On the Infinite Loch Ness monster

John A. Arredondo, Camilo Ramírez Maluendas

Abstract. In this paper we introduce the topological surface called *Infinite Loch Ness monster*, discussing how this name has evolved and how it has been historically understood. We give two constructions of this surface, one of them having translation structure and the other hyperbolic structure.

Keywords: Infinite Loch Ness monster; tame Infinite Loch Ness monster; hyperbolic Infinite Loch Ness monster

Classification: 51M15

Introduction

The term Loch Ness monster is well known around the world, specially in The Great Glen in the Scottish highlands, a rift valley which contains three important lochs for the region, called Lochy, Oich and Ness. The last one, people believe that a monster lives and lurks, baptized with the name of the loch. The existence of the monster is not farfetched, people say, taking into account that the Loch Ness is deeper than the North Sea and is very long, very narrow and has never been known to freeze (see Figure 1).

The earliest report of such a monster appeared in the Fifth century, and from that time different versions about the monster passed from generation to generation [Ste97]. A kind of modern interest in the monster was sparked by 1933 when George Spicer and his wife stated that they saw the monster crossing the road in front of their car. After that sighting, hundreds of different reports about the monster have been collected, including photos, portrayals and other descriptions. In spite of this evidence, without a body, a fossil or the monster in person, The Loch Ness monster is only part of the folklore.

In a different context, in mathematics, the term *Loch Ness monster* is also known, and not in folklore, in the study of topological surfaces, where this term makes reference to the surface obtained by gluing infinitely many torii along a ray (see Figure 4), actually, it is called *Infinite Loch Ness monster*.

In particular, we are interested in those topological surfaces having two kinds of structure, *translation* and *hyperbolic*. The first one of them have appeared naturally in different branches of the mathematics such as Dynamical System (see Steven Kerckhoff, Howard Masur and John Smillie [KMS86]), Teichmüller Theory (see [KZ03] by the authors Maxim Kontsevich and Anton Zorich), Riemann

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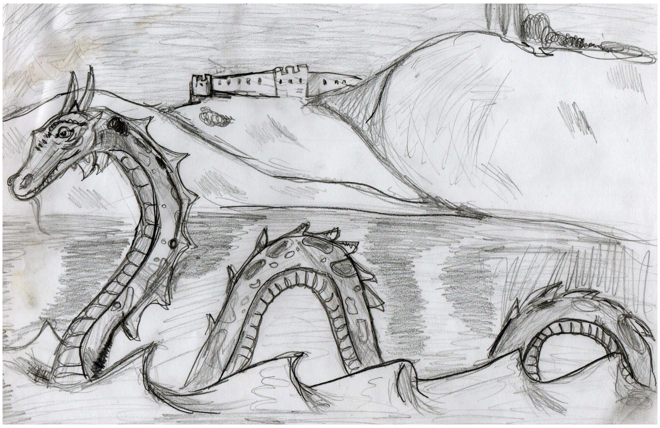


FIGURE 1. *Loch Ness monster in The Great Glen in the Scottish. Image by xKirinARTZx, taken from devianart.com*

Surfaces (see Howard Masur and Serge Tabachnikov [MT02]), Algebraic Geometry (see [Mol06] by Martin Möller), and others. Basically, a *translation structure* on a surface is an atlas of charts to the plane where the transition functions are translations. So, motivated by the Open problem 2.6.2 concerning construction of compact surface with translation structure introduced to the literature by Pascal Hubert and Thomas A. Schmidt [HS06], we present in Section 2 a surface topologically equivalent to the Infinite Loch Ness monster having tame translation structure.

On the other hand, the twenty-second problem of the Mathematical Problems published by David Hilbert [Hil00] was solved simultaneously in 1907 by Henri Poincaré and Paul Koebe, as reported by William Abikoff [Abi81]. They proved that:

Theorem 0.1 ([Bea84, p. 174]). *Let S be a Riemann surface, let $\widetilde{S}$ be the universal covering surface of S chosen from the surfaces $\hat{\mathbb{C}}$, $\mathbb{C}$, and Δ. Let Γ be the cover group of S. Then*

1. *S is conformally equivalent to $\widetilde{S}/\Gamma$;*
2. *Γ is a Möbius group which acts discontinuously on $\widetilde{S}$;*
3. *apart from the identity, the elements of Γ have no fixed points in $\widetilde{S}$;*
4. *the cover group Γ is isomorphic to $\pi(S)$.*

Encouraged by this valuable theorem, in Section 2, we construct explicitly an infinitely generated Fuchsian group $\Gamma < PSL(2, \mathbb{R})$, such that the quotient space $\mathbb{H}/\Gamma$ is a hyperbolic surface homeomorphic to the Infinite Loch Ness monster.

The paper is organized as follows: In Section 1 we present a review of some interesting mathematical situations where the Infinite Loch Ness monster appears. And in Section 2 we present two different constructions of the Infinite Loch Ness monster, with translation and hyperbolic structure, including all the necessary concepts to achieve this goal.

1. Some apparitions of the Loch Ness monster

From view of the Kerékjártó theorem of classification of noncompact surfaces (*e.g.*, Béla Kerékjártó [Ker23], Ian Richards [Ric63]), the *Infinite Loch Ness monster* is the name of the orientable surface which has infinite genus and only one end, such as Ferrán Valdez remarks [Val09]. Simply, Étienne Ghys [Ghy95] describes it as the orientable surface obtained from the Euclidean plane which is attached to an infinity of handles (see Figure 2). Or alternatively, from a geometric viewpoint one can think that the Infinite Loch Ness monster is the only orientable surface having infinitely many handles and only one way to go to infinity.

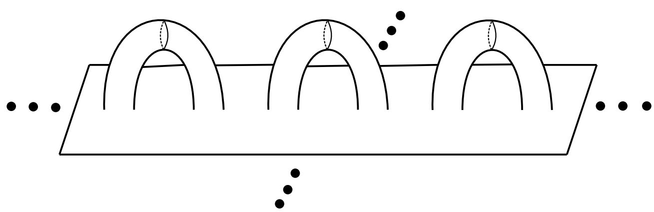


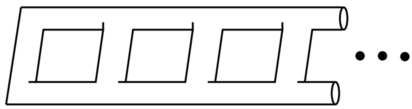
FIGURE 2. *The Infinite Loch Ness monster.*

In the seventies, the interest by several authors (*e.g.*, Jonathan D. Sondow [Son75], Toshiyuri Nishimori [Nis75], John Cantwell and Lawrence Conlon [CC78]) on the qualitative study in the noncompact leaves in foliations of closed manifolds had grown. Ongoing in this line of research, Anthony Phillips and Dennis Sullivan proved that the well known surfaces *Jacob's ladder*[1], the *Infinite jail cell windows* [Spi79, p. 24], and the *Infinite jangle gym* (see Figure 3) are diffeomorphic to the *Infinite Loch Ness monster* (see [PS81]).

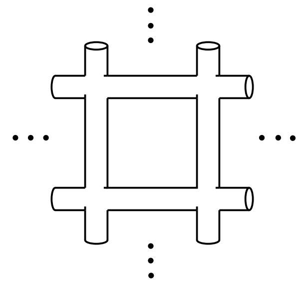
Roughly speaking, from the historical point of view, the name *Infinite Loch Ness monster* appeared published by first time in *Leaves with isolated ends in foliated 3-manifolds* ([CC77, 1977]), however the authors wedge this term to a preliminary manuscript of [PS81], which was published the following year. Under this evidence, one can consider to Anthony Phillips and Dennis Sullivan as the *Infinite Loch Ness monster*'s parents.

Remark 1.1. Perhaps the reader has found on the literature other names for this surface with infinite genus and only one end, for example, the *infinite-holed torus* (Spivak [Spi79, p. 23]). See Figure 4.

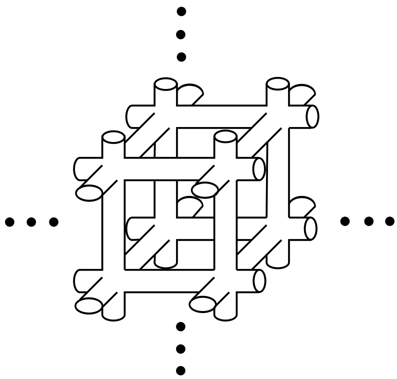
[1]Étienne Ghys calls Jacob's ladder to the surface with two ends and each ends having infinite genus (see [Ghy95]). However, Michael Spivak calls this surface the doubly infinite-holed torus (see [Spi79, p. 24]).



a. Jacob's ladder.



b. Infinite jail cell windows.



c. Infinite jangle gym.

FIGURE 3. *Surfaces having only one end and infinite genus.*

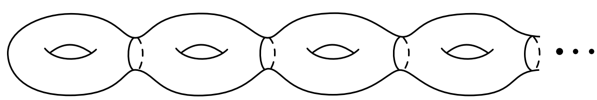
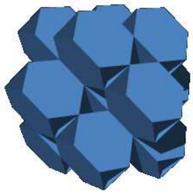
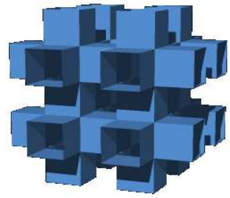


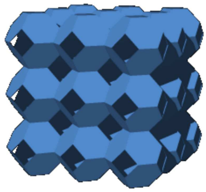
FIGURE 4. *The infinite-holed torus.*

The Infinite Loch Ness monster has also appeared in the area of Combinatorics. Its arrival was in 1926 when John Petrie told Harold Coxeter that he had found two new infinite regular polyhedra. As soon as Petrie began to describe them and Coxeter understood this, the second pointed out a third possible polyhedra. Later they wrote a paper calling this mathematical objects the *skew polyhedra* [Cox36], or also known today as the Coxeter-Petrie polyhedra. Indeed, they are topologically equivalent to the Infinite Loch Ness monster as shown by the authors jointly with Ferrán Valdez in [ARMV17]. Given that from a combinatorics view, one can think that skew polyhedra are multiple covers of the first three Platonic solids, John H. Conway and *et. al.,* [CBG08, p. 333] called them the *multiplied tetrahedron*, the *multiplied cube*, and the *multiplied octahedron*, and denoted them μT, μC, and μO, respectively. See Figure 5.





a. The multiplied tetrahedron μT. *b. The multiplied cube μC.*



c. The multiplied octahedron μO.

FIGURE 5. *Locally the skew polyhedra or Coxeter-Petrie polyhedra. Images by Tom Ruen, distributed under CC BY-SA 4.0.*

In billiards, an interesting area of Dynamical Systems, during 1936 the mathematicians *Ralph H. Fox* and *Richard B. Kershner* [FK36] associated to each *billiard* ϕ_P, coming from an Euclidean compact polygon $P \subset \mathbb{E}^2$, a surface S_P with translation structure, which they called *Überlagerungsfläche*[2], and a projection map $\pi_p : S_p \to \phi_P$, mapping each geodesic of S_P onto a *billiard trajectory* of ϕ_P (see Figure 6). Later, Ferrán Valdez published a paper [Val09], in which he proved that the surface S_P associated to the billiard ϕ_P, being $P \subset \mathbb{E}^2$ a polygon with at least an interior angle $\lambda\pi$ such that λ is an irrational number, is the Infinite Loch Ness monster.

Remark 1.2. In number theory there is a kind of series called exponential sums, which in general take the form

$$s_N = \sum_{n=1}^{N} e^{2\pi i f(n)}, \tag{1}$$

and for the special case in which

$$f(n) = (ln(n))^4 \tag{2}$$

the graph of the curve associated to the first N terms is called *Loch Ness monster* (see Figure 7), dubbed to the curve by John H. Loxton [Lox81], [Lox83].

[2] *Überlagerungsfläche* is a German term closer in meaning to the modern word covering, *i.e., covered surface* and it is also written as *Ueberlagerungsflaeche.*

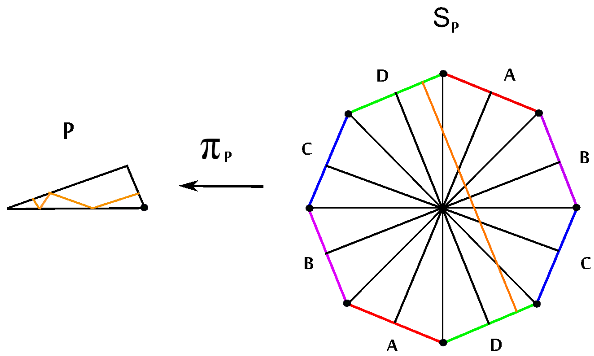


FIGURE 6. *Billiard associated to a rectangle triangle with interior angles* $(\pi/8, 3\pi/8)$.

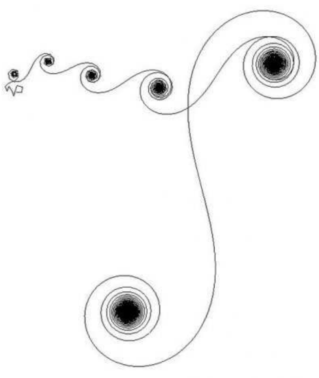


FIGURE 7. *Loch Ness monster curve depicted with* $N = 6000$.

2. Building the Infinite Loch Ness monster

We begin this section introducing the concept of end, one of the fundamental terms used in the classification theorem of orientable surfaces and in the definition of Infinite Loch Ness monster. After, we shall give the concept of tame translation and hyperbolic structure on any surface S. We then will build an Infinite Loch Ness monster having a tame translation and hyperbolic structure.

A *pre-end* of a connected surface S is a nested sequence $U_1 \supset U_2 \supset \cdots$ of connected open subsets of S such that the boundary of U_n in S is compact for every $n \in \mathbb{N}$ and for any compact subset K of S there exists $l \in \mathbb{N}$ such that $U_l \cap K = \emptyset$. We shall denote the pre-end $U_1 \supset U_2 \supset \cdots$ as $(U_n)_{n\in\mathbb{N}}$. Two such sequences $(U_n)_{n\in\mathbb{N}}$ and $(U'_n)_{n\in\mathbb{N}}$ are said to be equivalent if for any $i \in \mathbb{N}$ exists $j \in \mathbb{N}$ such that $U'_j \subset U_i$, and for any $k \in \mathbb{N}$ exists $l \in \mathbb{N}$ such that $U_l \subset U'_k$.

We denote by $Ends(S)$ the corresponding set of equivalence classes and call each equivalence class $[U_n]_{n\in\mathbb{N}} \in Ends(S)$ an *end* of S. The set $Ends(S)$ can be endowed with a topology by specifying a pre-basis as follows: for any open subset $W \subset S$ whose boundary is compact, we define $W^* := \{[U_n]_{n\in\mathbb{N}} \in Ends(S) : W \supset U_l$ for l sufficiently large$\}$. We call the corresponding topological space *the space of ends of S*.

Proposition 2.1 ([Ric63, Proposition 3])**.** *The space of ends of a connected surface S is totally disconnected, compact, and Hausdorff. In particular, $Ends(S)$ is homeomorphic to a closed subspace of the Cantor set.*

A surface is said to be *planar* if all of its compact subsurfaces are of genus zero. An end $[U_n]_{n\in\mathbb{N}}$ is called *planar* if there exists $l \in \mathbb{N}$ such that U_l is planar. The *genus* of a surface S is the maximum of the genera of its compact subsurfaces. Remark that if a surface S has *infinite genus* there exists no finite set $\mathcal{C}$ of mutually non-intersecting simple closed curves with the property that $S\setminus\mathcal{C}$ is *connected and planar*. We define $Ends_\infty(S) \subset Ends(S)$ as the set of all ends of S which are not planar. It follows from the definitions that $Ends_\infty(S)$ forms a closed subspace of $Ends(S)$ (see Ian Richards [Ric63] for details).

Theorem 2.2 (Classification of orientable surfaces. [Ker23, Chapter 5])**.** *Let S and S' be two orientable surfaces of the same genus. Then S and S' are homeomorphic if only if there exists a homeomorphism $f : Ends(S) \to Ends(S')$ such that $f(Ends_\infty(S)) = Ends_\infty(S')$.*

Definition 2.3 ([Val09])**.** Up to homeomorphism, the **Infinite Loch Ness monster** is the unique infinite genus surface with only one end.

We remark that a surface S has only one end if only if for all compact subset $K \subset S$ there exists a compact $K' \subset S$ such that $K \subset K'$ and $S \setminus K'$ is connected, see Ernst Specker [Spe49].

2.1 A tame Infinite Loch Ness monster. A surface S endowed with an atlas whose transition functions are translations is called a *translation surface*. Every translation surface inherits a natural flat metrics from the plane via pull back. We denote as $\hat{S}$ the *metric completion* of S with respect to this natural flat metric.

Definition 2.4 ([PSV11])**.** A translation surface S is called *tame* if for every point $x \in \widehat{S}$ there exists a neighborhood $U_x \subset \widehat{S}$ which is either isometric to some neighborhood of the Euclidean plane or to the neighborhood of the branching point of a cyclic branched covering of the unit disk in the Euclidean plane. In the later case we call x *a cone angle singularity of angle* $2n\pi$ if the cyclic covering is of (finite) order $n \in \mathbb{N}$ and an *infinite cone angle singularity* when the cyclic covering is infinite. We denote by $Sing(S) \subset \widehat{S}$ the set conformed by all cone angle singularities of S.

Based on the ideas above, the second author jointly with Ferrán Valdez have described a tame translation surface homeomorphic to the Infinite Loch Ness monster (see [RMV17, Construction 2.1]). However, they never formally proved that this object is indeed our object of interest. In order to complete the assertion we shall give a short and easy proof to this fact.

Theorem 2.5. *There exists an Infinite Loch Ness monster endowed with a tame translation structure.*

Proof: To build a tame Infinite Loch Ness monster we shall introduce the following definition, which is based on the principle *to glue translation surfaces along parallel marks*.

Definition 2.6 (Gluing marks. [RMV17, Definition 1.15]). A *mark* m on a translation surface S is finite length geodesic having no singular points in its interior. We can associate to each mark two vectors by developing the translation structure along them. Two marks on S are parallel if their respective vectors are parallel. Let m and m' be two disjoint parallel marks of same lengths on a translation surface S. We cut S along m and m', which turns S into a surface with boundary consisting of four straight segments. We glue this segments back using translations to obtain a tame translation surface S' *different* from the one we started from. We say that S' is obtained from S by *re-gluing* along m and m'.

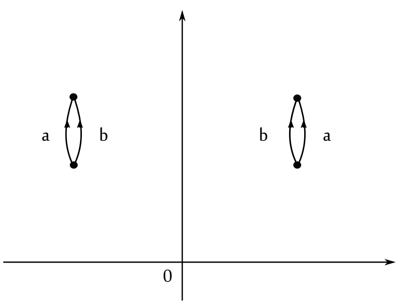


Figure 8. *Gluing marks.*

We denote by $m \sim_{glue} m'$ the operation of gluing the marks m and m' and $S' = S/(m \sim_{glue} m')$. In Figure 8 we depict the gluing of two marks on the plane. Remark that the operation of gluing marks can also be performed for marks on different surfaces. In any case, $Sing(S') \setminus Sing(S)$ is formed by two 4π cone angle singularities (see Figure 9), that is, S tame implies S' tame.

Let $\mathbb{E}^2$ be a copy of the Euclidean plane equipped with a fixed origin $\overline{0}$ and an orthogonal basis $\beta = \{e_1, e_2\}$. On $\mathbb{E}^2$ we draw[3] the following countable family of straight segments:

$$\mathcal{L} := \{l_i = ((4i-1)e_1,\, 4ie_1) : \forall i \in \mathbb{N}\} \text{ (see Figure 10).}$$

[3]Straight segments are given by their ends points.

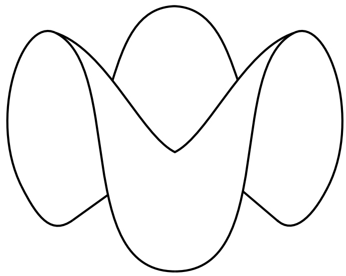


FIGURE 9. 4π *cone angle singularity.*

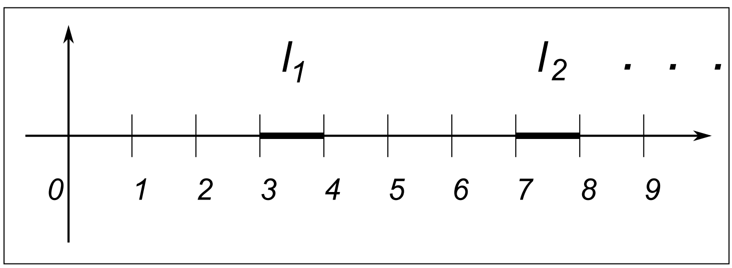


FIGURE 10. *Countable family of straight segments $\mathcal{L}$.*

Hence, we claim that the tame translation surface

$$S := \mathbb{E}^2/(l_{2i-1} \sim_{glue} l_{2i})_{i\in\mathbb{N}},$$

is the Infinite Loch Ness monster *i.e.*, it has infinite genus and only one end.

The surface S has only one end. Let $K \subset S$ be a compact set. We must prove that there exists a compact subset $K \subset K' \subset S$ such that the difference $S - K'$ is connected. We note that there exists a natural projection

$$\pi : (\mathbb{E}^2 - \mathcal{L}) \to S, \quad (x,y) \mapsto [x,y].$$

Then there exists a compact $\widetilde{K} \subset \mathbb{E}^2$ such that the closure of $\pi(\widetilde{K} - \mathcal{L})$ is K. In other words, we have $\overline{\pi(\widetilde{K} - \mathcal{L})} = K$. Given the Euclidean plane $\mathbb{E}^2$ has only one end, then there exists a compact $\widetilde{K}' \subset \mathbb{E}^2$ such that $\widetilde{K} \subset \widetilde{K}'$ and the difference $\mathbb{E}^2 - \widetilde{K}'$ is connected. Then the closure set $\overline{\pi(\widetilde{K}' - \mathcal{L})} := K' \subset S$ is a compact such that $K \subset K'$ and the difference $S - K'$ is connected. Hence, we conclude that S has only one end.

The surface S has infinite genus. For each $i \in \mathbb{N}$ we define the subset

$$\mathbb{E}_i := \{(x,y) \in \mathbb{E}^2 : (4(2i-1)-1)-1 < x < 4(2i)+1, \text{ and } -2 < y < 2\}.$$

We remark that the marks l_{2i-1} and l_{2i} belong to $\mathbb{E}_i$. Then $S_i := \mathbb{E}_i/(l_{2i-1} \sim_{glue} l_{2i}) \subset S$ is a subsurface with boundary homeomorphic to the torus punctured by only one point. Furthermore, for any two different $m \neq n$ the subsurfaces S_m and S_n are disjoint. Thus, we conclude that the translation surface S has infinite genus. $\square$

Remark 2.7. In [PSV11] and [RMV17] the reader can find different constructions of the tame Infinite Loch Ness monster and other non compact surfaces having tame translation structure.

2.2 Hyperbolic Infinite Loch Ness monster. Recall, an application of the Uniformization Theorem (see also Jesús Muciño-Raymundo [MR]) ensures the existence of a subgroup Γ of the isometries group of the hyperbolic plane $Isom(\mathbb{H})$ acting on the hyperbolic plane $\mathbb{H}$ performing the quotient space $\mathbb{H}/\Gamma$ in a hyperbolic surface homeomorphic to the Infinite Loch Ness monster. In other words, there exist a hyperbolic polygon $P \subset \mathbb{H}$, which is suitably identifying its sides by hyperbolic isometries to get the Infinite Loch Ness monster. An easy way to define the polygon P is as follows[4].

Theorem 2.8. *Let Γ be the group generated by the set of Möbius transformations $\{f_m(z), g_m(z), f_m^{-1}(z), g_m^{-1}(z) : m \in \mathbb{Z}\}$, where*

$$\begin{aligned}
f_m(z) &:= \frac{(16m+8)z - (1+16m(16m+8))}{z-16m},\\
g_m(z) &:= \frac{(16m+8)z - (1+(16m+4)(16m+8))}{z-(16m+4)},\\
f_m^{-1}(z) &:= \frac{-16mz + (1+16m(16m+8))}{-z+(16m+8)},\\
g_m^{-1}(z) &:= \frac{-(16m+4)z + (1+(16m+4)(16m+8))}{-z+(16m+8)}.
\end{aligned}$$

Then Γ is an infinitely generated Fuchsian group and the Riemann surface $\mathbb{H}/\Gamma$ is homeomorphic to the Infinite Loch Ness monster.

Proof: First, we consider the countable family conformed by the disjoint half-circles $\mathcal{C} = \{C_{4n} : n \in \mathbb{Z}\}$ with C_{4n} having center in $4n$ and radius equal to one, for every $n \in \mathbb{Z}$. See Figure 11. In other words, $C_{4n} := \{z \in \mathbb{H} : |z-4n| = 1\}$. Removing the half-circle C_{4n} of the hyperbolic plane $\mathbb{H}$ we get two connected component, which are called the *inside* of C_{4n} and the *outside* of C_{4n}, respectively (see Figure 12). They are denoted as $\check{C}_{4n}$ and $\hat{C}_{4n}$, respectively. Hence, our connected hyperbolic polygon $P \subset \mathbb{H}$ is the closure of the intersection of the outsides following (see Figure 13).

[4]The reader can also find in [ARM] a great variety of hyperbolic polygons that perform hyperbolic surfaces having infinite genus.

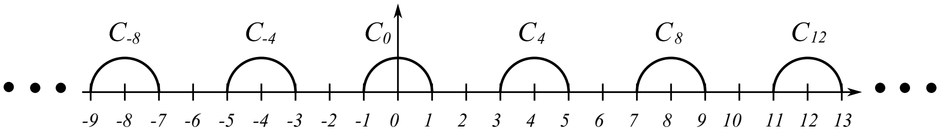


FIGURE 11. *Family of half-circles $\mathcal{C}$.*

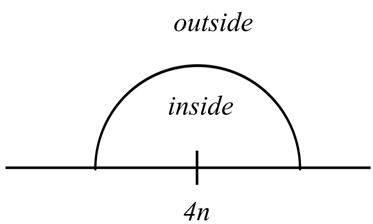


FIGURE 12. *Inside and outside.*

$$P := \overline{\bigcap_{n\in\mathbb{Z}} \hat{C}_{4n}} = \bigcap_{n\in\mathbb{Z}} \{z \in \mathbb{H} : |z - 4n| \geq 1\}. \tag{3}$$

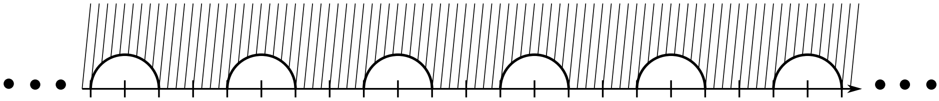


FIGURE 13. *Family of half-circles $\mathcal{C}$ and hyperbolic polygon P.*

The boundary of P is conformed by the half-circle belonged to the family $\mathcal{C}$. Then for every $m \in \mathbb{Z}$ the hyperbolic geodesics $C_{4(4m)}$ and $C_{4(4m+2)}$ are identified as it is shown in Figure 14 by some of the following Möbius transformations:

$$\begin{aligned} f_m(z) &:= \frac{(16m+8)z - (1+16m(16m+8))}{z-16m} \\ f_m^{-1}(z) &:= \frac{-16mz + (1+16m(16m+8))}{-z+(16m+8)}. \end{aligned} \tag{4}$$

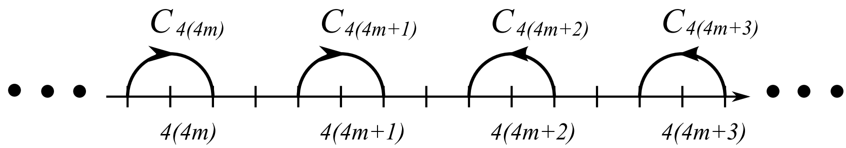


FIGURE 14. *Gluing the side of the hyperbolic polygon P, identifying $C_{4(4m+i)}$ with $C_{4(4m+2+i)}$ for $i \in \{0, 1\}$ and so on.*

Analogously, the hyperbolic geodesics $C_{4(4m+1)}$ and $C_{4(4m+3)}$ are identified as it is shown in Figure 14 by the Möbius transformations:

$$
\begin{aligned}
g_m(z) &:= \frac{(16m+8)z - (1 + (16m+4)(16m+8))}{z - (16m+4)}, \\
g_m^{-1}(z) &:= \frac{-(16m+4)z + (1 + (16m+4)(16m+8))}{-z + (16m+8)}.
\end{aligned} \tag{5}
$$

Remark 2.9. Through the Möbius transformations above, the inside of the half-circle $C_{4(4m)}$ (the half-circle $C_{4(4m+1)}$, respectively) is sent by the map $f_m(z)$ (the map $g_m(z)$, respectively) into the outside of the half-circle $C_{4(4m+2)}$ (the half-circle $C_{4(4m+3)}$, respectively). Furthermore, the outside of the half-circle $C_{4(4m)}$ (the half-circle $C_{4(4m+1)}$, respectively) is sent by $f_m(z)$ (the map $g_m(z)$, respectively) into the inside of the half-circle $C_{4(4m+2)}$ (the half-circle $C_{4(4m+3)}$, respectively).

Hence, the hyperbolic surface S that gets glued the side of the polygon P is the Infinite Loch Ness monster, *i.e.*, it has infinite genus and only one end. From the polygon P we deduce that noncompact quotient space S comes with a hyperbolic structure having infinite area.

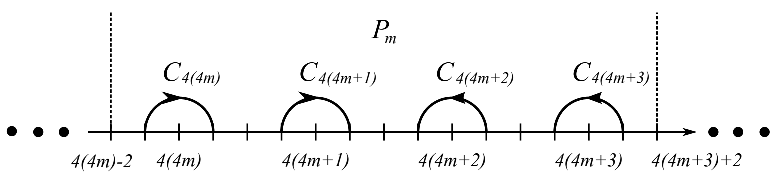


FIGURE 15. *Subregion P_m.*

Furthermore, for each integer number $m \in \mathbb{Z}$ we consider the subregion $P_m \subset P$, which is gotten by the intersection of P and the strip $\{z \in \mathbb{H} : 4(4m) - 2 < Re(z) < 4(4m+3) + 2\}$ (see Figure 15), then restricting to P_m the identification defined above turns it into a torus with one hole S_m (see Figure 16), which is a subsurface of S. Then the elements of the countable family $\{S_m : m \in \mathbb{Z}\}$ are pair disjoint subsurfaces of S and it performs infinite genus in the hyperbolic surface S. In other words, S is the Infinite Loch Ness monster.

From the analytic point of view, we have built a Fuchsian subgroup Γ of $PSL(2, \mathbb{Z})$, where Γ is infinitely generated by the set of Möbius transformations

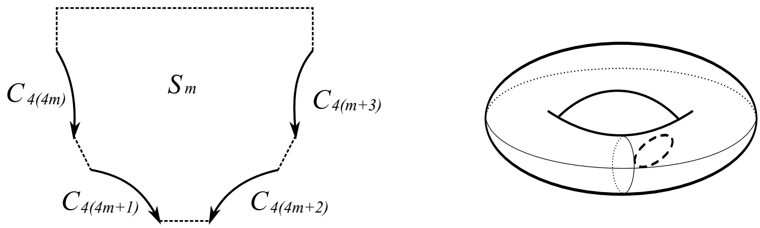


FIGURE 16. *Topological subregion P_m and torus with one hole S_m.*

$\{f_m(z), g_m(z), f_m^{-1}(z), g_m^{-1}(z) : \text{ for all } m \in \mathbb{Z}\}$ (see (4) and (5)), having the subset $P \subset \mathbb{H}$ as fundamental domain[5]. Then Γ acts on the hyperbolic plane $\mathbb{H}$. Defining the subset $K \subset \mathbb{H}$ as follows,

$$K := \{w \in \mathbb{H} : f(w) = w \text{ for any } f \in \Gamma - \{Id\}\} \subset \mathbb{H}, \tag{6}$$

the Fuchsian group Γ acts freely and properly discontinuously on the open subset $\mathbb{H} - K$, but we remark that to this case $K = \emptyset$ because of the intersection of any two different elements belonged to $\mathbb{C}$ is either empty or at infinity, that is, they meet in the same point in the real line $\mathbb{R}$. Hence, the quotient space

$$S := \mathbb{H}/\Gamma \tag{7}$$

is a well-defined surface homeomorphic to the Infinite Loch Ness monster, having hyperbolic structure via the projection map $p : \mathbb{H} \to S$, such as $z \mapsto [z]$. ☐

We conclude from Theorem 0.1.

Corollary 2.10. *The fundamental group $\pi_1(S)$ of the Infinite Loch Ness monster is isomorphic to Γ.*

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References

[Abi81] Abikoff W., *The uniformization theorem*, Amer. Math. Monthly **88** (1981), no. 8, 574–592.

[ARM] Arredondo J.A., Ramírez Maluendas C., *On infinitely generated Fuchsian groups of some infinite genus surfaces*, preliminary manuscript.

[ARMV17] Arredondo J.A., Ramírez Maluendas C., Valdez F., *On the topology of infinite regular and chiral maps*, Discrete Math. **340** (2017), no. 6, 1180–1186.

[Bea84] Beardon A.F., *A Premier on Riemann Surfaces*, London Mathematical Society Lecture Note Series, 78, Cambridge University Press, Cambridge, 1984.

[CC78] Cantwell J., Conlon L., *Leaf prescriptions for closed* 3*-manifolds*, Trans. Amer. Math. Soc. **236** (1978), 239–261.

[5] To deepen in these topics we suggest to reader [Ma88], [Kat92a].

[CC77] Cantwell J., Conlon L., *Leaves with isolated ends in foliated 3-manifolds*, Topology **16** (1977), no. 4, 311–322.

[CBG08] Conway J.H., Burgiel H., Goodman-Strauss C., *The Symmetries of Things*, A K Peters, Ltd., Wellesley, Massachusetts, 2008.

[Cox36] Coxeter H.S.M., *Regular skew polyhedra in three and four dimension, and their topological analogues*, Proc. London Math. Soc. (2) **43** (1937), no. 1, 33–62.

[FK36] Fox R.H., Kershner R.B., *Concerning the transitive properties of geodesics on a rational polyhedron*, Duke Math. J. **2** (1936), no. 1, 147–150.

[Ghy95] Ghys É., *Topologie des feuilles génériques*, Ann. of Math. (2) **141** (1995), no. 2, 387–422.

[Hil00] Hilbert D., *Mathematical problems*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 4, 407–436; reprinted from Bull. Amer. Math. Soc. **8** (1902), 437–479.

[HS06] Hubert P., Schmidt T.A., *An introduction to Veech surfaces*, Handbook of dynamical systems, 1B, Elsevier B.V., Amsterdam, 2006, pp. 501–526.

[Kat92a] Katok S., *Fuchsian Groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992.

[KZ75] Katok A.B., Zemljakov A.N., *Topological transitivity of billiards in polygons*, Mat. Zametki **18** (1975), no. 2, 291–300 (Russian).

[KMS86] Kerckhoff S., Masur H., Smillie J., *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. **124** (1986), no. 2, 293–311.

[Ker23] Kerékjártó B., *Vorlesungen über Topologie I*, Mathematics: Theory & Applications, Springer, Berlin, 1923.

[KZ03] Kontsevich M., Zorich A., *Connected components of the moduli space of abelian differentials with prescribed singularities*, Invent. Math. **153** (2003), no. 3, 631–678.

[Lox81] Loxton J.H., *Captain Cook and the Loch Ness Monster*, James Cook Mathematical Notes **27** (1981), no. 3, 3060–3064.

[Lox83] Loxton J.H., *The graphs of exponential sums*, Mathematika **30** (1983), no. 2, 153–163.

[Ma88] Maskit B., *Kleinian Groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 287, Springer, Berlin, 1988.

[MT02] Masur H., Tabachnikov S., *Rational billiards and flat structures*, Handbook of dynamical systems, 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089.

[Mol06] Möller M., *Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve*, Invent. Math. **165** (2006), 633–649.

[MR] Muciño-Raymundo J., *Superficies de Riemann y Uniformización*, `http://www.matmor.unam.mx/~muciray/articulos/Superficies_de_Riemann.pdf`

[Nis75] Nishimori T., *Isolated ends of open leaves of codimension-one foliations*, Quart. J. Math. **26** (1975), no. 1, 159–167.

[PS81] Phillips A., Sullivan D., *Geometry of leaves*, Topology **20** (1981), no. 2, 209–218.

[PSV11] Przytycki P., Schmithüsen G., Valdez F., *Veech groups of Loch Ness monsters*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 2, 673–687.

[RMV17] Ramírez Maluendas C., Valdez F., *Veech group of infinite-genus surfaces*, Algebr. Geom. Topol. **17** (2017), no. 1, 529–560.

[Ric63] Richards I., *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269.

[Son75] Sondow J., *When is a manifold a leaf of some foliation?*, Bull. Amer. Math. Soc. **81** (1975), no. 3, 622–624.

[Spe49] Specker E., *Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten*, Comment. Math. Helv. **23** (1949), 303–333.

[Spi79] Spivak M., *A comprehensive introduction to differential geometry, Vol. I*, second edition, Publish or Perish, Inc., Wilmington, Del., 1979.

[Ste97] Steuart C., *The Loch Ness Monster: The Evidence*, Prometheus Books, USA, 1997.

[Val09] Valdez F., *Infinite genus surfaces and irrational polygonal billiards*, Geom. Dedicata **143** (2009), 143–154.

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