Some applications of the point-open subbase game

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Abstract. Given a subbase $\mathcal{S}$ of a space X, the game $PO(\mathcal{S}, X)$ is defined for two players P and O who respectively pick, at the n-th move, a point $x_n \in X$ and a set $U_n \in \mathcal{S}$ such that $x_n \in U_n$. The game stops after the moves $\{x_n, U_n : n \in \omega\}$ have been made and the player P wins if $\bigcup_{n\in\omega} U_n = X$; otherwise O is the winner. Since $PO(\mathcal{S}, X)$ is an evident modification of the well-known point-open game $PO(X)$, the primary line of research is to describe the relationship between $PO(X)$ and $PO(\mathcal{S}, X)$ for a given subbase $\mathcal{S}$. It turns out that, for any subbase $\mathcal{S}$, the player P has a winning strategy in $PO(\mathcal{S}, X)$ if and only if he has one in $PO(X)$. However, these games are not equivalent for the player O: there exists even a discrete space X with a subbase $\mathcal{S}$ such that neither P nor O has a winning strategy in the game $PO(\mathcal{S}, X)$. Given a compact space X, we show that the games $PO(\mathcal{S}, X)$ and $PO(X)$ are equivalent for any subbase $\mathcal{S}$ of the space X.

Keywords: point-open game; subbase; winning strategy; players; discrete space; compact space; scattered space; measurable cardinal

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1. Introduction

The game we are going to study here is a slight variation of the well-known point-open game $PO(X)$ that was defined and studied independently by Galvin [4] and Telgársky [8]. Given a topological space X, the game $PO(X)$ is played on X as follows: the n-th move of the first player (from now on denoted by P) consists in taking a point $x_n \in X$. The second player (called O) answers by choosing an open set $U_n \subset X$ with $x_n \in U_n$. The play is finished after ω-many moves and P wins if $\bigcup_{n\in\omega} U_n = X$. If $\bigcup_{n\in\omega} U_n \neq X$, then O wins the play $\{x_n, U_n : n \in \omega\}$. The game $PO(X)$ is said to be determined on a space X if one of the players has a winning strategy.

In the paper [7] Pawlikowski gave a complete description of spaces X of countable pseudocharacter in which the game $PO(X)$ is undetermined: this happens if and only if X is uncountable and has the Rothberger property C''. In particular, the game $PO(X)$ is undetermined on an uncountable set $X \subset \mathbb{R}$ if and only if X is a C''-set. It follows from a result of Laver [6] that there exist models of ZFC

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in which every C''-subset of $\mathbb{R}$ is countable so it is consistent with ZFC that the game $PO(X)$ is determined on every set $X \subset \mathbb{R}$.

Telgársky established in [8] that if X is a σ-Čech-complete or pseudocompact space then $PO(X)$ is determined on X. Later in [9] he gave a ZFC example of a non-metrizable space X on which $PO(X)$ is undetermined. Daniels and Gruenhage [2] as well as Baldwin [1] studied the point-open game of uncountable length.

In this paper we consider a variation $PO(\mathcal{S}, X)$ of the game $PO(X)$ where $\mathcal{S}$ is a fixed subbase of the space X. The game $PO(\mathcal{S}, X)$ is played exactly as $PO(X)$ with the only difference that at every move Player O must pick an element of $\mathcal{S}$. Of course, the first question that must be answered about the game $PO(\mathcal{S}, X)$ is how different it is from $PO(X)$. We will show that, for any subbase $\mathcal{S}$, Player P has a winning strategy in $PO(\mathcal{S}, X)$ if and only if he has one in $PO(X)$. However, these games are not equivalent for Player O: there exists even a discrete space X with a subbase $\mathcal{S}$ such that neither P nor O has a winning strategy in the game $PO(\mathcal{S}, X)$. We also establish that a discrete space X of a measurable cardinality is determined: for any subbase $\mathcal{S}$ in X, Player O has a winning strategy in the game $PO(\mathcal{S}, X)$.

Given a compact space X and a subbase $\mathcal{S}$ in X, we prove that Player O has a winning strategy in $PO(\mathcal{S}, X)$ if and only if X is not scattered; since the same characterization holds for $PO(X)$, for any subbase $\mathcal{S}$ of the space X, the games $PO(\mathcal{S}, X)$ and $PO(X)$ are equivalent for both players.

2. Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the symbol $\tau(X)$ denotes the topology of X and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. If X is a space and $A \subset X$, then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. As usual, $\mathbb{R}$ is the set of reals; the set $\omega\backslash\{0\}$ is denoted by $\mathbb{N}$ and $\mathbb{I} = [0, 1] \subset \mathbb{R}$. The symbol $\mathbb{D}$ stands for the two-point space $\{0, 1\}$ with the discrete topology.

If $\mathcal{P} = \{x_n, U_n : n \in \omega\}$ is a play in the point-open game on a space X, then $\langle x_n, U_n : n \leq k\rangle$ is called an initial segment (or simply segment) of the play $\mathcal{P}$ for any $k \in \omega$.

A strategy of Player P in the point-open game $PO(X)$ on a space X is a function σ with values in X defined on the initial segments of $PO(X)$ called σ-admissible; they are inductively defined as follows. The empty segment is σ-admissible; if $n > 0$, then a segment $\langle x_0, U_0, \ldots, x_n, U_n\rangle$ is σ-admissible if $\langle x_0, U_0, \ldots, x_{n-1}, U_{n-1}\rangle$ is σ-admissible and $x_n = \sigma(x_0, U_0, \ldots, x_{n-1}, U_{n-1})$. The definition of a strategy s for Player O is analogous for s-admissible segments $\langle x_0, U_0, \ldots, x_{n-1}, U_{n-1}, x_n\rangle$. A play $\mathcal{P} = \{x_n, U_n : n \in \omega\}$ is called σ-admissible for a strategy σ of Player P if every initial segment of $\mathcal{P}$ is σ-admissible; in this case we will also say that P *applies the strategy* σ. An s-admissible play for a strategy s of Player O is defined analogously. A strategy σ of Player P is *winning* on X if P wins in any σ-admissible play. Analogously, a strategy s of Player O is winning on X if O is the winner in any s-admissible play.

A game $PO(X)$ or $PO(\mathcal{S}, X)$ is *undetermined* on a space X if neither of the players P and O has a winning strategy in the respective game on X. If a game is considered on a space X and A is one of the players, then X is called *A-favorable* if A has a winning strategy on X. We say that a space X is *crowded* if it has no isolated points. The space X is *scattered* if every non-empty subspace of X has an isolated point. The rest of our notation is standard and the unexplained notions can be found in the book [3].

3. Point-open subbase game

The point-open subbase game requires the player O to pick larger sets than in the point-open game so it is formally easier to win for the player P. Our main purpose is to establish that the point-open subbase game is equivalent to the point-open game for the player P while it might not be equivalent for the player O even in a discrete space.

The following statement is evident.

3.1 Proposition. *For any space X,*

(a) *if P has a winning strategy in the game $PO(X)$, then the same strategy is winning in the game $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ in the space X;*

(b) *if O has a winning strategy in the game $PO(\mathcal{S}, X)$ for some subbase $\mathcal{S}$ in the space X, then the same strategy is winning in the game $PO(X)$.*

Denote by $FO(X)$ the game in which the first player (called F) at the n-th move picks a finite set $F_n \subset X$ and the second player (called O) chooses an open set $U_n \supset F_n$. The play is finished after ω-many moves and F wins if $\bigcup_{n\in\omega} U_n = X$; otherwise O is the winner. The game $FO(X)$ is equivalent to $PO(X)$ for both players (see Corollary 4.3 and Corollary 4.4 of the paper [8]) so it can be used instead of $PO(X)$ when it is convenient.

3.2 Proposition. *Assume that X is a space and $\mathcal{S}$ is a subbase in X. If Player P has a winning strategy in the game $PO(\mathcal{S}, X)$, then Player F has a winning strategy in the game $FO(X)$.*

Proof: Let ρ be a winning strategy of P in $PO(\mathcal{S}, X)$. For any finite set $F \subset X$ and $U \in \tau(F, X)$ fix a finite family $\mathcal{A}(U, F) \subset \mathcal{S}$ such that for each $x \in F$ there exists a subfamily $\mathcal{B} \subset \mathcal{A}(U, F)$ with $x \in \bigcap \mathcal{B} \subset U$.

To construct a strategy σ for Player F in the game $FO(X)$ take the point $x_0 = \rho(\emptyset)$ and consider the set $F_0 = \{x_0\}$; letting $\sigma(\emptyset) = F_0$ we define the strategy for the first move of F. Given any $U_0 \in \tau(F_0, X)$ define $\sigma(F_0, U_0)$ to be the set $F_1 = \{\rho(x_0, S) : S \in \mathcal{A}(U_0, F_0)\}$.

Proceeding inductively, assume that $n \in \mathbb{N}$ and the strategy σ has been defined for every move $i \leq n$ in such a way that for any $i < n$ and any σ-admissible initial segment $\langle F_0, U_0, \dots, F_i, U_i\rangle$ we have the following property:

(1) if a point $x_j \in F_j$ and a set $S_j \in \mathcal{A}(U_j, F_j)$ are chosen for every $j \leq n$ in such a way that the segment $\langle x_j, S_j : j \leq i\rangle$ is ρ-admissible, then $\rho(x_0, S_0, \dots, x_i, S_i) \in F_{i+1}$.

Given an arbitrary σ-admissible segment $\langle F_0, U_0, \ldots, F_{n-1}, U_{n-1}, F_n\rangle$ take any $U_n \in \tau(F_n, X)$ and consider the family $\mathcal{E} = \{I : I = \langle x_0, S_0, \ldots, x_n, S_n\rangle$ is a ρ-admissible segment such that $x_i \in F_i$ and $S_i \in \mathcal{A}(U_i, F_i)$ for every $i \leq n\}$. It is clear that $\mathcal{E}$ is finite so letting $F_{n+1} = \sigma(F_0, U_0, \ldots, F_n, U_n) = \{\rho(I) : I \in \mathcal{E}\}$ we define our strategy σ for the move $n+1$ and it is straightforward that the property (1) holds if we replace n with $n+1$. Therefore the construction of our strategy σ is complete and the condition (1) is satisfied for any $n \in \mathbb{N}$.

To see that σ is winning, suppose that $\{F_i, U_i : i \in \omega\}$ is a play in which F applies the strategy σ and there exists a point $p \in X \setminus \bigcup_{n\in\omega} U_n$. It follows from $p \notin U_0$ and the definition of $\mathcal{A}(U_0, F_0)$ that there exists $S_0 \in \mathcal{A}(U_0, F_0)$ such that $x_0 \in U_0$ and $p \notin S_0$. Proceeding by induction assume that, for some $n \in \omega$, we have a ρ-admissible initial segment $\langle x_0, S_0, \ldots, x_n, S_n\rangle$ such that $x_i \in F_i \cap S_i$ while $S_i \in \mathcal{A}(U_i, F_i)$ and $p \notin S_i$ for every $i \leq n$. It follows from (1) that $x_{n+1} = \rho(x_0, S_0, \ldots, x_n, S_n) \in F_{n+1} \subset U_{n+1}$ so it follows from $p \notin U_{n+1}$ that we can choose $S_{n+1} \in \mathcal{A}(U_{n+1}, F_{n+1})$ such that $x_{n+1} \in S_{n+1}$ and $p \notin S_{n+1}$.

Therefore our inductive procedure can be continued to construct a play $\{x_i, S_i : i \in \omega\}$ in the game $PO(\mathcal{S}, X)$ where P applies the strategy ρ and $p \notin S_i$ for every $i \in \omega$. However, this implies that $p \notin \bigcup_{i\in\omega} S_i$ which is a contradiction with the fact that ρ is a winning strategy. This shows that $\bigcup_{n\in\omega} U_n = X$ and hence σ is also a winning strategy. □

3.3 Theorem. *If X is a space and $\mathcal{S}$ is a subbase in X, then the games $PO(X)$ and $PO(\mathcal{S}, X)$ are equivalent for P, i.e., Player P has a winning strategy in the game $PO(X)$ if and only if he has a winning strategy in the game $PO(\mathcal{S}, X)$.*

Proof: Since the game $FO(X)$ is equivalent to the game $PO(X)$ for both players, the games $PO(X)$ and $PO(\mathcal{S}, X)$ are equivalent for Player P by Proposition 3.1(a) and Proposition 3.2. □

3.4 Corollary. *If $PO(X)$ is undetermined on a space X, then so is $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ of the space X.*

Proof: It suffices to observe that, for such a space X, Player P does not have a winning strategy by Theorem 3.3 and Player O has no winning strategy by Proposition 3.1(b). □

Recall that X is a P-space if every G_δ-subset of X is open.

3.5 Observations. Telgársky constructed in [9] a Lindelöf P-space X on which $PO(X)$ is undetermined. By Corollary 3.4, on the same space X the game $PO(\mathcal{S}, X)$ is undetermined for any subbase $\mathcal{S}$.

In [7], a complete characterization was given by Pawlikowski for the game $PO(X)$ to be undetermined on a space X of countable pseudocharacter. In particular, the game $PO(M)$ is undetermined on a set $M \subset \mathbb{R}$ if and only if $|M| > \omega$ and M is a C''-set, i.e., for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of M, there exists a sequence $\{U_n : n \in \omega\} \subset \tau(X)$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\bigcup_{n\in\omega} U_n = M$. Therefore the game $PO(\mathcal{S}, M)$ is undetermined on a set

$M \subset \mathbb{R}$ for every subbase $\mathcal{S}$ of M if M is a C''-set. We will see later that the above implication cannot be reversed.

Telgársky proved in [8] that for every Lindelöf scattered space X, Player P has a winning strategy in the game $PO(X)$. He also established in [8] that a compact space X is scattered if and only if Player O has a winning strategy in $PO(X)$. As an immediate consequence, the game $PO(X)$ is determined on the class of compact spaces. We will show that the same is true for the game $PO(\mathcal{S}, X)$ whenever X is compact and $\mathcal{S}$ is a subbase in X.

3.6 Theorem. *Assume that a space X has a pseudocompact crowded subspace. Then Player O has a winning strategy in $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ in the space X.*

Proof: Let Y be a pseudocompact crowded subspace of X; since $\overline{Y}$ is also pseudocompact and crowded, we can consider that Y is closed in X. We will use the following trivial observation.

(2) If Z is a space and $\mathcal{G}$ is a finite family of closed subsets of Z such that the interior of $\bigcup \mathcal{G}$ is non-empty, then the interior of G is non-empty for some $G \in \mathcal{G}$.

The set Y is infinite being crowded, so for any point $x \in X$ we can find a set $U \in \tau^*(Y)$ such that $x \notin \overline{U}$. There exists a finite family $\mathcal{F} \subset \mathcal{S}$ such that $x \in \bigcap \mathcal{F} \subset X \backslash \overline{U}$. It follows from $\overline{U} \subset \bigcup\{X \backslash S : S \in \mathcal{F}\}$ that we can apply (2) to find a set $V \in \tau^*(Y)$ such that $\overline{V} \subset (X \backslash S) \cap U$. This proves that

(3) for any point $x \in X$, if U is a non-empty open subset of Y, then we can find $S \in \mathcal{S}$ and a non-empty open subset V of Y such that $x \in S$ and $\overline{V} \subset U \backslash S$.

Now it is easy to construct a winning strategy σ for Player O. If P chooses a point $x_0 \in X$, we can apply (3) to find a set $S_0 \in \mathcal{S}$ and $U_0 \in \tau^*(Y)$ such that $x_0 \in S_0 \subset X \backslash \overline{U}_0$; let $\sigma(x_0) = S_0$. Proceeding by induction assume that $n \in \omega$ and the strategy σ is constructed for the first n moves in such a way that

(4) for any σ-admissible segment $\langle x_0, S_0, \ldots, x_n, S_n \rangle$ we have defined a family $\{U_0, \ldots, U_n\}$ of non-empty open subsets of Y such that $U_i \cap S_i = \emptyset$ for every $i \leq n$ and $\overline{U}_{i+1} \subset U_i$ if $i < n$.

If the move of Player P is a point $x_{n+1} \in X$, then (3) can be applied again to find a set $S_{n+1} \in \mathcal{S}$ and $U_{n+1} \in \tau^*(Y)$ such that $x_{n+1} \in S_{n+1}$, $\overline{U}_{n+1} \subset U_n$ and $U_{n+1} \cap S_{n+1} = \emptyset$. Letting $\sigma(x_0, S_0, \ldots, x_n, S_n, x_{n+1}) = S_{n+1}$ we complete the definition of the strategy σ and it is immediate that (4) holds for all $n \in \omega$.

Finally, assume that $\{x_i, S_i : i \in \omega\}$ is a σ-admissible play. The definition of σ implies existence of a sequence $\{U_i : i \in \omega\} \subset \tau^*(Y)$ such that $\overline{U}_{i+1} \subset U_i$ and $U_i \cap S_i$ for every $i \in \omega$. It follows from pseudocompactness of Y that $\bigcap_{n\in\omega} U_n \neq \emptyset$. The property (4) guarantees that $(\bigcap_{n\in\omega} U_n) \cap (\bigcup_{n\in\omega} S_n) = \emptyset$ so $\bigcup_{n\in\omega} S_n \neq X$ and hence σ is a winning strategy. □

3.7 Corollary. *If X is a compact space and $\mathcal{S}$ is a subbase of X, then the following conditions are equivalent:*

(a) *X is scattered;*
(b) *Player P has a winning strategy in the game $PO(\mathcal{S}, X)$;*
(c) *Player O has no winning strategy in the game $PO(\mathcal{S}, X)$.*

Proof: We have already mentioned that the implication (a)$\Longrightarrow$(b) is true for the game $PO(X)$ (see [8, Corollary 9.5]) so it is true for $PO(\mathcal{S}, X)$ by Theorem 3.3. The implication (b)$\Longrightarrow$(c) is trivial and (c)$\Longrightarrow$(a) is an immediate consequence of Theorem 3.6. □

It follows from Theorem 2 of the paper [4] and Theorem 3.3 that Player P has no winning strategy in the game $PO(\mathcal{S}, X)$ if X is an uncountable space of countable pseudocharacter and $\mathcal{S}$ is a subbase of X. The same conclusion follows from the main result of the paper of Pawlikowski [7].

In particular, if X is a discrete uncountable space, then Player P has no winning strategy in the game $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ in X; for such an X, it is easy to see that Player O always has a winning strategy in the game $PO(X)$. We will show that this is not the case for the game $PO(\mathcal{S}, X)$.

3.8 Theorem. *Suppose that $X \subset \mathbb{I}$ and for any compact $K \subset \mathbb{I}$, if $K \subset X$ or $K \subset \mathbb{I}\backslash X$, then K is countable. Such X are called Bernstein sets and it is well known that they exist. Consider the families $\mathcal{S}_0 = \{[0, x] \cap X : x \in X\}$ and $\mathcal{S}_1 = \{[x, 1] \cap X : x \in X\}$; then $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ is a subbase for the discrete topology on X and neither of the players has a winning strategy in the game $PO(\mathcal{S}, X)$. In particular the discrete space X of cardinality $\mathfrak{c}$ admits a subbase $\mathcal{S}$ such that the game $PO(\mathcal{S}, X)$ is undetermined on X.*

Proof: It is trivial that $\mathcal{S}$ is a subbase for the discrete topology on X so, from now on we provide X with the discrete topology. Observe first that Player P has no winning strategy in the game $PO(\mathcal{S}, X)$, due to Theorem 3.3 and the fact that P has no winning strategy in the game $PO(X)$ by [4, Theorem 2]. Striving for a contradiction, assume that Player O has a winning strategy σ in the game $PO(\mathcal{S}, X)$. In what follows "initial segment" or simply "segment" will mean "a σ-admissible segment of a play in $PO(\mathcal{S}, X)$."

Given initial segments $I = \langle x_0, S_0, \ldots, x_n, S_n\rangle$ and $I' = \langle y_0, T_0, \ldots, y_m, T_m\rangle$ of a play in $PO(\mathcal{S}, X)$, we say that I' extends I if $I \subset I'$. For any initial segment I of a play in $PO(\mathcal{S}, X)$ let $\mathcal{E}(I) = \{J : J \supset I$ is an initial segment$\}$. Let $\mathcal{E} = \mathcal{E}(\emptyset)$ be the family of all initial segments of the game $PO(\mathcal{S}, X)$. Since the strategy σ is winning,

(5) if $I = \langle x_0, S_0, \ldots, x_n, S_n\rangle \in \mathcal{E}$, then $H(I) = X\backslash \bigcup\{S_i : i \leq n\}$ is dense (with respect to the natural topology) in a non-trivial closed interval.

We claim that

(6) for any initial segment $I \in \mathcal{E}$, there exist segments $I_0, I_1 \in \mathcal{E}(I)$ such that $\overline{H(I_0)} \cap \overline{H(I_1)} = \emptyset$ (the bar denotes the closure in $\mathbb{I}$).

To see that the statement (6) is true assume that there exists a segment $I = \langle x_0, S_0, \ldots, x_n, S_n\rangle \in \mathcal{E}$ such that $\overline{H(I_0)} \cap \overline{H(I_1)} \neq \emptyset$ for any $I_0, I_1 \in \mathcal{E}(I)$. It is easy to see that this implies that $F = \bigcap\{\overline{H(J)} : J \in \mathcal{E}(I)\} \neq \emptyset$; fix a point $r \in F$. We have two cases to consider.

Case 1. $r \in X$. Let $x_{n+1} = r$ and $S_{n+1} = \sigma(x_0, S_0, \dots, x_n, S_n, x_{n+1})$. We will inductively extend the segment $I_0 = \langle x_0, S_0, \dots, x_{n+1}, S_{n+1}\rangle$ to a play $\mathcal{P}$ in which O applies the strategy σ. We will only have to choose a point x_i and then the strategy σ will automatically give us the set $S_i = \sigma(x_0, S_0, \dots, x_{i-1}, S_{i-1}, x_i)$ for any $i > n+1$.

If $i \geq n+1$ and we have the segment $I = \langle x_0, S_0, \dots, x_i, S_i\rangle$, then it follows from $I \in \mathcal{E}(I_0)$ and the fact that the strategy σ is winning, that the set $H(I)$ is uncountable; since also $r \in \overline{H(I)}$, we can choose a point $x_{i+1} \in H(I)$ such that $|x_{i+1} - r| < 2^{-i}$. If $S_{i+1} = \sigma(x_0, S_0, \dots, x_i, S_i, x_{i+1})$ and S_{n+1} both belong to $\mathcal{S}_j$ for some $j \in \mathbb{D}$, then it follows from $x_{i+1} \in S_{i+1} \backslash S_{n+1}$ that $S_{n+1} \subset S_{i+1}$ and r is not the endpoint of the set S_{i+1}; this implies $r \notin \overline{H(J)}$ for the segment $J = \langle x_0, S_0, \dots, x_{i+1}, S_{i+1}\rangle$ which is a contradiction. Therefore, for some element $j \in \mathbb{D}$, we have $S_{n+1} \in \mathcal{S}_j$ and $S_{i+1} \in \mathcal{S}_{1-j}$; if $S_{n+1} \cap S_{i+1} \neq \emptyset$, then $S_{n+1} \cup S_{i+1} = X$ which is impossible because the strategy σ is winning so $S_{n+1} \cap S_{i+1} = \emptyset$ for any $i > n$.

Finally observe that the sequence $\{x_i : i > n+1\}$ converges to r and all of its elements remain on the same side from r; this easily implies that $\bigcup_{i \geq n+1} S_i = X$ which is again a contradiction with the fact that σ is a winning strategy.

Case 2. $r \notin X$. Choose a sequence $\{x_i : i \geq n+1\} \subset X$ which converges to r with the additional property that both sets $\{i \geq n+1 : x_i > r\}$ and $\{i \geq n+1 : x_i < r\}$ are infinite. If $\{x_i, S_i : i \in \omega\}$ is the play where O applies the strategy σ, then r cannot be the endpoint of any S_i. Therefore, if $i > n$ and $r \in S_i$, then r cannot belong to the closure of the set $H(I)$ for $I = \langle x_0, S_0, \dots, x_i, S_i\rangle$; this contradiction shows that $[x_i, 1] \cap X \subset S_i \subset (r, 1]$ if $x_i > r$ and $[0, x_i] \cap X \subset S_i \subset [0, r)$ if $x_i < r$. As an immediate consequence, $\bigcup_{i \in \omega} S_i = X$ which is once more a contradiction with the fact that σ is a winning strategy so the property (6) is proved.

Given any segment $I = \langle x_0, S_0, \dots, x_n, S_n\rangle \in \mathcal{E}$ observe that $\overline{H(I)}$ is an interval $[a, b]$ for some $a, b \in \mathbb{I}$ so we can choose a point $x_{n+1} \in H(I) \cap [a, b]$ in such a way that the length each of the intervals $[a, x_{n+1}]$ and $[x_{n+1}, b]$ does not exceed $\frac{2}{3}(b-a)$. Repeating such a choice the necessary number of times we can see that the following stronger version of the property (6) holds:

(7) for any $\varepsilon > 0$ and any initial segment $I \in \mathcal{E}$, there exist initial segments $I_0, I_1 \in \mathcal{E}(I)$ such that $\overline{H(I_0)} \cap \overline{H(I_1)} = \emptyset$ and the diameter of the set $\overline{H(I_j)}$ is less than ε for every $j \in \mathbb{D}$.

Take any point $z \in X$ and let $I_\emptyset = \{z, \sigma(z)\}$. Proceeding inductively, assume that $n \in \omega$ and we have constructed an initial segment I_s for any $s \in \bigcup\{\mathbb{D}^m : m \leq n\}$ in such a way that

(8) for any $m \leq n$, the family $\{\overline{H(I_s)} : s \in \mathbb{D}^m\}$ is disjoint;

(9) if $m \leq n$ and $s \in \mathbb{D}^m$, then the diameter of $\overline{H(I_s)}$ does not exceed 2^{-m};

(10) if $s \subset t$, then I_t is an extension of I_s.

For any $s \in \mathbb{D}^n$ apply the property (7) to find extensions I' and I'' of the segment I_s such that $\mathrm{diam}(\overline{H(I')}) < 2^{-n-1}$ and $\mathrm{diam}(\overline{H(I')}) < 2^{-n-1}$ while $\overline{H(I')} \cap \overline{H(I'')} = \emptyset$ and let $I_{s\frown 0} = I'$ and $I_{s\frown 1} = I''$. This gives us the family $\{I_s : s \in \bigcup\{\mathbb{D}^m : m \leq n+1\}\}$ and it is immediate that (8)–(10) are still fulfilled if we replace n with $n+1$. Therefore our inductive procedure can be continued to construct the family $\{I_s : s \in \mathbb{D}^{<\omega}\}$ such that the conditions (8)–(10) are satisfied for all $n \in \omega$.

The set $K_n = \bigcup\{\overline{H(I_s)} : s \in \mathbb{D}^n\}$ is compact and $K_{n+1} \subset K_n$ for all $n \in \omega$; it is standard to deduce from (8)–(10) that $K = \bigcap\{K_n : n \in \omega\}$ is homeomorphic to the Cantor set. If $x \in K$, then there is a unique function $f \in \mathbb{D}^\omega$ such that $\{x\} = \bigcap\{\overline{H(I_{f|n})} : n \in \omega\}$. The property (10) shows that there exists a play $\mathcal{P} = \{x_n, S_n : n \in \omega\}$ in which O applied the strategy σ and $I_{f|n}$ is an initial segment of $\mathcal{P}$ for any $n \in \omega$. The equality $\{x\} = \bigcap\{\overline{H(I_{f|n})} : n \in \omega\}$ shows that $X \backslash \{x\} \subset \bigcup_{n\in\omega} S_n$; since the strategy σ is winning, we must have $x \in X$. This proves that $K \subset X$ which is a contradiction. □

3.9 Corollary. *There exists a space $X \subset \mathbb{I}$ such that $PO(X)$ is determined on X but $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ of X. In particular, Pawlikowski's characterization does not hold for the game $PO(\mathcal{S}, X)$.*

Proof: Let $Z \subset \mathbb{I}$ be a set such that for any compact $K \subset \mathbb{I}$, if $K \subset Z$ or $K \subset \mathbb{I}\backslash Z$, then K is countable. Since the Rothberger property C'' is trivially preserved by finite unions, both sets Z and $\mathbb{I}\backslash Z$ cannot have the property C'' because $\mathbb{I}$ does not have it. So, one of them, let us call it X, is not a C''-set and hence Player O has a winning strategy in $PO(X)$ by Pawlikowski's theorem [7]. Therefore it suffices to show that Player O does not have a winning strategy in $PO(\mathcal{S}, X)$ for some subbase $\mathcal{S}$ in the space X.

Let $\mathcal{Q}_0 = \{[0, x] \cap X : x \in X\}$ and $\mathcal{Q}_1 = \{[x, 1] \cap X : x \in X\}$; by Theorem 3.8, Player O does not have a winning strategy in the game $PO(\mathcal{Q}, X)$ for the family $\mathcal{Q} = \mathcal{Q}_0 \cup \mathcal{Q}_1$. Let $\mathcal{S}_0 = \{[0, x) \cap X : x \in X\}$ and $\mathcal{S}_1 = \{(x, 1] \cap X : x \in X\}$; since X is dense in $\mathbb{I}$, the family $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ is easily seen to be a subbase of X. Suppose that σ is a winning strategy in $PO(\mathcal{S}, X)$. If $y \in X$ and $y \in U = [0, x) \cap X$ for some $x \in X$, then let $H(U, y) = [0, y] \cap X$. Analogously, if $y \in U = (x, 1] \cap X$ for some $x \in X$, then $H(U, y) = [y, 1] \cap X$.

Now, if we consider X to have the discrete topology, then it is easy to define inductively a strategy s for Player O in the game $PO(\mathcal{Q}, X)$ in such a way that for any s-admissible segment $I = \langle x_0, U_0, \ldots, x_{n-1}, U_{n-1}, x_n\rangle$ there exists a σ-admissible segment $J = \langle x_0, W_0, \ldots, x_{n-1}, W_{n-1}, x_n\rangle$ for which $U_i = H(W_i, x_i)$ for all $i \leq n-1$ and $s(I) = H(\sigma(J), x_n)$. The strategy s cannot be winning by Theorem 3.8 and hence there exists a σ-admissible play $\{x_n, W_n : n \in \omega\}$ such that $\bigcup_{n\in\omega} H(W_n, x_n) = X$. Observing that $H(W_n, x_n) \subset W_n$ for every $n \in \omega$, we conclude that $\bigcup_{n\in\omega} W_n = X$ which is a contradiction. □

The referee observed that it would be interesting to find out for a space X of countable pseudocharacter what conditions a subbase $\mathcal{S}$ in X must satisfy to guarantee that the game $PO(\mathcal{S}, X)$ is undetermined on X if and only if X is a C''-space; this would generalize Pawlikowski's theorem from [7]. We do not know the answer to this question. The referee also asked what replaces the Rothberger property if there are no restrictions on a subbase $\mathcal{S}$. We cannot answer this question either but it is worth noting that it follows from Theorem 3.8 that for a discrete space X of cardinality $\mathfrak{c}$, the game $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ in X. Therefore in this case the space X need not even be Lindelöf so if anything replaces the Rothberger property, it will be something very different.

Recall that a cardinal κ is called *measurable* if there exists a free σ-complete ultrafilter on κ.

3.10 Theorem. *If κ is a measurable cardinal and X is a discrete space of cardinality κ, then Player O has a winning strategy in the game $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ of the space X.*

PROOF: Fix a free ultrafilter μ on X which is σ-complete, i.e., closed under countable intersections and let $\mathcal{S}$ be any subbase for the discrete topology on X. Given any $n \in \omega$, if at the n-th move Player P picks a point $x_n \in X$, then there is a finite family $\mathcal{B}_n \subset \mathcal{S}$ such that $\bigcap \mathcal{B}_n = \{x_n\}$. If $\mathcal{B}_n \subset \mu$ then $\{x_n\} \in \mu$ which is a contradiction.

Therefore for any $n \in \omega$ there exists $S_n \in \mathcal{B}_n \backslash \mu$ and hence we can let $\sigma(x_0, S_0, \dots, x_{n-1}, S_{n-1}, x_n) = S_n$. If $\{x_n, S_n : n \in \omega\}$ is a play where O applies σ, then $X \backslash S_n \in \mu$ for any $n \in \omega$. The ultrafilter μ being σ-complete, the set $\bigcap_{n\in\omega} X \backslash S_n = X \backslash \bigcup_{n\in\omega} S_n$ belongs to μ and hence $X \neq \bigcup_{n\in\omega} S_n$ which shows that σ is a winning strategy for Player O. $\square$

In the paper [4] Galvin introduced a game $G^*(X)$ and proved that it is equivalent to $PO(X)$ for both players. In $G^*(X)$, at the n-th move Player P chooses an open cover $\mathcal{U}_n$ of the space X and O responds by taking a set $U_n \in \mathcal{U}_n$. As in $PO(X)$, Player P wins if $\bigcup_{n\in\omega} U_n = X$; otherwise O is the winner. The following game $CE(\mathcal{S}, X)$ is a modification of $G^*(X)$ such that $G^*(X) = CE(\mathcal{S}, X)$ for $\mathcal{S} = \tau(X)$. It follows from Theorem 3.8 that the games $PO(X)$ and $PO(\mathcal{S}, X)$ need not be equivalent for Player O so it is not immediately clear whether passing from $G^*(X)$ to $CE(\mathcal{S}, X)$ we must obtain a game equivalent to $PO(\mathcal{S}, X)$. However, we will show that the ideas from [4] still work for our modification and hence the game $CE(\mathcal{S}, X)$ is equivalent to $PO(\mathcal{S}, X)$ for both players.

3.11 Definition. Given a space X and a subbase $\mathcal{S}$ in X, in the game $CE(\mathcal{S}, X)$ we have Players C and E who at the n-th move take an open cover $\mathcal{U}_n \subset \mathcal{S}$ of the space X and an element $U_n \in \mathcal{U}_n$ respectively. The game stops after ω-many moves are made and the play $\{\mathcal{U}_n, U_n : n \in \omega\}$ is a win for Player E if $\bigcup_{n\in\omega} U_n = X$; otherwise C is the winner.

3.12 Theorem. *Given a space X and a subbase $\mathcal{S}$ of X,*

(a) *Player P has a winning strategy in $PO(\mathcal{S},X)$ if and only if E has a winning strategy in the game $CE(\mathcal{S},X)$;*
(b) *Player O has a winning strategy in $PO(\mathcal{S},X)$ if and only if C has a winning strategy in the game $CE(\mathcal{S},X)$.*

Proof: (a) If Player P has a winning strategy in $PO(\mathcal{S},X)$, then he has a winning strategy in $PO(X)$ by Theorem 3.3. By [4, Theorem 1], Player E has a winning strategy in $CE(\tau(X),X)$ which, evidently, implies that he has a winning strategy in $CE(\mathcal{S},X)$.

Next assume that X is E-favorable and fix a winning strategy s for Player E in the game $CE(\mathcal{S},X)$; let $\mathcal{S}(x)=\{S\in\mathcal{S}: x\in S\}$ for every $x\in X$. It turns out that

(11) if $I=\langle\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n\rangle$ is an s-admissible initial segment of $CE(\mathcal{S},X)$ (which can be empty), then there exists a point $p\in X$ such that for every set $S\in\mathcal{S}(p)$, there exists a cover $\mathcal{U}(S)\subset\mathcal{S}$ of the space X such that $S=s(I,\mathcal{U}(S))$.

Indeed, assume that for any $x\in X$ there exists a set $S_x\in\mathcal{S}(x)$ such that $S_x\neq s(I,\mathcal{U})$ for any cover $\mathcal{U}\subset\mathcal{S}$ of the space X. Then $\mathcal{U}=\{S_x: x\in X\}\subset\mathcal{S}$ is a cover of X and hence we have a point $p\in X$ such that $\sigma(I,\mathcal{U})=S_p$; this contradiction proves that (11) holds.

Apply (11) to find $x_0\in X$ such that $\mathcal{S}(x_0)\subset\{\sigma(\mathcal{U}):\mathcal{U}\subset\mathcal{S}$ and $\bigcup\mathcal{U}=X\}$ and let $\sigma(\emptyset)=x_0$. If O plays $U_0\in\mathcal{S}(x_0)$, then choose a cover $\mathcal{U}_0\subset\mathcal{S}$ such that $U_0=s(\mathcal{U}_0)$. Suppose that $n\in\omega$ and we have defined a strategy σ for the moves from 0 to n in such a way that for any σ-admissible initial segment $\langle x_0,U_0,\dots,x_n,U_n\rangle$ of the game $PO(\mathcal{S},X)$ we have covers $\mathcal{U}_0,\dots,\mathcal{U}_n$ of the space X such that the segment $\langle\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n\rangle$ is s-admissible. Apply (11) again to find $x_{n+1}\in X$ such that $\mathcal{S}(x_{n+1})\subset\{s(\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n,\mathcal{U}):\mathcal{U}\subset\mathcal{S}$ and $\bigcup\mathcal{U}=X\}$ and let $\sigma(x_0,U_0,\dots,x_n,U_n)=x_{n+1}$. If Player O takes a set $U_{n+1}\ni x_{n+1}$, then we can choose a cover $\mathcal{U}_{n+1}\subset\mathcal{S}$ of the space X such that $U_{n+1}=s(\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n,\mathcal{U}_{n+1})$. This completes the definition of the strategy σ.

To see that σ is winning note that to any σ-admissible play $\{x_n,U_n:n\in\omega\}$ we have associated an s-admissible play $\{\mathcal{U}_n,U_n:n\in\omega\}$ so $\bigcup_{n\in\omega}U_n=X$, i.e., the strategy σ is winning. Therefore every E-favorable space is P-favorable. This completes the proof of (a).

(b) If Player O has a winning strategy σ in the game $PO(\mathcal{S},X)$, then let $\mathcal{U}_0=\{\sigma(x):x\in X\}$ and $s(\emptyset)=\mathcal{U}_0$. If E chooses a set $U_0\in\mathcal{U}_0$, then there exists a point $x_0\in X$ such that $U_0=\sigma(x_0)$; consider the family $\mathcal{U}_1=\{\sigma(x_0,U_0,x): x\in X\}$ and let $s(\mathcal{U}_0,U_0)=\mathcal{U}_1$. Proceeding inductively, assume that $n\in\omega$ and the strategy s for Player C is defined for the moves from 0 to n in such a way that for any s-admissible initial segment $\langle\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n\rangle$ we have defined a set $\{x_0,\dots,x_n\}$ such that the segment $\langle x_0,U_0,\dots,x_n,U_n\rangle$ is σ-admissible.

Consider the family $\mathcal{U}_{n+1}=\{\sigma(x_0,U_0,\dots,x_n,U_n,x): x\in X\}$ and let $s(\mathcal{U}_0,U_0,\dots,\mathcal{U}_n,U_n)=\mathcal{U}_{n+1}$; if E answers with a set $U_{n+1}\in\mathcal{U}_{n+1}$, then choose the point $x_{n+1} \in X$ such that $U_{n+1} = \sigma(x_0, U_0, \dots, x_n, U_n, x_{n+1})$. This completes the construction of the strategy s.

To see that s is winning note that to any s-admissible play $\{\mathcal{U}_n, U_n : n \in \omega\}$ we have associated a σ-admissible play $\{x_n, U_n : n \in \omega\}$ so $\bigcup_{n\in\omega} U_n \neq X$, i.e., the strategy s is winning. Therefore every O-favorable space is C-favorable.

If s is a winning strategy for Player C, then for any point $x_0 \in X$ let $\sigma(x_0)$ be an element $U_0 \in \mathcal{U}_0 = s(\emptyset)$ that contains x_0. Suppose that $n \in \omega$ and we have defined a strategy σ for the moves from 0 to n in such a way that for any σ-admissible initial segment $\langle x_0, U_0, \dots, x_n, U_n\rangle$ of the game $PO(\mathcal{S}, X)$ we have constructed open covers $\mathcal{U}_0, \dots, \mathcal{U}_n \subset \mathcal{S}$ of the space X such that the segment $\langle \mathcal{U}_0, U_0, \dots, \mathcal{U}_n, U_n\rangle$ is s-admissible. For any point $x_{n+1} \in X$ choose an element $U_{n+1} \in \mathcal{U}_{n+1} = s(\mathcal{U}_0, U_0, \dots, \mathcal{U}_n, U_n)$ such that $x_{n+1} \in U_{n+1}$; letting $\sigma(x_0, U_0, \dots, x_n, U_n, x_{n+1}) = U_{n+1}$ we complete the definition of a strategy σ. To see that σ is winning observe that to any σ-admissible play $\{x_n, U_n : n \in \omega\}$ we have associated an s-admissible play $\{\mathcal{U}_n, U_n : n \in\ o\}$ so $\bigcup_{n\in\omega} U_n \neq X$, i.e., the strategy σ is winning. Therefore every C-favorable space is O-favorable. This completes the proof of (b). □

3.13 Corollary. *Given a space X and a subbase $\mathcal{S}$ in X, the games $CE(\mathcal{S}, X)$ and Galvin's game $G^*(X) = CE(\tau(X), X)$ are equivalent for Player E, i.e., E has a winnings strategy in $CE(\mathcal{S}, X)$ if and only if he has one in $G^*(X)$.*

Proof: It follows from [4, Theorem 1] that Player E has a winning strategy in the game $G^*(X)$ if and only if P has a winning strategy in $PO(X)$. By Theorem 3.3 the game $PO(X)$ is equivalent to $PO(\mathcal{S}, X)$ for Player P. Applying Theorem 3.12 we can see that $G^*(X)$ is equivalent to $CE(\mathcal{S}, X)$ for Player E. □

4. Open problems

A proof of a statement about discrete spaces usually involves no topology; it is all about set theory. Therefore most questions about discrete spaces belong more to set theory than to topology. In particular, this is the case when we consider the game $PO(\mathcal{S}, X)$ on discrete spaces. The most intriguing fact is that the point-open subbase game might be useful for a purely set-theoretic task of characterizing measurable cardinals.

4.1 Question. *Suppose that X is a discrete space such that Player O has a winning strategy in the game $PO(\mathcal{S}, X)$ for every subbase $\mathcal{S}$ in X. Must the cardinality of X be measurable?*

4.2 Question. *Suppose that X is a discrete space of cardinality $2^{\mathfrak{c}}$. Does there exist a subbase $\mathcal{S}$ in X for which Player O has no winning strategy in the game $PO(\mathcal{S}, X)$?*

4.3 Question. *Suppose that X is an uncountable discrete space whose cardinality is non-measurable. Does there exist a linear order $<$ on the set X such that, for the subbase*

$\mathcal{S} = \{\{y \in X : y \le x\} : x \in X\} \cup \{\{y \in X : x \le y\} : x \in X\}$,
Player O has no winning strategy in the game $PO(\mathcal{S}, X)$?

4.4 Question. *Suppose that X is a discrete space of uncountable cardinality such that Player O has no winning strategy in the game $PO(\mathcal{B}, X)$ for some subbase $\mathcal{B}$ in X. Does there exist a linear order $<$ on the set X such that, for the subbase*

$\mathcal{S} = \{\{y \in X : y \le x\} : x \in X\} \cup \{\{y \in X : x \le y\} : x \in X\}$,
Player O has no winning strategy in the game $PO(\mathcal{S}, X)$?

4.5 Question. *Does there exist a pseudocompact space X such that the games $PO(X)$ and $PO(\mathcal{S}, X)$ are not equivalent for Player O for some subbase $\mathcal{S}$ in the space X?*

4.6 Question. *Does there exist a countably compact space X such that the games $PO(X)$ and $PO(\mathcal{S}, X)$ are not equivalent for Player O for some subbase $\mathcal{S}$ in the space X?*

4.7 Question. *Given a maximal almost disjoint family $\mathcal{N}$ on ω let $X = \omega \cup \mathcal{N}$ be the Mrowka space determined by $\mathcal{N}$ (see* [3, Example 3.6.I(a)]*). Does there exist a subbase $\mathcal{S}$ in X such that Player O has no winning strategy in the game $PO(\mathcal{S}, X)$?*

4.8 Question. *Suppose that X is an uncountable second countable space such that every compact subspace of X is countable. Is it true that the game $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ of the space X?*

4.9 Question. *Suppose that X is an uncountable space with a countable network such that every compact subspace of X is countable. Is it true that the game $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ of the space X?*

4.10 Question. *Suppose that X is an uncountable hereditarily Lindelöf space such that every compact subspace of X is countable. Is it true that the game $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ of the space X?*

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