \in (-1,1), \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

and $\delta > 0$ is sufficiently small. Choose a Lipschitz function $h = h_i \colon \mathbb{R}^d \to \mathbb{R}$ such that h is nowhere differentiable in $E \cap \mathrm{Int}(U_i)$. By rescaling h if necessary, we may ensure that $\|h\|_\infty \le \eta/2$ and $\mathrm{Lip}(h) \le \eta/2$. We define g on $\mathrm{Int}(U_i)$ by $g(x) = \varphi(x)h(x)$. The smoothness of φ and the fact that $\varphi > 0$ on $\mathrm{Int}(U_i)$ ensure that g inherits all non-differentiability points of h in $\mathrm{Int}(U_i)$. The assertions of the lemma are now readily verified. $\square$

The previous two lemmas admit the following corollary:

Corollary 3.4. *A set $E \subseteq \mathbb{R}^d$ is a non-universal differentiability set if and only if for every $x \in E$, there exists $\varepsilon = \varepsilon_x > 0$ such that $B(x, \varepsilon) \cap E$ is a non-universal differentiability set.*

Proof: We prove only the non-trivial implication. Let $E \subseteq \mathbb{R}^d$ be a set for which every point $x \in E$ admits $\varepsilon_x > 0$ as in the statement of Corollary 3.4. Let $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ denote the collection of all boxes $W_i \subseteq \mathbb{R}^d$ with rational vertices and the property that $E \cap W_i$ is a non-universal differentiability set. By hypothesis $\mathcal{W}$ is a cover of E. We now define a collection $\{U_i\}_{i=1}^{\infty}$ of boxes as follows: set $U_1 = W_1$ and $p_1 = 1$. If $n \geq 1$ and the boxes $U_1, \ldots, U_{p_n}$ are already defined, we choose $p_{n+1} \geq p_n$ so that the set $W_{n+1} \setminus \operatorname{Int}(\bigcup_{i=1}^{p_n} U_i)$ may be expressed as a finite union of boxes $U_{p_n+1}, \ldots, U_{p_{n+1}}$ with pairwise disjoint interiors. To see that such a choice of p_{n+1} is possible, consider the minimal (finite) collection $\mathcal{H}$ of hyperplanes such that $\bigcup_{i=1}^{p_n} \partial U_i \subseteq \bigcup \mathcal{H}$. Then $\mathcal{H}$ determines a 'partition' of W_{n+1} (in the natural sense) into boxes with pairwise disjoint interiors. More precisely, the set

$$\left\{\mathrm{Cl(K)} \colon K \text{ is a connected component of } W_{n+1} \setminus \bigcup \mathcal{H}\right\}$$

consists of a finite number of boxes with pairwise disjoint interiors whose union is the box W_{n+1}. We may now define $U_{p_n+1}, \ldots, U_{p_{n+1}}$ as the elements of this set which do not intersect $\operatorname{Int}(\bigcup_{i=1}^{p_n} U_i)$. The construction ensures that $E \subseteq \bigcup_{i=1}^{\infty} U_i$, the boxes $\{U_i\}_{i=1}^{\infty}$ have pairwise disjoint interiors and that $E \cap U_i$ is a non-universal differentiability set for each i. Finally, we apply Lemma 3.3 and Lemma 3.2 to complete the proof. $\square$

Lemma 3.5. *Let $\{U_i\}_{i=1}^{\infty}$ be a collection of boxes in $\mathbb{R}^d$ with pairwise disjoint interiors, $h \colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function and $\sigma > 0$. Then, there exists a Lipschitz function $\widehat{h} \colon \mathbb{R}^d \to \mathbb{R}$ such that*

$$\left\|\widehat{h} - h\right\|_{\infty} \leq \sigma \text{ and } \operatorname{Lip}(\widehat{h}) \leq \operatorname{Lip}(h) + \sigma, \tag{3.5}$$

$$\widehat{h} \text{ is everywhere differentiable inside } \bigcup_{i=1}^{\infty} \operatorname{Int}(U_i), \text{ and} \tag{3.6}$$

$$\widehat{h}(x) = h(x) \quad \text{for all } x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \operatorname{Int}(U_i). \tag{3.7}$$

Proof: We may assume that $\operatorname{Lip}(h) > 0$. Outside of $\bigcup_{i=1}^{\infty} \operatorname{Int}(U_i)$, we define the function $\widehat{h}$ according to (3.7). Given $i \in \mathbb{N}$, we use the following proposition to define $\widehat{h}$ on $\operatorname{Int}(U_i)$:

Proposition (Corollary 88 [11, p. 448])**.** *Let X be a Hilbert space and $\Omega \subseteq X$ be an open set. Then for any L-Lipschitz function $f \colon \Omega \to \mathbb{R}$, any continuous function $\zeta \colon \Omega \to (0, \infty)$, and any $\eta > 1$ there exists an ηL-Lipschitz function $g \in C^1(\Omega)$ such that $|f(x) - g(x)| < \zeta(x)$ for all $x \in \Omega$.*

We define $\widehat{h}$ on $\operatorname{Int}(U_i)$ as the function g given by the conclusion of the above proposition applied to $X = \mathbb{R}^d$, $\Omega = \operatorname{Int}(U_i)$, $L = \operatorname{Lip}(h)$, $f = h$, $\eta = 1 + \frac{\sigma}{\operatorname{Lip}(h)}$ and ζ, where $\zeta \colon \operatorname{Int}(U_i) \to (0, \infty)$ is any continuous function with the properties $\|\zeta\|_{\infty} \leq \sigma$ and $\lim_{x \to z} \zeta(x) = 0$ for all $z \in \partial U_i$.

This completes the construction of the function $\widehat{h}\colon \mathbb{R}^d \to \mathbb{R}$, whose properties (3.5), (3.6) and (3.7) are readily verified; the second inequality of (3.5) can be quickly proved using [12, Lemma 2]. □

Lemma 3.6. *Let $F \subseteq \mathbb{R}^d$, A be a relatively closed subset of F, $h\colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function and $\{U_i\}_{i=1}^{\infty}$ be a collection of boxes with pairwise disjoint interiors such that*

$$F \cap \bigcup_{i=1}^{\infty} U_i = F \setminus A, \quad \operatorname{dist}(U_i, A) > 0,\ i \in \mathbb{N} \qquad \text{and} \quad \frac{\operatorname{diam}(U_i)}{\operatorname{dist}(U_i, A)} \to 0 \text{ as } i \to \infty. \tag{3.8}$$

Then the following two statements hold.

1. *Suppose that $F \setminus A$ is a non-universal differentiability set and let $\varepsilon > 0$. Then there exists a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ with the following properties:*

$$f \text{ is nowhere differentiable in } \quad (F \setminus A) \setminus \left(\bigcup_{i=1}^{\infty} \partial U_i \right), \tag{3.9}$$

$$\|f - h\|_\infty \le \varepsilon \quad \text{and} \quad \operatorname{Lip}(f) \le \operatorname{Lip}(h) + \varepsilon, \tag{3.10}$$

$$f(y) = h(y) \quad \text{whenever} \quad y \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \operatorname{Int}(U_i). \tag{3.11}$$

2. *Let $x \in A$ and suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is any Lipschitz function satisfying the condition* (3.11)*. Then f is differentiable at x if and only if h is differentiable at x.*

Proof: Let us first verify statement 1. Note that the boxes $\{U_i\}_{i=1}^{\infty}$, the function h and $\sigma = \varepsilon/2$ satisfy the conditions of Lemma 3.5. Let $\widehat{h}\colon \mathbb{R}^d \to \mathbb{R}$ be the function given by the conclusion of Lemma 3.5.

The conditions of Lemma 3.3 are satisfied for $E = F \setminus A$, the collection $\{U_i\}_{i=1}^{\infty}$ and $\eta = \varepsilon/2$. Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be given by the conclusion of Lemma 3.3. We define $f = \widehat{h} + g$. The assertions (3.9), (3.10) and (3.11) now follow easily from the properties of $\widehat{h}$ and g.

Finally, we prove statement 2. Suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfies (3.11) and let $x \in A$. We show that the function $(h - f)$ has derivative zero at x, which suffices. Given $\varepsilon > 0$ we use (3.8) to choose $N \in \mathbb{N}$ such that

$$\frac{\operatorname{diam}(U_i)}{\operatorname{dist}(U_i, A) - \operatorname{diam}(U_i)} < \frac{\varepsilon}{\operatorname{Lip}(h) + \operatorname{Lip}(f)}$$

for all $i \geq N$. Let $r := \min\{\operatorname{dist}(x, U_i) \colon 1 \leq i \leq N\}$ and $y \in B(x,r) \setminus \{x\}$. If $y \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \operatorname{Int}(U_i)$ then $(h-f)(y) = (h-f)(x) = 0$. In the remaining case, there exists $i \in \mathbb{N}$ with $i > N$ such that $y \in \operatorname{Int}(U_i)$. We choose $y' \in \partial U_i$ and observe, using $h(y') = f(y')$ and $h(x) = f(x)$, that

$$\begin{aligned}\frac{|(h-f)(y)-(h-f)(x)|}{\|y-x\|} &\leq \frac{|h(y)-h(y')|+|f(y')-f(y)|}{\|y-x\|}\\ &\leq \frac{(\operatorname{Lip}(h)+\operatorname{Lip}(f))\operatorname{diam}(U_i)}{\operatorname{dist}(U_i,A)-\operatorname{diam}(U_i)} < \varepsilon.\end{aligned}$$

The above argument proves that the derivative of $(h-f)$ at x is zero. □

We are now ready to combine the results of the present section in proofs of our main results:

Proof of Lemma 2.1: Suppose the contrary for some $A \subseteq F \subseteq \mathbb{R}^d$. Then there exists a Lipschitz function $h \colon \mathbb{R}^d \to \mathbb{R}$ such that h is nowhere differentiable in A. Applying Lemma 3.1, part (i), we find a collection $\{U_i\}_{i=1}^{\infty}$ of boxes with pairwise disjoint interiors such that (3.8) holds. The conditions of Lemma 3.6 are satisfied for F, A, h and $\{U_i\}_{i=1}^{\infty}$. Further, the hypothesis of Lemma 3.6 part 1 is satisfied for F, A and arbitrary $\varepsilon > 0$. Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be the Lipschitz function given by the conclusion of Lemma 3.6, part 1.

By Lemma 3.6, part 2 the differentiability of f and h coincides at all points of A. Thus f is nowhere differentiable in A. Moreover, f is nowhere differentiable in $(F \setminus A) \setminus \bigcup_{i=1}^{\infty} \partial U_i$ by (3.9). Hence $F \setminus \bigcup_{i=1}^{\infty} \partial U_i$ is a non-universal differentiability set and Lemma 3.2 asserts that F is also a non-universal differentiability set. □

Proof of Theorem 2.3: Note that $\ker(F)$ is a closed subset of F and $F \setminus \ker(F)$ is a non-universal differentiability set by Corollary 3.4. Therefore, we may apply Lemma 2.1 with $A = \ker(F)$ to deduce that $\ker(F)$ is a universal differentiability set. This proves (i). For (ii), it only remains to check that $\ker(\ker(F)) = \ker(F)$. Let $x \in \ker(F)$ and $\varepsilon > 0$. Then we observe that

$$B(x,\varepsilon) \cap \ker(F) = \ker(B(x,\varepsilon) \cap F),$$

and the latter set is a universal differentiability set by part (i) and $x \in \ker(F)$. □

Proof of Theorem 2.4: Suppose that the contrary holds for some universal differentiability set $E \subseteq \mathbb{R}^d$. This means that there exist relatively closed subsets A_i of E such that $E = \bigcup_{i=1}^{\infty} A_i$ and each A_i is a non-universal differentiability set. An inductive argument based on Lemma 2.1 yields that for each $k \in \mathbb{N}$ the set $\bigcup_{i=1}^{k} A_i$ is a non-universal differentiability set and relatively closed in E. Hence, we may assume that $A_k \subseteq A_{k+1}$ for each $k \geq 1$. We will obtain a contradiction by proving that E is a non-universal differentiability set.

We begin the construction by using Lemma 3.1, part (i) to find a collection of boxes $\{U_{i,1}\}_{i=1}^{\infty}$ with pairwise disjoint interiors such that

$$E \cap \bigcup_{i=1}^{\infty} U_{i,1} = E \setminus A_1, \quad \operatorname{dist}(U_{i,1}, A_1) > 0,\ i \in \mathbb{N}$$
$$\text{and} \quad \frac{\operatorname{diam}(U_{i,1})}{\operatorname{dist}(U_{i,1}, A_1)} \to 0 \text{ as } i \to \infty.$$

Choose a Lipschitz function $f_1\colon \mathbb{R}^d \to \mathbb{R}$ such that f_1 is nowhere differentiable in A_1.

Suppose $n \geq 1$, the Lipschitz function $f_n\colon \mathbb{R}^d \to \mathbb{R}$ and the collections $\{U_{i,l}\}_{i=1}^{\infty}$ of boxes with pairwise disjoint interiors are defined for $l = 1, \ldots, n$ such that

$$f_n \text{ is nowhere differentiable in the set} \quad A_n \setminus \left(\bigcup_{l=1}^{n-1} \bigcup_{i=1}^{\infty} \partial U_{i,l} \right), \tag{3.12}$$

$$E \cap \bigcup_{i=1}^{\infty} U_{i,n} = E \setminus A_n, \quad \operatorname{dist}(U_{i,n}, A_n) > 0,\ i \in \mathbb{N} \tag{3.13}$$
$$\text{and} \quad \frac{\operatorname{diam}(U_{i,n})}{\operatorname{dist}(U_{i,n}, A_n)} \to 0 \ \text{ as } \ i \to \infty.$$

Let the Lipschitz function $f_{n+1}\colon \mathbb{R}^d \to \mathbb{R}$ be given by the conclusion of Lemma 3.6, part 1 when we take $A = A_n$, $F = A_{n+1}$, $h = f_n$, $U_i = U_{i,n}$ and $\varepsilon = 2^{-(n+1)}$. Then f_{n+1} is nowhere differentiable in $(A_{n+1} \setminus A_n) \setminus \bigcup_{i=1}^{\infty} \partial U_{i,n}$. From part 2 of Lemma 3.6, the differentiability of f_{n+1} and f_n coincides at all points of A_n. Hence, using (3.12), f_{n+1} is nowhere differentiable in the set

$$A_{n+1} \setminus \big(\bigcup_{l=1}^{n} \bigcup_{i=1}^{\infty} \partial U_{i,l} \big).$$

Let the collection of boxes $\{U_{i,n+1}\}_{i=1}^{\infty}$ be given by the conclusion of Lemma 3.1, part (ii) when we take $F = E$, $A = A_{n+1}$ and $V_i = U_{i,n}$. This ensures the validity of (3.13) with n replaced by $n+1$.

We have defined, for each integer $n \geq 1$, a Lipschitz function $f_n\colon \mathbb{R}^d \to \mathbb{R}$ and a collection of boxes $\{U_{i,n}\}_{i=1}^{\infty}$ with pairwise disjoint interiors. In addition to (3.12) and (3.13), the construction ensures that the following conditions hold for each $n \geq 2$:

$$\|f_n - f_{n-1}\|_{\infty} \leq 2^{-n} \text{ and } \operatorname{Lip}(f_n) \leq \operatorname{Lip}(f_{n-1}) + 2^{-n}. \tag{3.14}$$

$$f_n(y) = f_{n-1}(y) \text{ whenever } y \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \operatorname{Int}(U_{i,n-1}) \right). \tag{3.15}$$

$$\text{For each } i \in \mathbb{N} \text{ there exists } j \in \mathbb{N} \text{ such that } U_{i,n} \subseteq U_{j,n-1}. \tag{3.16}$$

For the sake of future reference we point out that

$$(3.17)\qquad f_m(y) = f_n(y) \quad \text{whenever} \quad m \geq n \quad \text{and} \quad y \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \operatorname{Int}(U_{i,n}) \right).$$

This follows from (3.15) and (3.16). By (3.14) the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$. Using (3.17), we deduce that the function f satisfies

$$(3.18)\qquad f(y) = f_n(y) \quad \text{whenever} \quad y \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \operatorname{Int}(U_{i,n}) \right).$$

We are now ready to prove that E is a non-universal differentiability set. In view of Lemma 3.2, it is sufficient to show that $E' = E \setminus \left(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \partial U_{i,k} \right)$ is a non-universal differentiability set. We will prove that f is nowhere differentiable in E'.

Fix $x \in E'$ and choose n such that $x \in A_n$. The condition (3.13) ensures that the conditions of Lemma 3.6 are satisfied for $F = E$, $A = A_n$, $h = f_n$ and $U_i = U_{i,n}$. Further, from (3.18), the hypothesis of Lemma 3.6, part 2 is satisfied for the function $f\colon \mathbb{R}^d \to \mathbb{R}$. Therefore, the differentiability of f at x coincides with that of f_n at x and, by (3.12), the proof is complete. □

4. Differentiability inside sets of positive measure

In this section we give an application of Theorem 2.4 to differentiability inside sets of positive Lebesgue measure.

Theorem 4.1. *Let $d \geq 2$ and suppose $P_1, P_2, \ldots, P_d \subseteq \mathbb{R}$ are sets of positive one-dimensional Lebesgue measure. Then $P_1 \times \ldots \times P_d$ contains a compact universal differentiability set with Lebesgue measure zero.*

Proof: We may assume that each set P_i is closed. For $k = 0, 1, \ldots, d$, let Π_k be the statement that $P_1 \times P_2 \times \ldots \times P_k \times \mathbb{R}^{d-k}$ contains a compact universal differentiability set C_k with Lebesgue measure zero. The statement Π_0 is proved in [4]. Suppose now that $0 < k \leq d$ and that the statement Π_{k-1} holds. Let us prove the statement Π_k and thus, by induction, Theorem 4.1.

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q}$ and consider the set

$$F_k = \bigcup_{n=1}^{\infty} (\mathbb{R}^{k-1} \times (P_k + r_n) \times \mathbb{R}^{d-k}).$$

Writing $F_{k,n} = \mathbb{R}^{k-1} \times (P_k + r_n) \times \mathbb{R}^{d-k}$ for each n, we have $F_k = \bigcup_{n=1}^{\infty} F_{k,n}$ and each set $F_{k,n}$ is closed. Further, observe that $p_k(\mathbb{R}^d \setminus F_k) = \mathbb{R} \setminus \bigcup_{n=1}^{\infty}(P_k + r_n)$ is a Lebesgue measurable subset of $\mathbb{R}$, which is closed under rational translations and whose complement has positive Lebesgue measure. Thus, the next (known) proposition (see [2, Theorem A]) immediately implies that $p_k(\mathbb{R}^d \setminus F_k)$ has Lebesgue measure zero. The proof is based on the Lebesgue density theorem [16, p. 235].

Proposition. *Let $U \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $U + q \subseteq U$ for all $q \in \mathbb{Q}$. Then either U or $\mathbb{R} \setminus U$ has Lebesgue measure zero.*

Proof: We write $\mathcal{L}$ for the Lebesgue measure on $\mathbb{R}$. If U has no density points then the Lebesgue Density Theorem implies that U has Lebesuge measure zero. If U has a density point $y \in U$, then we may choose $s_0 > 0$ such that $\mathcal{L}((\mathbb{R} \setminus U) \cap (y - s, y + s)) < \frac{1}{4}s$ for all $s \in (0, s_0)$. Given $x \in \mathbb{R}$ and $s \in (0, s_0)$, let $\lambda \in \mathbb{Q}$ be such that $|\lambda - (y - x)| \leq \frac{1}{4}s$. Using that $\mathbb{R} \setminus U$ is closed under rational translations, we deduce

$$\begin{aligned}\mathcal{L}((\mathbb{R} \setminus U) \cap (x - s, x + s)) &= \mathcal{L}((\mathbb{R} \setminus U) \cap (x + \lambda - s, x + \lambda + s)) \\ &\leq \mathcal{L}((\mathbb{R} \setminus U) \cap (y - s, y + s)) + |\lambda - (y - x)| < \frac{1}{2}s.\end{aligned}$$

Hence every point $x \in \mathbb{R}$ is not a density point of $\mathbb{R} \setminus U$ and the Lebesgue density theorem implies that $\mathbb{R} \setminus U$ has Lebesgue measure zero. □

Let us return to the proof of Theorem 4.1. We can write

$$C_{k-1} = (C_{k-1} \cap F_k) \cup (C_{k-1} \cap (\mathbb{R}^d \setminus F_k)).$$

Since $\mathbb{R}^d \setminus F_k$ projects to a set of one-dimensional Lebesgue measure zero, we may apply [8, Lemma 2.1] to conclude that $C_{k-1} \cap F_k$ is a universal differentiability set. Next, using Theorem 2.4, we deduce that there exists n such that $C_{k-1} \cap F_{k,n}$ is a universal differentiability set. Setting

$$C_k = (C_{k-1} - r_n e_k) \cap (\mathbb{R}^{k-1} \times P_k \times \mathbb{R}^{d-k}),$$

we observe that

$$C_k = (C_{k-1} \cap F_{k,n}) - r_n e_k.$$

C_k is a universal differentiability set, due to the easily verified fact that any translate of a universal differentiability set is a universal differentiability set. Moreover, C_k is also compact and has Lebesgue measure zero. Note that $(C_{k-1} - r_n e_k) \subseteq P_1 \times \ldots \times P_{k-1} \times \mathbb{R}^{d-k+1}$. Hence, $C_k \subseteq P_1 \times \ldots \times P_k \times \mathbb{R}^{d-k}$ and the proof of statement Π_k is complete. □

The above Theorem 4.1 provides a partial answer to the following question of Godefroy: does every subset of $\mathbb{R}^d$ with positive Lebesgue measure contain a universal differentiability set of Lebesgue measure zero? This question was asked following a talk of Maleva at the 2012 conference 'Geometry of Banach spaces' in CIRM, Luminy, and remains open. Theorem 4.1 also builds on an observation of Doré and Maleva: a consequence of Lemma 3.5 in [5] is that every set of the form $P \times \mathbb{R}^{d-1} \subseteq \mathbb{R}^d$, where $P \subseteq \mathbb{R}$ is a set of positive Lebesgue measure, contains a Lebesgue null universal differentiability set.

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