Some constructions of biharmonic maps on the warped product manifolds

Abdelmadjid Bennouar, Seddik Ouakkas

Abstract. In this paper, we characterize a class of biharmonic maps from and between product manifolds in terms of the warping function. Examples are constructed when one of the factors is either Euclidean space or sphere.

Keywords: harmonic map; biharmonic map; warped product

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1. Introduction

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ, one can refer to [7], [8] for background on harmonic maps. As the generalizations of harmonic maps, biharmonic maps are defined as follows. The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv_g.$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame field on M, then

$$Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) = \left(\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla^M_{e_i} e_i}\right) \tau(\phi),$$

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where we sum over repeated indices. We will call the operator $\tau_2(\phi)$ the bi-tension field of the map ϕ. We can refer the reader to [16], for a survey of biharmonic maps.

Clearly any harmonic map is biharmonic, therefore it is interesting to construct non-harmonic biharmonic maps (see [4]–[7] and [17]–[20] for some constructions of non-harmonic biharmonic maps). Since we consider the identity map between different warped products (Theorem 1), one could mention that in [4], Baird and Kamissoko studied biharmonicity of the identity map with respect to conformal deformations of the domain metric, and that Balmus in [6] did so with respect to conformal deformations of the codomain metric. In [7], Balmus, Montaldo and Oniciuc studied biharmonic maps between warped products where they gave the condition for the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_f N$ and of the projection $\overline{\pi} : M \times_f N \longrightarrow M$ (see Corollary 5 and 6). Moreover, in [20] Perktas and Kilic gave some extensions of the results in [7] together with some further constructions of biharmonic maps, and some characterizations for non-harmonic biharmonic maps are given by using product of harmonic maps and warping metric. Lu in [16] studied the f-harmonicity of some special maps from or into a doubly warped product manifold, he obtained some similar results given in [20], such as the conditions for f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps, furthermore, he investigated non-trivial f-harmonicity of the product of two harmonic maps. In this paper, we give other constructions of biharmonic maps on the warped product. In the first instance, we characterize the biharmonicity of the map $\phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times_\beta N^n, G_\beta)$ defined by $\phi(x, y) = (x, y)$ (Theorem 1). With this setting, we obtain new examples of biharmonic non-harmonic maps. As a second result, we study the biharmonicity of the map $\widetilde{\phi} : (M^m \times_f N^n, G_f) \longrightarrow (P_1^{p_1} \times P_2^{p_2}, G)$ defined by $\widetilde{\phi}(x, y) = (\phi(x), \psi(y))$, where $\phi : (M, g) \longrightarrow (P_1^{p_1}, k_1)$ and $\psi : (N, h) \longrightarrow (P_2^{p_2}, k_2)$ are two harmonic maps (Theorem 2). In the last part, we study the biharmonicity of some maps on the warped product (Theorem 2 and 3) where we give some special cases.

2. The main results

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The warped product $M \times_f N$ is the product manifold $M \times N$ endowed with the Riemannian metric G_f defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_f(X, Y) = g\left(d\pi(X), d\pi(Y)\right) + (f \circ \pi)^2\, h\left(d\eta(X), d\eta(Y)\right),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The function f is called the warping function of the warped product. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$. The Levi-Civita connection $\widetilde{\nabla}$ of the warped product $M \times_f N$ is given by

$$\tag{1} \begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y + X_1 (\ln f)(0, Y_2) + Y_1 (\ln f)(0, X_2) \\ - f^2 h(X_2, Y_2)(\operatorname{grad} \ln f, 0).\end{aligned}$$

The relation between the curvature tensor fields of G_f and G is
(2)

$$\begin{aligned}\widetilde{R}(X,Y) - R(X,Y) = & \frac{1}{2f^2}\left(\nabla_{Y_1}\operatorname{grad} f^2 - \frac{1}{2f^2} Y_1\left(f^2\right)\operatorname{grad} f^2, 0\right) \wedge_{G_f} (0, X_2) \\ & - \frac{1}{2f^2}\left(\nabla_{X_1}\operatorname{grad} f^2 - \frac{1}{2f^2} X_1\left(f^2\right)\operatorname{grad} f^2, 0\right) \wedge_{G_f} (0, Y_2) \\ & - \frac{1}{4f^4}\left|\operatorname{grad} f^2\right|^2 (0, X_2) \wedge_{G_f} (0, Y_2),\end{aligned}$$

where

$$\left(X \wedge_{G_f} Y\right) Z = G_f(Z,Y)X - G_f(Z,X)Y,$$

for all $X, Y, Z \in \Gamma(T(M \times N))$ (see [8] and [15]). At first, we can simplify the formula given by (2), and we get the following proposition:

Proposition 1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The relation between the curvature tensor fields of G_f and G is given by the following formula:*

$$\tag{3} \begin{aligned}\widetilde{R}(X,Y) - R(X,Y) = & \left(\nabla_{Y_1} \mathit{grad} \ln f + Y_1 (\ln f)\, \mathit{grad} \ln f, 0\right) \wedge_{G_f} (0, X_2) \\ & - \left(\nabla_{X_1} \mathit{grad} \ln f + X_1 (\ln f)\, \mathit{grad} \ln f, 0\right) \wedge_{G_f} (0, Y_2) \\ & - |\mathit{grad} \ln f|^2 (0, X_2) \wedge_{G_f} (0, Y_2),\end{aligned}$$

for all $X, Y \in \Gamma(T(M \times N))$, where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$.

Proof of Proposition 1: By equation (2), we have
(4)

$$\begin{aligned}\widetilde{R}(X,Y) - R(X,Y) = & \frac{1}{2f^2}\left(\nabla_{Y_1}\operatorname{grad} f^2 - \frac{1}{2f^2} Y_1\left(f^2\right)\operatorname{grad} f^2, 0\right) \wedge_{G_f} (0, X_2) \\ & - \frac{1}{2f^2}\left(\nabla_{X_1}\operatorname{grad} f^2 - \frac{1}{2f^2} X_1\left(f^2\right)\operatorname{grad} f^2, 0\right) \wedge_{G_f} (0, Y_2) \\ & - \frac{1}{4f^4}\left|\operatorname{grad} f^2\right|^2 (0, X_2) \wedge_{G_f} (0, Y_2).\end{aligned}$$

A simple calculation gives

$$\begin{aligned}\nabla_{Y_1}\operatorname{grad} f^2 & = 2\nabla_{Y_1} f^2 \operatorname{grad} \ln f \\ & = 2f^2 \nabla_{Y_1} \operatorname{grad} \ln f + 2Y_1\left(f^2\right)\operatorname{grad} \ln f \\ & = 2f^2 \nabla_{Y_1} \operatorname{grad} \ln f + 4f^2 Y_1 (\ln f) \operatorname{grad} \ln f\end{aligned}$$

and $Y_1\left(f^2\right)\operatorname{grad}f^2 = 4f^4 Y_1(\ln f)\operatorname{grad}\ln f$, this gives us

$$\text{(5)}\quad \nabla_{Y_1}\operatorname{grad}f^2 - \frac{1}{2f^2}Y_1\left(f^2\right)\operatorname{grad}f^2 = 2f^2\nabla_{Y_1}\operatorname{grad}\ln f + 2f^2 Y_1(\ln f)\operatorname{grad}\ln f.$$

By a similar calculation, we get

$$\text{(6)}\quad \nabla_{X_1}\operatorname{grad}f^2 - \frac{1}{2f^2}X_1\left(f^2\right)\operatorname{grad}f^2 = 2f^2\nabla_{X_1}\operatorname{grad}\ln f + 2f^2 X_1(\ln f)\operatorname{grad}\ln f$$

and

$$\text{(7)}\qquad \left|\operatorname{grad}f^2\right|^2 = \left|2f^2\operatorname{grad}\ln f\right|^2 = 4f^4\left|\operatorname{grad}\ln f\right|^2.$$

If we replace (5), (6) and (7) in (4), we obtain

$$\begin{aligned}\widetilde{R}(X,Y) - R(X,Y) = {} & (\nabla_{Y_1}\operatorname{grad}\ln f + Y_1(\ln f)\operatorname{grad}\ln f, 0)\wedge_{G_f}(0,X_2) \\ & - (\nabla_{X_1}\operatorname{grad}\ln f + X_1(\ln f)\operatorname{grad}\ln f, 0)\wedge_{G_f}(0,Y_2) \\ & - \left|\operatorname{grad}\ln f\right|^2 (0,X_2)\wedge_{G_f}(0,Y_2).\end{aligned}$$

This completes the proof of Proposition 1. □

As a first result, we consider (M^m, g) and (N^n, h) two Riemannian manifolds and $\alpha, \beta \in C^\infty(M)$, the biharmonicity of the map $\phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times_\beta N^n, G_\beta)$ defined by $\phi(x,y) = (x,y)$ is given by the following theorem.

Theorem 1. *Let $\phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times_\beta N^n, G_\beta)$ be defined by $\phi(x,y) = (x,y)$. The map ϕ is biharmonic if and only if*

$$\begin{aligned} & grad\Delta f + 2Ricci\,(gradf) - 2\left(\Delta\ln\alpha + (n-2)\left|grad\ln\alpha\right|^2\right) gradf \\ \text{(8)}\qquad & + (n-4)\left(\nabla^M_{grad\ln\alpha}gradf\right) - 2n\frac{\beta^2}{\alpha^2}df\,(grad\ln\beta)\,grad\ln\beta \\ & - n\frac{\beta^2}{\alpha^2}\left(\nabla^M_{gradf}grad\ln\beta\right) = 0,\end{aligned}$$

where $f = \alpha^2 - \beta^2$ and $\alpha, \beta \in C^\infty(M)$ are positive functions.

Proof of Theorem 1: Let us choose $(e_i)_{1\le i\le m}$ to be an orthonormal frame on M and $(f_j)_{1\le j\le n}$ to be an orthonormal frame on N. An orthonormal frame on $M \times_\alpha N$ (respectively on $M \times_\beta N$) is given by $\{(e_i,0), \frac{1}{\alpha}(0,f_j)\}$ (respectively by $\{(e_i,0), \frac{1}{\beta}(0,f_j)\}$). Note that in this case we have $d\phi(X,Y) = (X,Y)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. By definition of the tension field, we have

$$\begin{aligned}\tau(\phi) &= Tr_{G_\alpha}\widetilde{\nabla}d\phi \\ &= \widetilde{\nabla}^\phi_{(e_i,0)}d\phi(e_i,0) - d\phi\left(\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)\right) \\ &\quad + \frac{1}{\alpha^2}\widetilde{\nabla}^\phi_{(0,f_j)}d\phi(0,f_j) - \frac{1}{\alpha^2}d\phi\left(\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)\right),\end{aligned}$$

where we sum over repeated indices. Using equation (2), we obtain

$$\widetilde{\nabla}^{\phi}_{(e_i,0)} d\phi\,(e_i,0) = \widetilde{\nabla}^{M\times_\beta N}_{(e_i,0)}\,(e_i,0) = \left(\nabla^M_{e_i} e_i, 0\right),$$

$$d\phi\left(\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}\,(e_i,0)\right) = d\phi\left(\nabla^M_{e_i} e_i, 0\right) = \left(\nabla^M_{e_i} e_i, 0\right),$$

$$\widetilde{\nabla}^{\phi}_{(0,f_j)} d\phi\,(0,f_j) = \widetilde{\nabla}^{M\times_\beta N}_{(0,f_j)}\,(0,f_j) = \left(0, \nabla^N_{f_j} f_j\right) - n\beta^2\,(\mathrm{grad}\ln\beta, 0)\,,$$

and

$$\begin{aligned} d\phi\left(\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}\,(0,f_j)\right) &= d\phi\left(\left(0, \nabla^N_{f_j} f_j\right) - n\alpha^2\,(\mathrm{grad}\ln\alpha, 0)\right) \\ &= \left(0, \nabla^N_{f_j} f_j\right) - n\alpha^2\,(\mathrm{grad}\ln\alpha, 0)\,, \end{aligned}$$

then

$$\tau\,(\phi) = n\,(\mathrm{grad}\ln\alpha, 0) - n\frac{\beta^2}{\alpha^2}\,(\mathrm{grad}\ln\beta, 0) = \frac{n}{2\alpha^2}\left(\mathrm{grad}\left(\alpha^2-\beta^2\right), 0\right).$$

Note that the map ϕ is harmonic if and only if the function $\alpha^2-\beta^2$ is constant. By definition, the map ϕ is biharmonic if and only if

$$Tr_{G_\alpha}\widetilde{\nabla}^2\tau\,(\phi) + Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}\,(\tau\,(\phi)\,, d\phi)\,d\phi = 0.$$

Let $f=\alpha^2-\beta^2$, then ϕ is biharmonic if and only if

$$Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}\,(\mathrm{grad}f, 0) + \frac{1}{\alpha^2}Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}\,((\mathrm{grad}f,0)\,, d\phi)\,d\phi = 0. \tag{9}$$

Let us start with the calculation of the first term $Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}(\mathrm{grad}f,0)$ of (9), we have

(10)

$$\begin{aligned} Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}\,(\mathrm{grad}f,0) &= \widetilde{\nabla}^{\phi}_{(e_i,0)}\widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\,(\mathrm{grad}f,0) - \widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)}\frac{1}{\alpha^2}\,(\mathrm{grad}f,0) \\ &\quad + \frac{1}{\alpha^2}\left(\widetilde{\nabla}^{\phi}_{(0,f_j)}\widetilde{\nabla}^{\phi}_{(0,f_j)}\frac{1}{\alpha^2}\,(\mathrm{grad}f,0) - \widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\frac{1}{\alpha^2}\,(\mathrm{grad}f,0)\right). \end{aligned}$$

We will study term by term the right-hand of this expression. Using the equation (2), we have

$$\begin{aligned} \widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\,(\mathrm{grad}f,0) &= \frac{1}{\alpha^2}\widetilde{\nabla}^{\phi}_{(e_i,0)}\,(\mathrm{grad}f,0) + e_i\left(\frac{1}{\alpha^2}\right)(\mathrm{grad}f,0) \\ &= \frac{1}{\alpha^2}\widetilde{\nabla}^{M\times_\beta N}_{(e_i,0)}\,(\mathrm{grad}f,0) + e_i\left(\frac{1}{\alpha^2}\right)(\mathrm{grad}f,0) \\ &= \frac{1}{\alpha^2}\left(\nabla^M_{e_i}\mathrm{grad}f, 0\right) - \frac{2}{\alpha^2}e_i\,(\ln\alpha)\,(\mathrm{grad}f,0)\,. \end{aligned}$$

It follows that

$$\widetilde{\nabla}^{\phi}_{(e_i,0)}\widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)=\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}\left(\nabla^M_{e_i}\mathrm{grad}f,0\right)-\frac{2}{\alpha^2}e_i\left(\ln\alpha\right)\left(\mathrm{grad}f,0\right)\right)$$
$$=\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}\left(\nabla^M_{e_i}\mathrm{grad}f,0\right)\right)-2\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}e_i\left(\ln\alpha\right)\left(\mathrm{grad}f,0\right)\right).$$

By (2), we have

$$\widetilde{\nabla}^{\phi}_{(e_i,0)}\widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)=\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}\left(\nabla^M_{e_i}\mathrm{grad}f,0\right)-\frac{2}{\alpha^2}e_i\left(\ln\alpha\right)\left(\mathrm{grad}f,0\right)\right)$$
$$=\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}\left(\nabla^M_{e_i}\mathrm{grad}f,0\right)\right)-2\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}e_i\left(\ln\alpha\right)\left(\mathrm{grad}f,0\right)\right)$$

and

$$\begin{aligned}
&\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(\frac{1}{\alpha^2}e_i\left(\ln\alpha\right)\left(\mathrm{grad}f,0\right)\right)\\
&=\frac{1}{\alpha^2}e_i\left(\ln\alpha\right)\left(\nabla^M_{e_i}\mathrm{grad}f,0\right)+e_i\left(\frac{1}{\alpha^2}e_i\left(\ln\alpha\right)\right)\left(\mathrm{grad}f,0\right)\\
&=\frac{1}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)+\left(\frac{1}{\alpha^2}e_i\left(e_i\left(\ln\alpha\right)\right)+e_i\left(\frac{1}{\alpha^2}\right)e_i\left(\ln\alpha\right)\right)\left(\mathrm{grad}f,0\right)\\
&=\frac{1}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)+\frac{1}{\alpha^2}\left(e_i\left(e_i\left(\ln\alpha\right)\right)-2e_i\left(\ln\alpha\right)e_i\left(\ln\alpha\right)\right)\left(\mathrm{grad}f,0\right)\\
&=\frac{1}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)+\frac{1}{\alpha^2}\left(e_i\left(e_i\left(\ln\alpha\right)\right)-2\left|\mathrm{grad}\ln\alpha\right|^2\right)\left(\mathrm{grad}f,0\right).
\end{aligned}$$

We deduce that

(11)
$$\begin{aligned}\widetilde{\nabla}^{\phi}_{(e_i,0)}\widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)=&\frac{1}{\alpha^2}\left(\left(\nabla^M_{e_i}\nabla^M_{e_i}\mathrm{grad}f,0\right)-4\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)\right)\\
&-\frac{2}{\alpha^2}\left(e_i\left(e_i(\ln)\right)-2\left|\mathrm{grad}\ln\alpha\right|^2\right)\left(\mathrm{grad}f,0\right).\end{aligned}$$

For the term $\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)$ and using the equation (2), we have

$$\begin{aligned}\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)&=\widetilde{\nabla}^{\phi}_{\left(\nabla^M_{e_i}e_i,0\right)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)\\
&=\widetilde{\nabla}^{M\times_\beta N}_{\left(\nabla^M_{e_i}e_i,0\right)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)\\
&=\frac{1}{\alpha^2}\widetilde{\nabla}^{M\times_\beta N}_{\left(\nabla^M_{e_i}e_i,0\right)}\left(\mathrm{grad}f,0\right)\\
&\quad+\left(\nabla^M_{e_i}e_i\right)\left(\frac{1}{\alpha^2}\right)\left(\mathrm{grad}f,0\right),\end{aligned}$$

then
(12)
$$\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)=\frac{1}{\alpha^2}\left(\nabla^M_{\nabla^M_{e_i}e_i}\mathrm{grad}f,0\right)-\frac{2}{\alpha^2}\left(\nabla^M_{e_i}e_i\right)(\ln\alpha)\left(\mathrm{grad}f,0\right).$$

Equations (11) and (12) give us
(13)
$$\begin{aligned}\widetilde{\nabla}^{\phi}_{(e_i,0)}\widetilde{\nabla}^{\phi}_{(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)-\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(e_i,0)}(e_i,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)&\\ =\frac{1}{\alpha^2}\left(\left(Tr_g\nabla^2\mathrm{grad}f,0\right)-4\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)\right)&\\ -\frac{2}{\alpha^2}\left(\Delta\ln\alpha-2\left|\mathrm{grad}\ln\alpha\right|^2\right)\left(\mathrm{grad}f,0\right).&\end{aligned}$$

Similarly, we will calculate the term $\widetilde{\nabla}^{\phi}_{(0,f_j)}\widetilde{\nabla}^{\phi}_{(0,f_j)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)$, and we get

$$\begin{aligned}\widetilde{\nabla}^{\phi}_{(0,f_j)}\widetilde{\nabla}^{\phi}_{(0,f_j)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)&=\widetilde{\nabla}^{\phi}_{(0,f_j)}\left(\frac{1}{\alpha^2}\widetilde{\nabla}^{M\times_\beta N}_{(0,f_j)}\left(\mathrm{grad}f,0\right)\right)\\ &=\widetilde{\nabla}^{\phi}_{(0,f_j)}\left(\frac{1}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)(0,f_j)\right)\\ &=\frac{1}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)\widetilde{\nabla}^{M\times_\beta N}_{(0,f_j)}\left(0,f_j\right)\\ &=\frac{1}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)\left(\left(0,\nabla^N_{f_j}f_j\right)-n\beta^2\left(\mathrm{grad}\ln\beta,0\right)\right).\end{aligned}$$

It follows that
(14)
$$\widetilde{\nabla}^{\phi}_{(0,f_j)}\widetilde{\nabla}^{\phi}_{(0,f_j)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)=\frac{1}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)\left(\left(0,\nabla^N_{f_j}f_j\right)-n\beta^2\left(\mathrm{grad}\ln\beta,0\right)\right).$$

For the last term $\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)$, we have

$$\begin{aligned}&\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)\\ &=\widetilde{\nabla}^{\phi}_{\left(0,\nabla^N_{f_j}f_j\right)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)-n\alpha^2\widetilde{\nabla}^{\phi}_{(\mathrm{grad}\ln\alpha,0)}\frac{1}{\alpha^2}\left(\mathrm{grad}f,0\right)\\ &=\frac{1}{\alpha^2}\widetilde{\nabla}^{M\times_\beta N}_{\left(0,\nabla^N_{f_j}f_j\right)}\left(\mathrm{grad}f,0\right)-n\widetilde{\nabla}^{M\times_\beta N}_{(\mathrm{grad}\ln\alpha,0)}\left(\mathrm{grad}f,0\right)\\ &\quad-n\alpha^2\left(\mathrm{grad}\ln\alpha\right)\left(\frac{1}{\alpha^2}\right)\left(\mathrm{grad}f,0\right)\\ &=\frac{1}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)\left(0,\nabla^N_{f_j}f_j\right)-n\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)\\ &\quad+2n\left|\mathrm{grad}\ln\alpha\right|^2\left(\mathrm{grad}f,0\right),\end{aligned}$$

then we obtain
(15)
$$\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)=\frac{1}{\alpha^2}df(\mathrm{grad}\ln\beta)\left(0,\nabla^N_{f_j}f_j\right)-n\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)$$
$$+2n\left|\mathrm{grad}\ln\alpha\right|^2(\mathrm{grad}f,0).$$

Equations (14) and (15) give
(16)
$$\widetilde{\nabla}^{\phi}_{(0,f_j)}\widetilde{\nabla}^{\phi}_{(0,f_j)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)-\widetilde{\nabla}^{\phi}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\frac{1}{\alpha^2}(\mathrm{grad}f,0)$$
$$=n\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)-2n\left|\mathrm{grad}\ln\alpha\right|^2(\mathrm{grad}f,0)$$
$$-n\frac{\beta^2}{\alpha^2}df(\mathrm{grad}\ln\beta)(\mathrm{grad}\ln\beta,0).$$

If we replace (13) and (16) in (10), we arrive at the following formula

$$Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}(\mathrm{grad}f,0)=\frac{1}{\alpha^2}\left(Tr_g\nabla^2\mathrm{grad}f,0\right)+\frac{n-4}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)$$
$$-\frac{2}{\alpha^2}\left(\Delta\ln\alpha+(n-2)\left|\mathrm{grad}\ln\alpha\right|^2\right)(\mathrm{grad}f,0)$$
$$-n\frac{\beta^2}{\alpha^4}df(\mathrm{grad}\ln\beta)(\mathrm{grad}\ln\beta,0).$$

Finally, using the fact that (see [17])

$$Tr_g\nabla^2\mathrm{grad}f=\mathrm{grad}\Delta f+Ricci(\mathrm{grad}f),$$

we conclude that

$$\begin{aligned}Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}(\mathrm{grad}f,0)&=\frac{1}{\alpha^2}(\mathrm{grad}\Delta f,0)+\frac{1}{\alpha^2}(Ricci(\mathrm{grad}f),0)\\&-\frac{2}{\alpha^2}\left(\Delta\ln\alpha+(n-2)\left|\mathrm{grad}\ln\alpha\right|^2\right)(\mathrm{grad}f,0)\\&+\frac{n-4}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)\\&-n\frac{\beta^2}{\alpha^4}df(\mathrm{grad}\ln\beta)(\mathrm{grad}\ln\beta,0).\end{aligned}\tag{17}$$

To complete the proof, it remains to investigate the term $Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),d\phi)d\phi$, we have
(18)
$$Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),d\phi)d\phi=\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),(e_i,0))(e_i,0)$$
$$+\frac{1}{\alpha^2}\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),(0,f_j))(0,f_j).$$

By (3), a simple calculation gives

$$\widetilde{R}^{M\times_\beta N}\left((\mathrm{grad}f,0),(e_i,0)\right)=R^{M\times_\beta N}\left((\mathrm{grad}f,0),(e_i,0)\right)=\left(R^M\left(\mathrm{grad}f,e_i\right),0\right),$$

then

$$\widetilde{R}^{M\times_\beta N}\left((\mathrm{grad}f,0),(e_i,0)\right)(e_i,0)=\left(Ricci\left(\mathrm{grad}f\right),0\right). \tag{19}$$

For the term $\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),(0,f_j))(0,f_j)$ and by (3), we have

$$\begin{aligned}\widetilde{R}^{M\times_\beta N}\left((\mathrm{grad}f,0),(0,f_j)\right)=&-\left(\nabla_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right)\wedge_{G_f}(0,f_j)\\&-df\left(\mathrm{grad}\ln\beta\right)\left(\mathrm{grad}\ln\beta,0\right)\wedge_{G_f}(0,f_j).\end{aligned}$$

For this expression, we have

$$\begin{aligned}&\left(\left(\nabla_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right)\wedge_{G_f}(0,f_j)\right)(0,f_j)\\&=G_\beta\left((0,f_j),(0,f_j)\right)\left(\nabla_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right)\\&\quad-G_\beta\left((0,f_j),\left(\nabla_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right)\right)(0,f_j)\\&=n\beta^2\left(\nabla_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right)\end{aligned}$$

and

$$\begin{aligned}\left(\left(\mathrm{grad}\ln\beta,0\right)\wedge_{G_f}(0,f_j)\right)(0,f_j)&=G_\beta\left((0,f_j),(0,f_j)\right)\left(\mathrm{grad}\ln\beta,0\right)\\&\quad-G_\beta\left((0,f_j),\left(\mathrm{grad}\ln\beta,0\right)\right)(0,f_j)\\&=n\beta^2\left(\mathrm{grad}\ln\beta,0\right),\end{aligned}$$

then

$$\begin{aligned}\widetilde{R}^{M\times_\beta N}\left((\mathrm{grad}f,0),(0,f_j)\right)(0,f_j)=&-n\beta^2 df\left(\mathrm{grad}\ln\beta\right)\left(\mathrm{grad}\ln\beta,0\right)\\&-n\beta^2\left(\nabla^M_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right).\end{aligned} \tag{20}$$

If we replace (19) and (20) in (18), we obtain

$$\begin{aligned}&Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}\left((\mathrm{grad}f,0),d\phi\right)d\phi\\&=\left(Ricci\left(\mathrm{grad}f\right),0\right)-n\frac{\beta^2}{\alpha^2}df\left(\mathrm{grad}\ln\beta\right)\left(\mathrm{grad}\ln\beta,0\right)\\&\quad-n\frac{\beta^2}{\alpha^2}\left(\nabla^M_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right).\end{aligned} \tag{21}$$

Finally, the equations (17) and (21) give us the following formula:

$$\begin{aligned}
Tr_{G_\alpha}\widetilde{\nabla}^2\frac{1}{\alpha^2}(\mathrm{grad}f,0)+\frac{1}{\alpha^2}Tr_{G_\alpha}\widetilde{R}^{M\times_\beta N}((\mathrm{grad}f,0),d\phi)\,d\phi \\
=\frac{1}{\alpha^2}(\mathrm{grad}\Delta f,0)+\frac{2}{\alpha^2}(Ricci(\mathrm{grad}f),0) \\
-\frac{2}{\alpha^2}\left(\Delta\ln\alpha+(n-2)\left|\mathrm{grad}\ln\alpha\right|^2\right)(\mathrm{grad}f,0) \\
+\frac{n-4}{\alpha^2}\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f,0\right)-n\frac{\beta^2}{\alpha^4}\left(\nabla^M_{\mathrm{grad}f}\mathrm{grad}\ln\beta,0\right) \\
-2n\frac{\beta^2}{\alpha^4}df(\mathrm{grad}\ln\beta)(\mathrm{grad}\ln\beta,0).
\end{aligned}$$

Then ϕ is biharmonic if and only if

$$\begin{aligned}
&\mathrm{grad}\Delta f+2Ricci(\mathrm{grad}f)-2\left(\Delta\ln\alpha+(n-2)\left|\mathrm{grad}\ln\alpha\right|^2\right)\mathrm{grad}f \\
&+(n-4)\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}f\right)-n\frac{\beta^2}{\alpha^2}\left(\nabla^M_{\mathrm{grad}f}\mathrm{grad}\ln\beta\right) \\
&-2n\frac{\beta^2}{\alpha^2}df(\mathrm{grad}\ln\beta)\,\mathrm{grad}\ln\beta=0.
\end{aligned}$$

The proof of Theorem 1 is complete. □

As consequences, we obtain

Corollary 1. *The map $\phi:(M\times_\alpha N,G_\alpha)\longrightarrow(M\times N,G)$ defined by $\phi(x,y)=(x,y)$ is biharmonic if and only if*

$$grad(\Delta\ln\alpha)+\frac{n}{2}grad\left(\left|grad\ln\alpha\right|^2\right)+2Ricci(grad\ln\alpha)=0.$$

Proof of Corollary 1: By Theorem 1, if we replace $f=\alpha^2$, we deduce that the map $\phi:(M\times_\alpha N,G_\alpha)\longrightarrow(M\times N,G)$ defined by $\phi(x,y)=(x,y)$ is biharmonic if and only if

$$\begin{aligned}
&\mathrm{grad}\Delta\alpha^2+2Ricci\left(\mathrm{grad}\alpha^2\right)-2\left(\Delta\ln\alpha+(n-2)\left|\mathrm{grad}\ln\alpha\right|^2\right)\mathrm{grad}\alpha^2 \\
&\quad+(n-4)\left(\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}\alpha^2\right)=0.
\end{aligned}$$

A simple calculation gives

$$\mathrm{grad}\alpha^2=2\alpha^2\mathrm{grad}\ln\alpha,$$

$$\begin{aligned}
\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}\alpha^2&=2\nabla^M_{\mathrm{grad}\ln\alpha}\alpha^2\mathrm{grad}\ln\alpha \\
&=2\alpha^2\nabla^M_{\mathrm{grad}\ln\alpha}\mathrm{grad}\ln\alpha+2\mathrm{grad}\ln\alpha\left(\alpha^2\right)\mathrm{grad}\ln\alpha \\
&=\alpha^2\mathrm{grad}\left(\left|\mathrm{grad}\ln\alpha\right|^2\right)+4\alpha^2\left|\mathrm{grad}\ln\alpha\right|^2\mathrm{grad}\ln\alpha.
\end{aligned}$$

It is known that

$$\Delta\alpha^2 = 2\alpha^2\Delta\ln\alpha + 4\alpha^2 \left|\operatorname{grad}\ln\alpha\right|^2,$$

then

$$\begin{aligned}
\operatorname{grad}\Delta\alpha^2 &= \operatorname{grad}\left(2\alpha^2\Delta\ln\alpha + 4\alpha^2\left|\operatorname{grad}\ln\alpha\right|^2\right)\\
&= 2\operatorname{grad}\left(\alpha^2\Delta\ln\alpha\right) + 4\operatorname{grad}\left(\alpha^2\left|\operatorname{grad}\ln\alpha\right|^2\right)\\
&= 2\alpha^2\operatorname{grad}\left(\Delta\ln\alpha\right) + 4\alpha^2\left(\Delta\ln\alpha\right)\operatorname{grad}\ln\alpha\\
&\quad + 4\alpha^2\operatorname{grad}\left(\left|\operatorname{grad}\ln\alpha\right|^2\right) + 8\alpha^2\left|\operatorname{grad}\ln\alpha\right|^2\operatorname{grad}\left(\ln\alpha\right).
\end{aligned}$$

Finally, we conclude that the map $\phi : (M \times_\alpha N, G_\alpha) \longrightarrow (M \times N, G)$ defined by $\phi(x,y) = (x,y)$ is biharmonic if and only if

$$\operatorname{grad}\left(\Delta\ln\alpha\right) + \frac{n}{2}\operatorname{grad}\left(\left|\operatorname{grad}\ln\alpha\right|^2\right) + 2Ricci\left(\operatorname{grad}\ln\alpha\right) = 0.$$

□

In the following we shall present some examples of biharmonic non-harmonic maps.

Example 1. Let $\phi : \mathbb{R}^m \setminus \{0\} \times_\alpha N^n \longrightarrow \mathbb{R}^m \setminus \{0\} \times N^n$ $(m \neq 2)$ be defined by $\phi(x,y) = (x,y)$ when we suppose that $\ln\alpha$ is radial ($\ln\alpha = f(r)$). Then by Corollary 1, we deduce that the map ϕ is biharmonic if and only if the function f satisfies the following differential equation

$$f''' + \frac{m-1}{r}f'' - \frac{m-1}{r^2}f' + nf'f'' = 0.$$

Let $\beta = f'$, this equation becomes

$$\beta'' + \frac{m-1}{r}\beta' - \frac{m-1}{r^2}\beta + n\beta\beta' = 0.$$

Looking for particular solutions of type $\beta = a/r$ $(a \in \mathbb{R}^*)$, then ϕ is biharmonic if and only if

$$a = \frac{4-2m}{n}.$$

We obtain $\alpha(r) = Cr^{\frac{4-2m}{n}}$ $(C > 0)$ and in this case the map $\phi : \mathbb{R}^m \setminus \{0\} \times_\alpha N^n \longrightarrow \mathbb{R}^m \setminus \{0\} \times N^n$ defined by $\phi(x,y) = (x,y)$ is biharmonic non-harmonic.

Example 2. Consider $M = S^m$ provided parametrization $x = (\cos s, \sin s.z)$ $s \in [0,\pi]$, $z \in S^{m-1}$. An orthonormal basis for S^m is given by $e_1 = \frac{\partial}{\partial s}$, $e_i = (0, f_i)$ for $i = 2,\dots,m$, where f_i are tangent to the sphere S^{m-1}. We have $\sum_{i=2}^m \nabla_{e_i}e_i = -(m-1)\cot s\frac{\partial}{\partial s}$. We consider the map $\phi : S^m \times_\alpha N^n \longrightarrow S^m \times N^n$ defined by $\phi(x,y) = (x,y) = ((\cos s, \sin s.z), y)$ when we suppose that the function $f = \ln\alpha$

depends only on s. Then by Corollary 1, we deduce that the map ϕ is biharmonic if and only if the function f satisfies the following differential equation

$$f''' + nf'f'' + (m-1)\left((\cot s)f'' - \left(1-\cot^2 s\right)f'\right) = 0.$$

Let $\gamma(s) = f'(s)$, then the last equation becomes

$$\gamma'' + n\gamma\gamma' + (m-1)\left((\cot s)\gamma' - \left(1-\cot^2 s\right)\gamma\right) = 0.$$

For example, if $m = 1$, the function $\gamma(s) = \frac{1}{ns+C}$ is a solution of this equation, and we obtain

$$\alpha(s) = \sqrt[n]{(ns+C)^2}.$$

In this case, the map $\phi : S^1 \times_\alpha N^n \longrightarrow S^1 \times N^n$ defined by $\phi(x,y) = (x,y) = ((\cos s, \sin s), y)$ is biharmonic non-harmonic.

A similar result is given by the following corollary:

Corollary 2. *The map $\phi : (M \times N, G) \longrightarrow (M \times_\beta N, G_\beta)$ defined by $\phi(x,y) = (x,y)$ is biharmonic if and only if*

$$\begin{aligned} &grad\Delta \ln\beta + 2(\Delta\ln\beta)\, grad\ln\beta + \left(4 - 2n\beta^2\right)|grad\ln\beta|^2\, grad\ln\beta \\ &+ \left(2 - \frac{n}{2}\beta^2\right) grad\left(|grad\ln\beta|^2\right) + 2Ricci\,(grad\ln\beta) = 0. \end{aligned}$$

Equivalently, ϕ is biharmonic if and only if the function $f = \beta^2$ satisfies the following equation

$$grad\Delta f + 2Ricci\,(gradf) - \frac{n}{4} grad\left(|gradf|^2\right) = 0.$$

As a second result, we will study the biharmonicity of the map $\widetilde{\phi} : (M^m \times_f N^n, G_\alpha) \longrightarrow (P_1^{p_1} \times P_2^{p_2}, G)$ defined by $\widetilde{\phi}(x,y) = (\phi(x), \psi(y))$. We have the following theorem:

Theorem 2. *Let $\widetilde{\phi} : (M^m \times_f N^n, G_\alpha) \longrightarrow (P_1^{p_1} \times P_2^{p_2}, G)$ be defined by $\widetilde{\phi}(x,y) = (\phi(x), \psi(y))$ where $\phi : (M,g) \longrightarrow (P_1,k_1)$ and $\psi : (N,h) \longrightarrow (P_2,k_2)$ are two harmonic maps. Then the map $\widetilde{\phi}$ is biharmonic if and only if*

$$\begin{aligned} (22)\quad &Tr_g\left(\nabla^\phi\right)^2 d\phi\,(\mathrm{grad}\ln f) \\ &+ Tr_g R^{P_1}\left(d\phi\,(\mathrm{grad}\ln f), d\phi\right) d\phi + n\nabla^\phi_{\mathrm{grad}\ln f} d\phi\,(\mathrm{grad}\ln f) = 0. \end{aligned}$$

Proof of Theorem 2: Let us choose $(e_i)_{1\le i\le m}$ to be an orthonormal frame on M and $(f_j)_{1\le j\le n}$ to be an orthonormal frame on N. An orthonormal frame on $M \times_f N$ is given by $\{(e_i,0), \frac{1}{f}(0,f_j)\}$. Note that in this case we have $d\widetilde{\phi}(X,Y) =$

$(d\phi(X), d\psi(Y))$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. By definition of the tension field, we have

$$\begin{aligned}
\tau\left(\widetilde{\phi}\right) &= Tr_{G_f} \nabla d\widetilde{\phi} \\
&= \nabla^{\widetilde{\phi}}_{(e_i,0)} d\widetilde{\phi}\left(e_i,0\right) + \frac{1}{f^2\circ\pi} \nabla^{\widetilde{\phi}}_{(0,f_j)} d\widetilde{\phi}\left(0,f_j\right) \\
&\quad - d\widetilde{\phi}\left(\widetilde{\nabla}_{(e_i,0)}\left(e_i,0\right)\right) - \frac{1}{f^2\circ\pi} d\widetilde{\phi}\left(\widetilde{\nabla}_{(0,f_j)}\left(0,f_j\right)\right).
\end{aligned}$$

Using equation (1), a direct calculation gives $\widetilde{\nabla}_{(e_i,0)}\left(e_i,0\right) = \left(\nabla_{e_i} e_i, 0\right)$ and

$$\widetilde{\nabla}_{(0,f_j)}\left(0,f_j\right) = \left(0, \nabla_{f_j} f_j\right) - n\left(f^2\circ\pi\right)\left(\operatorname{grad}\ln f, 0\right),$$

then

$$\begin{aligned}
\tau\left(\widetilde{\phi}\right) &= \left(\nabla^{\phi}_{e_i} d\phi\left(e_i\right), 0\right) - \left(d\phi\left(\nabla_{e_i} e_i\right), 0\right) \\
&\quad + \frac{1}{f^2\circ\pi}\left(0, \nabla^{\psi}_{f_j} d\psi\left(f_j\right)\right) - \frac{1}{f^2\circ\pi}\left(0, d\psi\left(\nabla_{f_j} f_j\right)\right) + n\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) \\
&= \left(\tau\left(\phi\right), 0\right) + \frac{1}{f^2\circ\pi}\left(0, \tau\left(\psi\right)\right) + n\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right).
\end{aligned}$$

Since ϕ and ψ are harmonic, we deduce that

$$\tau\left(\widetilde{\phi}\right) = n(d\phi(\operatorname{grad}\ln f), 0).$$

By definition, the map $\widetilde{\phi}$ is biharmonic if and only if

(23)
$$Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) + Tr_{G_f} R^{P_1\times P_2}\left((d\phi(\operatorname{grad}\ln f), 0), d\widetilde{\phi}\right) d\widetilde{\phi} = 0.$$

Let us start with the simplification of the term $Tr_{G_f}(\nabla^{\widetilde{\phi}})^2(d\phi(\operatorname{grad}\ln f), 0)$, we have

$$\begin{aligned}
&Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) = \nabla^{\tilde{\phi}}_{(e_i,0)} \nabla^{\tilde{\phi}}_{(e_i,0)}\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) \\
&- \nabla^{\tilde{\phi}}_{\tilde{\nabla}^{M\times_f N}_{(e_i,0)}(e_i,0)}\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) \\
&+ \frac{1}{f^2\circ\pi}\left(\nabla^{\tilde{\phi}}_{(0,f_j)} \nabla^{\tilde{\phi}}_{(0,f_j)}\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right) - \nabla^{\tilde{\phi}}_{\tilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\left(d\phi\left(\operatorname{grad}\ln f\right), 0\right)\right).
\end{aligned}$$

Term by term for this expression, we have

$$\begin{aligned}
\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\left(d\phi\left(\operatorname{grad}\ln f\right),0\right) &= \left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(\operatorname{grad}\ln f\right),0\right),\\
\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}^{M\times_f N}_{(e_i,0)}(e_i,0)}\left(d\phi(\operatorname{grad}\ln f),0\right) &= \nabla^{\phi}_{\left(\nabla_{e_i}e_i,0\right)}\left(d\phi\left(\operatorname{grad}\ln f\right),0\right)\\
&= \left(\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(\operatorname{grad}\ln f\right),0\right),
\end{aligned}$$

$$\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\left(d\phi\left(\operatorname{grad}\ln f\right),0\right)=0$$

and

$$\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}^{M\times_\alpha N}_{(0,f_j)}(0,f_j)}\left(d\phi\left(\operatorname{grad}\ln f\right),0\right) = -n\left(f^2\circ\pi\right)\left(\nabla^{\phi}_{\operatorname{grad}\ln f}d\phi\left(\operatorname{grad}\ln f\right),0\right).$$

Then we have

$$\begin{aligned}
(24)\quad Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\left(d\phi\left(\operatorname{grad}\ln f\right),0\right) = \Big(&Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(\operatorname{grad}\ln f\right)\\
&+ n\nabla^{\phi}_{\operatorname{grad}\ln f}d\phi\left(\operatorname{grad}\ln f\right),0\Big).
\end{aligned}$$

For the term $Tr_{G_f}R^{P_1\times P_2}((d\phi(\operatorname{grad}\ln f),0),d\widetilde{\phi})d\widetilde{\phi}$, we have

$$\begin{aligned}
&Tr_{G_f}R^{P_1\times P_2}\left(\left(d\phi(\operatorname{grad}\ln f),0\right),d\widetilde{\phi}\right)d\widetilde{\phi}\\
&= R^{P_1\times P_2}\left(\left(d\phi(\operatorname{grad}\ln f),0\right),d\widetilde{\phi}\left(e_i,0\right)\right)d\widetilde{\phi}\left(e_i,0\right)\\
&\quad+\frac{1}{f^2\circ\pi}R^{P_1\times P_2}\left(\left(d\widetilde{\phi}(\operatorname{grad}\ln f),0\right),d\widetilde{\phi}\left(0,f_j\right)\right)d\widetilde{\phi}\left(0,f_j\right).
\end{aligned}$$

It is very simple to see that

$$\begin{aligned}
&R^{P_1\times P_2}\left(\left(d\phi\left(\operatorname{grad}\ln f\right),0\right),d\widetilde{\phi}\left(e_i,0\right)\right)d\widetilde{\phi}\left(e_i,0\right)\\
&\qquad= \left(Tr_gR^{P_1}(d\phi(\operatorname{grad}\ln f),d\phi)d\phi,0\right)
\end{aligned}$$

and

$$R^{P_1\times P_2}\left(\left(d\phi(\operatorname{grad}\ln f),0\right),d\widetilde{\phi}\left(0,f_j\right)\right)d\widetilde{\phi}\left(0,f_j\right)=0,$$

then

$$\begin{aligned}
(25)\qquad &Tr_{G_f}R^{P_1\times P_2}\left(\left(d\phi\left(\operatorname{grad}\ln f\right),0\right),d\widetilde{\phi}\right)d\widetilde{\phi}\\
&\qquad= \left(Tr_gR^{P_1}\left(d\phi\left(\operatorname{grad}\ln f\right),d\phi\right)d\phi,0\right).
\end{aligned}$$

Both equations (24) and (25) give us

$$\begin{aligned}
&Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2(d\phi(\operatorname{grad}\ln f),0)+Tr_{G_f}R^{P_1\times P_2}\left((d\phi(\operatorname{grad}\ln f),0),d\widetilde{\phi}\right)d\widetilde{\phi}\\
&=\Big(Tr_g\left(\nabla^{\phi}\right)^2 d\phi(\operatorname{grad}\ln f)+n\nabla^{\phi}_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f)\\
&\quad+Tr_gR^{P_1}(d\phi(\operatorname{grad}\ln f),d\phi)\,d\phi,0\Big).
\end{aligned}$$

We conclude that the map $\widetilde{\phi}$ is biharmonic if and only if ϕ satisfies the following equation

$$\begin{aligned}
Tr_g\left(\nabla^{\phi}\right)^2 d\phi(\operatorname{grad}\ln f)+Tr_gR^{P_1}(d\phi(\operatorname{grad}\ln f),d\phi)\,d\phi&\\
+n\nabla^{\phi}_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f)&=0.
\end{aligned}$$

The proof of Theorem 2 is complete. □

As an application of this theorem, we have the following example:

Example 3. Let $\phi:\mathbb{R}^4\longrightarrow\mathbb{R}^3$ be the projection defined by $\phi(t,x_2,x_3,x_4)=(t,x_2,x_3)$ and we consider the map $\widetilde{\phi}:\mathbb{R}^4\times_f N^n\longrightarrow\mathbb{R}^3\times N^n$ defined by $\widetilde{\varphi}((t,x_2,x_3,x_4),y)=((t,x_2,x_3),y)$ when we suppose that the function $\alpha=\ln f$ depends only on t. By Theorem 2, $\widetilde{\phi}$ is biharmonic if and only if $\alpha=\ln f$ satisfies the following differential equation of the third order

$$\alpha'''+n\alpha'\alpha''=0.$$

Let $\beta(t)=\alpha'(t)$, then the last equation becomes

$$\beta''+n\beta\beta'=0.$$

For example, the function $\beta=\frac{2}{nt+C}$ is a solution of this equation, we obtain $f(t)=\sqrt[n]{(nt+C)^2}$ and in this case, the map $\widetilde{\phi}$ is biharmonic non-harmonic.

An immediate consequence of Theorem 2 is given by the following corollary:

Corollary 3. *Let $\widetilde{\phi}:(M^m\times_f N^n,G_\alpha)\longrightarrow(M^m\times P^p,G)$ be defined by $\widetilde{\phi}(x,y)=(x,\psi(y))$ where $\psi:N\longrightarrow P$ is a harmonic map. Then $\widetilde{\phi}$ is biharmonic if and only if*

$$grad\Delta\ln f+2Ricci^M(grad\ln f)+\frac{n}{2}grad\left(|grad\ln f|^2\right)=0.$$

In particular, if $\psi=Id_N$, we obtain

Corollary 4. *Let $\phi:(M^m\times_f N^n,G_\alpha)\longrightarrow(M^m\times N^n,G)$ be defined by $\phi(x,y)=(x,y)$. Then ϕ is biharmonic if and only if*

$$grad\Delta\ln f+2Ricci^M(grad\ln f)+\frac{n}{2}grad\left(|grad\ln f|^2\right)=0.$$

Now, we consider a harmonic map $\phi : (N^n, h) \longrightarrow (N^n, \widetilde{h})$ and we study the biharmonicity of the map $\widetilde{\phi} : N \longrightarrow (M \times_f N, G_f)$ defined by $\widetilde{\phi}(y) = (x_0, \phi(y))$.

Theorem 3. *Let $\phi : (N^n, h) \longrightarrow (P^p, k)$ be a harmonic map, then the map $\widetilde{\phi} : N \longrightarrow (M \times_f P, G_f)$ defined by $\widetilde{\phi}(y) = (x_0, \phi(y))$ is biharmonic if and only if*

$$\left\{ \begin{array}{c} (e(\phi))^2 \, grad \left(\left| \mathrm{grad} f^2 \right|^2 \right) - 2 \left(\Delta e(\phi) \right) \mathrm{grad} f^2 = 0, \\ d\phi \left(\mathrm{grad} \left(e(\phi) \right) \right) = 0. \end{array} \right.$$

Proof of Theorem 3: Let us choose $(f_j)_{1 \leq j \leq n}$ to be an orthonormal frame on N. The tension field of $\widetilde{\phi}$ is given by

$$\begin{aligned} \tau\left(\widetilde{\phi}\right) &= Tr_h \widetilde{\nabla} d\widetilde{\phi} \\ &= \widetilde{\nabla}^{\widetilde{\phi}}_{f_j} d\widetilde{\phi}(f_j) - d\widetilde{\phi}\left(\nabla^N_{f_j} f_j\right) \\ &= \widetilde{\nabla}^{\widetilde{\phi}}_{f_j} (0, d\phi(f_j)) - \left(0, d\phi\left(\nabla^N_{f_j} f_j\right)\right) \\ &= \left(0, \nabla^{\phi}_{f_j} d\phi(f_j)\right) - 2f^2 e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} - \left(0, d\phi\left(\nabla^N_{f_j} f_j\right)\right) \\ &= (0, \tau(\phi)) - 2f^2 e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi}. \end{aligned}$$

Since ϕ is harmonic, it follows that

$$\tau\left(\widetilde{\phi}\right) = -2f^2 e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi}.$$

Then $\widetilde{\phi}$ is biharmonic if and only if

$$Tr_h \left(\nabla^{\widetilde{\phi}}\right)^2 e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} + e(\phi) Tr_h \widetilde{R}^{M \times_f P} \left(\left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi}, d\widetilde{\phi} \right) d\widetilde{\phi} = 0. \tag{26}$$

For the first term $Tr_h (\nabla^{\widetilde{\phi}})^2 e(\phi) (\mathrm{grad} \ln f, 0) \circ \widetilde{\phi}$ of (26), we have by definition

$$\begin{aligned} (27) \quad & Tr_h \left(\nabla^{\widetilde{\phi}}\right)^2 e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} \\ & = \widetilde{\nabla}^{\widetilde{\phi}}_{f_j} \widetilde{\nabla}^{\widetilde{\phi}}_{f_j} e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} - \widetilde{\nabla}^{\widetilde{\phi}}_{\nabla^N_{f_j} f_j} e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi}. \end{aligned}$$

(Here henceforth we sum over repeated indices.) Calculate the first term $\widetilde{\nabla}^{\widetilde{\phi}}_{f_j} \widetilde{\nabla}^{\widetilde{\phi}}_{f_j} e(\phi) (\mathrm{grad} \ln f, 0) \circ \widetilde{\phi}$ of (27). Using (1), we obtain

$$\begin{aligned} & \nabla^{\widetilde{\phi}}_{f_j} e(\phi) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} \\ & = e(\phi) \nabla^{\widetilde{\phi}}_{f_j} \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} + f_j \left(e(\phi)\right) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi} \\ & = e(\phi) \left|\mathrm{grad} \ln f\right|^2 (0, d\phi(f_j)) \circ \widetilde{\phi} + f_j \left(e(\phi)\right) \left(\mathrm{grad} \ln f, 0\right) \circ \widetilde{\phi}, \end{aligned}$$

which gives us

$$
\begin{aligned}
&\nabla^{\widetilde{\phi}}_{f_j}\nabla^{\widetilde{\phi}}_{f_j} e(\phi)(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&=\nabla^{\widetilde{\phi}}_{f_j}\left(e(\phi)\left|\operatorname{grad}\ln f\right|^2(0,d\phi(f_j))\right)\circ\widetilde{\phi}+\nabla^{\widetilde{\phi}}_{f_j}\left(f_j(e(\phi))(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\right)\\
&=e(\phi)\left|\operatorname{grad}\ln f\right|^2\nabla^{\widetilde{\phi}}_{f_j}(0,d\phi(f_j))\circ\widetilde{\phi}+\left|\operatorname{grad}\ln f\right|^2 f_j(e(\phi))(0,d\phi(f_j))\circ\widetilde{\phi}\\
&\quad+f_j(e(\phi))\nabla^{\widetilde{\phi}}_{f_j}(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}+f_j(f_j(e(\phi)))(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\nabla^{\widetilde{\phi}}_{f_j}\nabla^{\widetilde{\phi}}_{f_j} e(\phi)(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}=&\left|\operatorname{grad}\ln f\right|^2 e(\phi)\left(0,\nabla^{\phi}_{f_j}d\phi(f_j)\right)\circ\widetilde{\phi}\\
&-2f^2(e(\phi))^2\left|\operatorname{grad}\ln f\right|^2(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&+2\left|\operatorname{grad}\ln f\right|^2(0,d\phi(\operatorname{grad}e(\phi)))\circ\widetilde{\phi}\\
&+f_j(f_j(e(\phi)))(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}.
\end{aligned}
\tag{28}
$$

Always using the equation (1), a simple calculation gives us

$$
\begin{aligned}
&\nabla^{\widetilde{\phi}}_{\nabla^N_{f_j}f_j} e(\phi)(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&=e(\phi)\nabla^{\widetilde{\phi}}_{\nabla^N_{f_j}f_j}(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}+\left(\nabla^N_{f_j}f_j\right)(e(\phi))(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&=\left|\operatorname{grad}\ln f\right|^2 e(\phi)\left(0,d\phi\left(\nabla^N_{f_j}f_j\right)\right)\circ\widetilde{\phi}\\
&\quad+\left(\nabla^N_{f_j}f_j\right)(e(\phi))(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}.
\end{aligned}
\tag{29}
$$

By replacing (28) and (29) in (27) and using the fact that ϕ is harmonic, we obtain

$$
\begin{aligned}
&Tr_h\left(\nabla^{\widetilde{\phi}}\right)^2 e(\phi)(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&=-2f^2(e(\phi))^2\left|\operatorname{grad}\ln f\right|^2(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}\\
&\quad+2\left|\operatorname{grad}\ln f\right|^2(0,d\phi(\operatorname{grad}(e(\phi))))\circ\widetilde{\phi}\\
&\quad+\Delta e(\phi)(\operatorname{grad}\ln f,0)\circ\widetilde{\phi}.
\end{aligned}
\tag{30}
$$

To complete the proof, it remains to investigate the term
$Tr_h\widetilde{R}^{M\times_f P}((\operatorname{grad}\ln f,0)\circ\widetilde{\phi},d\widetilde{\phi})d\widetilde{\phi}$, we have

$$
\begin{aligned}
&Tr_h\widetilde{R}^{M\times_f P}\left((\operatorname{grad}\ln f,0)\circ\widetilde{\phi},d\widetilde{\phi}\right)d\widetilde{\phi}\\
&\qquad=\widetilde{R}^{M\times_f P}((\operatorname{grad}\ln f,0),(0,d\phi(f_j)))(0,d\phi(f_j))\circ\widetilde{\phi}.
\end{aligned}
$$

By (3), a simple calculation gives

$$\begin{aligned}\widetilde{R}^{M\times_f P}\left((\operatorname{grad}\ln f,0),(0,d\phi\left(f_j\right))\right) = &-\frac{1}{2}\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\wedge_{G_f}\left(0,d\phi\left(f_j\right)\right)\\ &-\left|\operatorname{grad}\ln f\right|^2\left(\operatorname{grad}\ln f,0\right)\wedge_{G_f}\left(0,d\phi\left(f_j\right)\right).\end{aligned}$$

To simplify this expression, we have

$$\begin{aligned}&\left(\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\wedge_{G_f}\left(0,d\phi\left(f_j\right)\right)\right)\left(0,d\phi\left(f_j\right)\right)\\ &\qquad= 2f^2 e\left(\phi\right)\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\end{aligned}$$

and

$$\left(\left(\operatorname{grad}\ln f,0\right)\wedge_{G_f}\left(0,d\phi\left(f_j\right)\right)\right)\left(0,d\phi\left(f_j\right)\right) = 2f^2 e\left(\phi\right)\left(\operatorname{grad}\ln f,0\right).$$

Then

$$\begin{aligned}&Tr_h\widetilde{R}^{M\times_f P}\left(\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi},d\widetilde{\phi}\right)d\widetilde{\phi}\\ &= -f^2 e\left(\phi\right)\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\circ\widetilde{\phi}\\ &\quad - 2f^2\left|\operatorname{grad}\ln f\right|^2 e(\phi)\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi}.\end{aligned}\tag{31}$$

Finally, we conclude that

$$\begin{aligned}&Tr_h\left(\nabla^{\widetilde{\phi}}\right)^2 e\left(\phi\right)\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi} + e\left(\phi\right)Tr_h\widetilde{R}^{M\times_f P}\left(\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi},d\widetilde{\phi}\right)d\widetilde{\phi}\\ &= -4f^2\left(e\left(\phi\right)\right)^2\left|\operatorname{grad}\ln f\right|^2\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi} + \Delta e\left(\phi\right)\left(\operatorname{grad}\ln f,0\right)\circ\widetilde{\phi}\\ &\quad - f^2\left(e(\phi)\right)^2\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\circ\widetilde{\phi}\\ &\quad + 2\left|\operatorname{grad}\ln f\right|^2\left(0,d\phi\left(\operatorname{grad}\left(e(\phi)\right)\right)\right)\circ\widetilde{\phi}.\end{aligned}$$

We deduce that $\widetilde{\phi}$ is biharmonic if and only if

$$\begin{cases} f^2\left(e(\phi)\right)^2\left(4\left|\operatorname{grad}\ln f\right|^2\operatorname{grad}\ln f + \operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)\right),\\ -\left(\Delta e\left(\phi\right)\right)\operatorname{grad}\ln f = 0,\\ d\phi\left(\operatorname{grad}\left(e\left(\phi\right)\right)\right) = 0.\end{cases}$$

It is easy to see that

$$\begin{aligned}\operatorname{grad}\left(\left|\operatorname{grad}f^2\right|^2\right) &= \operatorname{grad}\left(\left|2f^2\operatorname{grad}\ln f\right|^2\right)\\ &= 4\operatorname{grad}\left(f^4\left|\operatorname{grad}\ln f\right|^2\right)\\ &= 4f^4\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right) + 16f^4\left|\operatorname{grad}\ln f\right|^2\operatorname{grad}\ln f\\ &= 4f^4\left(4\left|\operatorname{grad}\ln f\right|^2\operatorname{grad}\ln f + \operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)\right),\end{aligned}$$

which gives us

$$4\left|\operatorname{grad}\ln f\right|^2\operatorname{grad}\ln f + \operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right) = \frac{1}{4f^4}\operatorname{grad}\left(\left|\operatorname{grad}f^2\right|^2\right).$$

Then $\widetilde{\phi}$ is biharmonic if and only if

$$\begin{cases}(e(\phi))^2\operatorname{grad}\left(\left|\operatorname{grad}f^2\right|^2\right) - 2(\Delta e(\phi))\operatorname{grad}f^2 = 0,\\ d\phi(\operatorname{grad}(e(\phi))) = 0.\end{cases}$$

The proof of Theorem 3 is complete. □

By application of Theorem 3, if the function $e(\phi)$ is constant, we get the following result.

Corollary 5. *Let $\phi : (N^n, h) \longrightarrow (P^p, k)$ be a harmonic map when we suppose that the function $e(\phi)$ is constant. Then the map $\widetilde{\phi} : N \longrightarrow (M \times_f P, G_f)$ defined by $\widetilde{\phi}(y) = (x_0, \phi(y))$ is biharmonic if and only if*

$$\mathit{grad}\left(\left|\mathit{grad}f^2\right|^2\right) = 0.$$

In particular, if $\phi = Id_N$, we obtain (see [7]):

Corollary 6. *The inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$ defined by $i_{x_0}(y) = (x_0, y)$ is biharmonic if and only if*

$$\mathit{grad}\left(\left|\mathit{grad}f^2\right|^2\right) = 0.$$

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Department of Mathematics, U.S.T. Oran, Algeria

E-mail: abdelmadjid.bennouar@univ-usto.dz

Laboratory of Geometry, Analysis, Control and Applications, University of Saida, Algeria

E-mail: seddik.ouakkas@univ-saida.dz

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