Characterization of functions whose forward differences are exponential polynomials

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Abstract. Given $\{h_1, \cdots, h_t\}$ a finite subset of $\mathbb{R}^d$, we study the continuous complex valued functions and the Schwartz complex valued distributions f defined on $\mathbb{R}^d$ with the property that the forward differences $\Delta_{h_k}^{m_k} f$ are (in distributional sense) continuous exponential polynomials for some natural numbers $m_1, \cdots, m_t$.

Keywords: functional equations; exponential polynomials; generalized functions; forward differences

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1. Introduction

Let X_d indistinctly denote either the set of continuous complex valued functions $C(\mathbb{R}^d)$ or the set of Schwartz complex valued distributions $\mathcal{D}(\mathbb{R}^d)'$. Let $f \in X_d$ and let us denote by τ_y and Δ_h^m the translation operator and the forward differences operator defined on X_d, respectively. In formulas, $(\tau_y f)(x) = f(x+y)$ and $(\Delta_h^m f)(x) = \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k}\tau_{kh}(f)$ if f is an ordinary function, and $\tau_y f\{\varphi\} = f\{\tau_{-y}(\varphi)\}$, $(\Delta_h^m f)\{\varphi\} = f\{\Delta_{-h}^m(\varphi)\}$ if $f \in \mathcal{D}(\mathbb{R}^d)'$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

We prove that, if $\{h_1, \cdots, h_t\}$ spans a dense subgroup of $\mathbb{R}^d$, $f \in X_d$ and there exist natural numbers $\{m_k\}_{k=1}^t$ such that, for every $k \in \{1, \dots, t\}$, $\Delta_{h_k}^{m_k} f$ is (in distributional sense) a continuous exponential polynomial, then f is (in distributional sense) a continuous exponential polynomial. Moreover, for the case of continuous functions and for arbitrary sets $\{h_1, \cdots, h_t\}$, we characterize the functions f satisfying that $\Delta_{h_k}^{m_k} f$ is an exponential polynomial for some natural numbers $m_1, \cdots, m_t$.

2. The case of finitely generated dense subgroups of $\mathbb{R}^d$

Let us state two technical results, which are important for our arguments in this section. These results were, indeed, recently introduced by the author, and have proved their usefulness for the study of several Montel-type theorems for polynomial and exponential polynomial functions (see, e.g., [1]–[5]). We include the proofs for the sake of completeness.

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Definition 1. Let t be a positive integer, E a vector space, $L_1, L_2, \cdots, L_t : E \to E$ pairwise commuting linear operators. Given a subspace $V \subseteq E$, we denote by $\diamond_{L_1,L_2,\cdots,L_t}(V)$ the smallest subspace of E containing V which is L_i-invariant for $i = 1, 2, \dots, t$.

Lemma 2. *With the notation we have just introduced, if V is an L^n-invariant subspace of E, then the linear space*

$$V_L^{[n]} = V + L(V) + \cdots + L^{n-1}(V)$$

is L-invariant. Furthermore $V_L^{[n]} = \diamond_L(V)$. In other words, $V_L^{[n]}$ is the smallest L-invariant subspace of E containing V.

Proof: Let v be in $V_L^{[n]}$, then

$$v = v_0 + Lv_1 + \cdots + L^{n-2}v_{n-2} + L^{n-1}v_{n-1} \tag{1}$$

with some elements $v_0, v_1, \dots, v_{n-1}$ in V. By the L^n-invariance of V, we have that $L^n v_{n-1} = u$ is in V, hence it follows

$$Lv = u + L(v_0) + L^2 v_1 + \cdots + L^{n-1}v_{n-2}\,,$$

and the right hand side is clearly in $V_L^{[n]}$. This proves that $V_L^{[n]}$ is L-invariant. On the other hand, if W is an L-invariant subspace of E, which contains V, then $L^k(V) \subseteq W$ for $k = 1, 2, \dots, n-1$, hence the right hand side of (1) is in W. $\square$

Lemma 3. *Let t be a positive integer, E a vector space, $L_1, L_2, \cdots, L_t : E \to E$ pairwise commuting linear operators, and let $s_1, \cdots, s_t$ be natural numbers. Given a subspace $V \subseteq E$ we form the sequence of subspaces*

$$V_0 = V, \;\; V_i = (V_{i-1})_{L_i}^{[s_i]}, \;\; i = 1, 2, \dots, t\,. \tag{2}$$

If for $i = 1, 2, \dots, t$ the subspace V is $L_i^{s_i}$-invariant, then V_t is L_i-invariant, and it contains V. Furthermore $V_t = \diamond_{L_1,L_2,\cdots,L_t}(V)$ and $\dim(V) < \infty$ if and only if $\dim(\diamond_{L_1,L_2,\cdots,L_t}(V)) < \infty$.

Proof: First we prove by induction on i that V_i is $L_j^{s_j}$-invariant and it contains V for each $i = 0, 1, \dots, t$ and $j = 1, 2, \dots, t$. For $i = 0$ we have $V_0 = V$, which is $L_j^{s_j}$-invariant for $j = 1, 2, \dots, t$, by assumption.

Suppose that $i \geq 1$, and we have proved the statement for V_{i-1}. Now we prove it for V_i. If v is in V_i, then we have

$$v = u_0 + L_i u_1 + \cdots + L_i^{s_i-1} u_{s_i-1}\,,$$

where u_j is in V_{i-1} for $j = 0, 1, \dots, s_i - 1$. It follows for $j = 1, 2, \dots, t$

$$L_j^{s_j} v = (L_j^{s_j} u_0) + L_i(L_j^{s_j} u_1) + \cdots + L_i^{s_i-1}(L_j^{s_j} u_{s_i-1})\,.$$

Here we used the commuting property of the given operators, which obviously holds for their powers, too. By the induction hypothesis, the elements in the brackets on the right hand side belong to V_{i-1}, hence $L_j^{s_j}v$ is in V_i, that is, V_i is $L_j^{s_j}$-invariant. As V_i includes V_{i-1}, we also conclude that V is in V_i, and our statement is proved.

Now we have

$$V_t = V_{t-1} + L_t(V_{t-1}) + \cdots + L_t^{s_t-1}(V_{t-1}),$$

and we apply the previous lemma: as V_{t-1} is $L_t^{s_t}$-invariant, we have that V_t is L_t-invariant.

Let us now prove the invariance of V_t under the operators L_j $(j < t)$. Since V_1 is clearly L_1-invariant, by Lemma 2, an induction process gives that V_{t-1} is L_i-invariant for $1 \leq i \leq t-1$. Thus, if we take $1 \leq i \leq t-1$, then we can use that $L_iL_t = L_tL_i$ and $L_i(V_{t-1})$ is a subset of V_{t-1} to conclude that

$$\begin{aligned} L_i(V_t) &= L_i(V_{t-1}) + L_t(L_i(V_{t-1})) + \cdots + L_t^{s_t-1}(L_i(V_{t-1})) \\ &\subseteq V_{t-1} + L_t(V_{t-1}) + \cdots + L_t^{s_t-1}(V_{t-1}) = V_t\,, \end{aligned}$$

which completes this part of the proof.

Suppose that W is a subspace in E such that $V \subseteq W$, and W is L_j-invariant for $j = 1, 2, \ldots, t$. Then, obviously, all the subspaces V_i for $i = 1, 2, \ldots, t$ are included in W. In particular, V_t is included in W. This proves that V_t is the smallest subspace in E, which includes V, and which is invariant with respect to the family of operators L_i. In particular, $V_t = \diamond_{L_1,L_2,\cdots,L_t}(V)$ is uniquely determined by V, and by the family of the operators L_i, no matter how we label these operators. $\square$

The following result generalizes Anselone-Korevaar's Theorem [6].

Lemma 4. *Assume that $h_1\mathbb{Z} + h_2\mathbb{Z} + \cdots + h_t\mathbb{Z}$ is dense in $\mathbb{R}^d$. Assume, furthermore, that $\Delta_{h_k}^{m_k}(H) \subseteq H$, $k = 1, \cdots, t$, for certain positive integral numbers m_k and certain finite dimensional subspace H of X_d. Then all elements of H are, in distributional sense, continuous exponential polynomials.*

Proof: We apply Lemma 3 with $E = X_d$, $L_i = \Delta_{h_i}$, $s_i = m_i$, $i = 1, \cdots, t$, and $V = H$, since

$$\Delta_{h_i}^{m_i}(H) \subseteq H, \text{ for all } i = 1, \cdots, t$$

so that $H \subseteq Z = \diamond_{\Delta_{h_1},\Delta_{h_2},\cdots,\Delta_{h_t}}(H)$ and Z is a finite dimensional subspace of X_d satisfying $\Delta_{h_i}(Z) \subseteq Z$, $i = 1, 2, \cdots, t$. Hence Z is invariant by translations, since $h_1\mathbb{Z} + h_2\mathbb{Z} + \cdots + h_t\mathbb{Z}$ is dense in $\mathbb{R}^d$. Applying Anselone-Korevaar's theorem, we conclude that all elements of Z (hence, also all elements of H) are continuous exponential polynomials. $\square$

Now we can demonstrate the main result of this section:

Theorem 5. *Assume that* $\{h_1, \cdots, h_t\}$ *spans a dense subgroup of* $\mathbb{R}^d$, $f \in X_d$ *and there exist natural numbers* $\{m_k\}_{k=1}^t$ *such that, for every* $k \in \{1, \ldots, t\}$, $\Delta_{h_k}^{m_k} f$ *is a continuous exponential polynomial. Then* f *is a continuous exponential polynomial.*

PROOF: Let $g_k = \Delta_{h_k}^{m_k} f$ be an exponential polynomial for $k = 1, \cdots, t$ and set $H = \tau(\{g_k\}_{k=1}^t) + \mathbf{span}\{f\}$, were $\tau(S)$ denotes the smallest subspace of X_d which is translation invariant and contains the set S. Obviously, H is a finite dimensional subspace of X_d and satisfies $\Delta_{h_i}^{m_i}(H) \subseteq H$ for $i = 1, \cdots, t$. Thus, we can apply Lemma 4 to H. □

3. Finitely generated nondense subgroups of $\mathbb{R}^d$

In this section we demonstrate that density of $G = h_1\mathbb{Z} + h_2\mathbb{Z} + \cdots + h_t\mathbb{Z}$ in $\mathbb{R}^d$ is a necessary hypothesis in Theorem 5. Moreover, under the hypothesis that this group G is not a dense subgroup of $\mathbb{R}^d$, we characterize the continuous functions f satisfying that, for a certain finite dimensional space $H \subseteq C(\mathbb{R}^d)$ and certain natural numbers n_k, m_k, $k = 1, \cdots, t$, the relations $\bigcup_{k=1}^t \Delta_{h_k}^{m_k}(H) \subseteq H$ and $\Delta_{h_k}^{n_k} f \in H$, $k = 1, \cdots, t$, hold.

Let d be a positive integer. If G denotes the additive subgroup of $\mathbb{R}^d$ generated by the elements $\{h_1, \cdots, h_t\}$, then it is well-known [8, Theorem 3.1] that $\overline{G}$, the topological closure of G with the Euclidean topology, satisfies $\overline{G} = V \oplus \Lambda$, where V is a vector subspace of $\mathbb{R}^d$ and Λ is a discrete additive subgroup of $\mathbb{R}^d$. Furthermore, the case when G is dense in $\mathbb{R}^d$, or, what is the same, the case whenever $V = \mathbb{R}^d$, has been characterized in several different ways (see e.g., [7, Theorem 442, p. 382], [8, Proposition 4.3]).

Assume that V is a proper subspace of $\mathbb{R}^d$ (equivalently, G is not dense in $\mathbb{R}^d$). It is well known that, in this case, there exist a hyperplane U and an element $h \notin U$ such that $G \subset W = \{u + nh : u \in U, n \in \mathbb{Z}\}$. Then every element $x \in \mathbb{R}^d$ can be written uniquely as $x = u + s(x)h$, where $u \in U$ and $s(x) \in \mathbb{R}$. Let $g(x) = \varphi(s(x))$, where φ is 1-periodic and non-smooth (so that it is not an exponential polynomial). For example, we can take $\varphi(t)$ equal to the 1-periodic extension to the real line of the absolute value restricted to the period interval $I = [-1/2, 1/2]$. Then, for every $y \in W$ (and henceforth for every $y \in G$), we have that

$$\begin{aligned}
\Delta_y g(x) &= g(x + y) - g(x) = g(u + s(x)h + v + nh) - g(u + s(x)h) \\
&= g(u + v + (s(x) + n)h) - g(u + s(x)h) \\
&= \varphi(s(x) + n) - \varphi(s(x)) = 0
\end{aligned}$$

and obviously g is not an exponential polynomial.

We now give a description, for the case when V is a proper subspace of $\mathbb{R}^d$ (arbitrary d), of the sets of continuous functions f satisfying that $\Delta_{h_k}^{n_k} f \in H$ for $k = 1, \cdots, t$, for a certain finite dimensional subspace H of $C(\mathbb{R}^d)$ which is $\Delta_{h_k}^{m_k}$-invariant for some m_k, $k = 1, \cdots, t$.

Theorem 6. *Let t be a positive integer, let $n_1, n_2, \ldots, n_t, m_1, m_2, \cdots, m_t$ be natural numbers, let $H \subseteq C(\mathbb{R}^d)$ be a linear finite dimensional subspace satisfying*

$$\bigcup_{k=1}^{t} \Delta_{h_k}^{m_k}(H) \subseteq H.$$

Further let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function satisfying

$$\Delta_{h_k}^{n_k} f \in H$$

for $k = 1, \cdots, t$. If the subgroup G in $\mathbb{R}^d$ generated by $\{h_1, h_2, \ldots, h_t\}$ satisfies $\overline{G} = V \oplus \Lambda$, where V is a vector subspace of $\mathbb{R}^d$, and Λ is a discrete additive subgroup of $\mathbb{R}^d$, then for each λ in Λ there exist a continuous exponential polynomial $e_\lambda : \mathbb{R}^d \to \mathbb{R}$ such that

$$f(x + \lambda) = e_\lambda(x) \text{ for all } x \in V.$$

For the proof of this result, we need firstly to introduce the following technical result:

Lemma 7. *Let $(G, +)$ be a commutative topological group, $h_1, \cdots, h_t \in G$ and $G' = h_1\mathbb{Z} + \cdots + h_t\mathbb{Z}$. Let H be a vector subspace of $\mathbb{C}^G$ such that $\tau_h(H) \subseteq H$ for all $h \in G'$, and assume that $f : G \to \mathbb{C}$ satisfies $\Delta_{h_i}^{n_i} f \in H$ for certain natural numbers n_i and $i = 1, \cdots, t$. Take $N = n_1 + \cdots + n_t$. Then $\Delta_h^N f \in H$ for all $h \in G'$. Moreover, if H is a closed subspace of $C(G)$, then $\Delta_h^N f \in H$ for all $h \in \overline{G'}$.*

Proof: The proof follows similar arguments to those used in [4, Theorem 2] and, in fact, this lemma is a generalization of that result, which follows as a corollary just imposing $H = \{0\}$.

Take $N = n_1 + \cdots + n_t$ and let $h \in G'$. Then there exist $m_1, \cdots, m_t \in \mathbb{Z}$ such that $h = m_1 h_1 + \cdots + m_t h_t$ and

$$\begin{aligned}
\Delta_h^N f &= \Delta_{m_1h_1+\cdots+m_th_t}^N f = (\tau_{m_1h_1+\cdots+m_th_t} - 1_d)^N (f) \\
&= (\tau_{h_1}^{m_1}\tau_{h_2}^{m_2}\ldots\tau_{h_t}^{m_t} - 1_d)^N (f) \\
&= \big[(\tau_{h_1}^{m_1}\tau_{h_2}^{m_2}\ldots\tau_{h_t}^{m_t} - \tau_{h_2}^{m_2}\ldots\tau_{h_t}^{m_t}) + (\tau_{h_2}^{m_2}\ldots\tau_{h_t}^{m_t} - \tau_{h_3}^{m_3}\ldots\tau_{h_t}^{m_t}) + \\
&\quad \cdots \\
&\quad +(\tau_{h_{t-2}}^{m_{t-2}}\tau_{h_{t-1}}^{m_{t-1}}\tau_{h_t}^{m_t} - \tau_{h_{t-1}}^{m_{t-1}}\tau_{h_t}^{m_t}) + (\tau_{h_{t-1}}^{m_{t-1}}\tau_{h_t}^{m_t} - \tau_{h_t}^{m_t}) + (\tau_{h_t}^{m_t} - 1_d)\big]^N (f) \\
&= \big[(\tau_{h_1}^{m_1} - 1_d)\tau_{h_2}^{m_2}\ldots\tau_{h_t}^{m_t} + (\tau_{h_2}^{m_2} - 1_d)\tau_{h_3}^{m_3}\ldots\tau_{h_t}^{m_t} \\
&\quad + \cdots + (\tau_{h_{t-1}}^{m_{t-1}} - 1_d)\tau_{h_t}^{m_t} + (\tau_{h_t}^{m_t} - 1_d)\big]^N (f)\,.
\end{aligned}$$

We have that $\Delta_h^N f = A^N(f)$, where $A^N(f)$ is the last expression of the displayed formula. Here A is a sum of t terms such that each term is a multiple of one of $(\tau_{h_i} - 1_d)$, $i = 1, \cdots, t$. Note that this claim is obvious for the terms where $m_i > 0$. If, on the contrary, $m_i < 0$, the claim follows from the identity

$$\tau_h^{-1} - 1_d = \tau_{-h} - 1_d = -\tau_{-h}(\tau_h - 1_d)\,.$$

Since $N = n_1 + \cdots + n_t$, it follows that A^N is a sum each term of which is a multiple of one of the $(\tau_{h_i} - 1_d)^{n_i} = \Delta_{h_i}^{n_i}$. Since $\Delta_{h_i}^{n_i} f \in H$ for every i, and H is invariant under translations by elements of G', all summands of $A^N(f)$ belong to the vector space H, which proves that $\Delta_h^N f \in H$.

If H is a closed subspace of $C(G)$ and $h \in \overline{G'}$ then there exists a sequence $\{g_n\} \subset G'$ converging to h and the sequence given by $f_n = \Delta_{g_n}^N f$ converges to $\Delta_h^N f$, which belongs to H since H is closed and $g_n \in H$ for all n. □

Proof of Theorem 6: Lemma 3 allows us to substitute H satisfying

$$\bigcup_{k=1}^{t} \Delta_{h_k}^{m_k}(H) \subseteq H$$

by a finite dimensional subspace $\widetilde{H}$ of $C(\mathbb{R}^d)$ which contains H and satisfies

$$\bigcup_{k=1}^{t} \Delta_{h_k}(\widetilde{H}) \subseteq \widetilde{H}.$$

If we take $W = \mathbf{span}\{f\}$, we can apply again Lemma 3 with $E = C(\mathbb{R}^d)$, $L_i = \Delta_{h_i}$, $i = 1, \cdots, t$, to the vector space $M = W + \widetilde{H}$, since

$$\begin{aligned} \Delta_{h_i}^{n_i}(W + \widetilde{H}) &= \Delta_{h_i}^{n_i}(W) + \Delta_{h_i}^{n_i}(\widetilde{H}) \\ &\subseteq H + \widetilde{H} \\ &\subseteq \widetilde{H} \subseteq W + \widetilde{H}. \end{aligned}$$

Hence $M \subseteq Z = \diamond_{\Delta_{h_1}, \Delta_{h_2}, \cdots, \Delta_{h_t}}(M)$ and Z is a finite dimensional subspace of $C(\mathbb{R}^d)$ satisfying $\Delta_{h_i}(Z) \subseteq Z$, $i = 1, 2, \cdots, t$. Let $V^\perp$ denote the orthogonal complement of V in $\mathbb{R}^d$ with respect to the standard scalar product and let $P_V : \mathbb{R}^d \to \mathbb{R}^d$ denote the orthogonal projection onto V with respect to the standard scalar product of $\mathbb{R}^d$. We define the function $F : \mathbb{R}^d \to \mathbb{R}$ by $F(x) = f(P_V(x))$. Obviously, F is a continuous extension of $f_{|V}$. Furthermore, if $\widetilde{H}$ admits a basis $\{g_k\}_{k=1}^m$, we introduce the new vector space $\widetilde{\widetilde{H}} = \mathbf{span}\{G_k\}_{k=1}^m$, where $G_k(x) = g_k(P_V(x))$ for all $x \in \mathbb{R}^d$. Then, if $x = v + w \in \mathbb{R}^d$ with $v \in V$, $w \in V^\perp$, $k \in \{1, \cdots, t\}$, and $j \in \{1, \cdots, m\}$, we have that

$$\begin{aligned} \Delta_{h_k} G_j(x) &= G_j(v + w + h_k) - G_j(v + w) \\ &= g_j(v + h_k) - g_j(v) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \alpha_{i,j} g_i(v), \text{ since } \Delta_{h_k}(\widetilde{H}) \subseteq \widetilde{H} \\
&= \sum_{i=1}^{m} \alpha_{i,j} G_i(v+w) \\
&= \sum_{i=1}^{m} \alpha_{i,j} G_i(x).
\end{aligned}
$$

Thus, $\Delta_{h_k}(\widetilde{\widetilde{H}}) \subseteq \widetilde{\widetilde{H}}$ for $k = 1, \cdots, t$.

Take $\{h_1^*, \cdots, h_s^*\} \subseteq V^\perp$ such that $\{h_1, \cdots, h_t, h_1^*, \cdots, h_s^*\}$ spans a dense subgroup of $\mathbb{R}^d$. Then if $x = v + w \in \mathbb{R}^d$ with $v \in V$, $w \in V^\perp$, $k \in \{1, \cdots, s\}$, and $j \in \{1, \cdots, m\}$, we have that

$$
\begin{aligned}
\Delta_{h_k^*} G_j(x) &= G_j(v + w + h_k^*) - G_j(v + w) \\
&= G_j(v) - G_j(v) = 0.
\end{aligned}
$$

Hence we also have that $\Delta_{h_k^*}(\widetilde{\widetilde{H}}) \subseteq \widetilde{\widetilde{H}}$ for $k = 1, \cdots, s$. Anselone-Korevaar's theorem implies that all elements of $\widetilde{\widetilde{H}}$ are continuous exponential polynomials on $\mathbb{R}^d$.

Let us now do the computations for F. Take $N = n_1 + \cdots + n_t$. Then

$$
\begin{aligned}
&\Delta_{h_i}^N F(x) = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} F(x + kh_i) \\
&= \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} F(P_V(x) + kP_V(h_i) + [(x - P_V(x)) + k(h_i - P_V(h_i))]) \\
&= \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} f(P_V(x) + kP_V(h_i)) \\
&= \Delta_{P_V(h_i)}^N f(P_V(x)) \\
&= \sum_{j=1}^{m} a_{i,j} g_j(P_V(x)) \text{ since } \Delta_{P_V(h_i)}^N f \in \widetilde{H} \\
&= \sum_{j=1}^{m} a_{i,j} G_j(x) \in \widetilde{\widetilde{H}}.
\end{aligned}
$$

Here we have used that $\widetilde{H}$ is G-invariant and Lemma 7 to conclude that $\Delta_{P_V(h_i)}^N f \in \widetilde{H}$ since $P_V(h_i) \in V \subseteq \overline{G}$ and $\widetilde{H}$ is a closed subspace of $C(\mathbb{R}^d)$, since it is finite dimensional. On the other hand, $\Delta_{h_j^*} F = 0 \in \widetilde{\widetilde{H}}$ for $j = 1, \cdots, s$.

Hence we can apply Theorem 5 to F to prove that F is an exponential polynomial whose restriction to V is $f_{|V}$. Thus, if we set $e_0 = F$, we have that e_0 is a continuous exponential polynomial on $\mathbb{R}^d$ and $f(x) = e_0(x)$ for all x in V.

Now let λ be arbitrary in Λ and we consider the function $g_\lambda : V \to \mathbb{R}$ defined by $g_\lambda(x) = f(x+\lambda)$ for x in V and λ in Λ. Then $g_\lambda = \tau_\lambda(f)$, so that

$$\Delta_{h_i}^{n_i} g_\lambda = \Delta_{h_i}^{n_i} \tau_\lambda(f) = \tau_\lambda(\Delta_{h_i}^{n_i}(f)) \in \tau_\lambda(H) \subseteq \tau_\lambda(\widetilde{H}) \subseteq \widetilde{H}.$$

Thus we can repeat all arguments above with g_λ instead of f to get that, for some continuous exponential polynomial e_λ defined on $\mathbb{R}^d$ we have that $g_\lambda(x) = e_\lambda(x)$ for all $x \in V$. Hence, if $x \in V$ and $\lambda \in \Lambda$,

$$f(x+\lambda) = g_\lambda(x) = e_\lambda(x).$$

This ends the proof. $\square$

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