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Sufficient conditions for the solvability of some third order functional boundary value problems on the half-line

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Abstract. This paper is concerned with the existence of bounded or unbounded solutions to third-order boundary value problem on the half-line with functional boundary conditions. The arguments are based on the Green functions, a Nagumo condition, Schauder fixed point theorem and lower and upper solutions method. An application to a Falkner-Skan equation with functional boundary conditions is given to illustrate our results.

Keywords: functional boundary conditions; unbounded solutions; half-line; upper and lower solutions; Nagumo condition; Green's function; fixed point theory; Falkner-Skan equation

Classification: 34B10, 34B15, 34B27, 34B40, 34B60, 45G10

1. Introduction

In this paper we consider a third order boundary value problem, composed by the fully differential equation

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), \quad t \in [0, +\infty), \tag{1}$$

where $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function, and the functional boundary conditions on the half-line

$$\begin{aligned} L_0(u, u(0)) &= 0, \\ L_1\left(u, u'(0)\right) &= 0, \\ L_2\left(u, u''(+\infty)\right) &= 0, \end{aligned} \tag{2}$$

with $L_i : C(\mathbb{R}_0^+) \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, 2$, continuous functions verifying some monotone assumptions (see (H_4)) and

$$u''(+\infty) := \lim_{t \to +\infty} u''(t).$$

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There is an extensive literature on Boundary Value Problems (BVP) in bounded domains, as this type of problems is an adequate tool to describe countless phenomena of real life, such as models on chemical engineering, heat conduction, thermo-elasticity, plasma physics, fluids flow,... (see, for instance, [2], [6], [8], [11], [12], [13], [14], [16]). However, on the real line or half-line the results are scarcer (see, for example, [1], [17] and the references therein).

In some backgrounds the models require different kinds of nonlocal or integral boundary conditions. In this way, it is useful to consider generalized boundary data, which include usual and non classic boundary conditions. In fact, if BVP contains a functional dependence on the unknown functions, or in its derivatives, either in the differential equation, or in the boundary data, these functional BVP allow a much more variety of problems such as separated, multi-point, nonlocal, integro-differential, with maximum or minimum arguments,..., as it can be seen, for instance, in [3], [4], [7], [9], [10], [15].

To the authors' best knowledge, it is the first time where this type of functional boundary conditions are applied to third order BVP on the half-line. From the different arguments used we highlight the weighted norms, fixed point theory and lower and upper solutions method. This last technique provides a location result, which is particularly useful to get some qualitative properties on the solution, such as positivity, monotony and convexity, among others.

The paper is organized as it follows: in Section 2 some auxiliary results are stated such as the adequate space of admissible functions, weighted norms, an existence result for a linear BVP via Green's functions, an *a priori* bound for the second derivative from a Nagumo-type condition, a criterion to overcome the lack of compactness, and the definition of lower and upper solutions. Section 3 contains the main result: an existence and localization theorem whose proof combines lower and upper solution technique with the fixed point theory. Finally an application to a Falkner-Skan equation is shown to illustrate our results, which are not covered by previous works in the literature, as far as we know.

2. Definitions and preliminary results

Consider the space

$$X = \left\{ x \in C^2\left(\mathbb{R}_0^+\right) : \lim_{t\to+\infty} \frac{x(t)}{1+t^2} \in \mathbb{R},\ \lim_{t\to+\infty} \frac{x'(t)}{1+t} \in \mathbb{R},\ \lim_{t\to+\infty} x''(t) \in \mathbb{R} \right\}$$

with the norm $\|x\|_X = \max\{\|x\|_0, \|x'\|_1, \|x''\|_2\}$, where

$$\|\omega\|_0 := \sup_{0\le t<+\infty} \frac{|\omega(t)|}{1+t^2},\ \|\omega\|_1 := \sup_{0\le t<+\infty} \frac{|\omega(t)|}{1+t} \text{ and } \|\omega\|_2 := \sup_{0\le t<+\infty} \|\omega(t)\|.$$

In this way $(X, \|.\|_X)$ is a Banach space.

Definition 1. A function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is L^1-Carathéodory if it verifies

(i) for each $x, y, z \in \mathbb{R}$, $t \mapsto f(t, x, y, z)$ is measurable on $\mathbb{R}_0^+$;

(ii) for almost every $t \in [0, +\infty), (x, y, z) \mapsto f(t, x, y, z)$ is continuous in $\mathbb{R}^3$;
(iii) for each $\rho > 0$, there exists a positive function ϕ_ρ such that $\phi_\rho, t\phi_\rho, t^2\phi_\rho \in L^1(\mathbb{R}_0^+)$ and for $(x(t), y(t), z(t)) \in \mathbb{R}^3$ with

$$\sup_{0 \le t < +\infty} \left\{ \frac{|x(t)|}{1+t^2}, \frac{|y(t)|}{1+t}, |z(t)| \right\} < \rho,$$

one has

$$|f(t, x, y, z)| \le \phi_\rho(t), \quad \text{a.e. } t \in [0, +\infty).$$

The solutions of the linear problem associated to (1), with the usual two-point boundary conditions, can be defined with Green's function:

Lemma 2. *Let $t^2h, th, h \in L^1(\mathbb{R}_0^+)$. Then the linear boundary value problem*

$$\begin{cases} u'''(t) = h(t), \ \text{a.e. } t \in [0, +\infty), \\ u(0) = A, \\ u'(0) = B, \\ u''(+\infty) = C \end{cases} \tag{3}$$

with $A, B, C \in \mathbb{R}$, has a unique solution given by

$$u(t) = A + Bt + \frac{Ct^2}{2} + \int_0^{+\infty} G(t, s)h(s)\, ds \tag{4}$$

where

$$G(t, s) = \begin{cases} \frac{s^2}{2} - ts, & 0 \le s \le t \\ -\frac{t^2}{2}, & 0 \le t \le s < +\infty. \end{cases} \tag{5}$$

Proof: If u is a solution of problem (3), then the general solution for the differential equation is:

$$u(t) = c_1 + c_2 t + c_3 t^2 + \int_0^t \left(\frac{s^2}{2} - ts + \frac{t^2}{2} \right) h(s)\, ds,$$

where c_1, c_2, c_3 are real constants. Since $u(t)$ should satisfy the boundary conditions, we get

$$c_1 = A, c_2 = B, c_3 = \frac{C}{2} - \frac{1}{2} \int_0^{+\infty} h(s)\, ds,$$

and, therefore,

$$u(t) = A + Bt + \frac{Ct^2}{2} - \frac{t^2}{2} \int_0^{+\infty} h(s)\, ds + \int_0^t \left(\frac{s^2}{2} - ts + \frac{t^2}{2} \right) h(s)\, ds,$$

which can be written as (4), with $G(t, s)$ given by (5). $\square$

Some trivial properties of (5) will play an important role forward:

Lemma 3. *Function $G(t,s)$ defined by* (5) *verifies*

(i) $\lim_{t\to+\infty}\frac{G(t,s)}{1+t^2}\in\mathbb{R}$, $\forall s\in\mathbb{R}$;

(ii) $G_1(t,s):=\frac{\partial G(t,s)}{\partial t}:=\begin{cases}-s, & 0\le s\le t\\ -t, & 0\le t\le s<+\infty.\end{cases}$;

(iii) $\lim_{t\to+\infty}\frac{G_1(t,s)}{1+t}\in\mathbb{R}$, $\forall s\in\mathbb{R}$.

Let $\gamma,\Gamma\in X$ be such that $\gamma(t)\le\Gamma(t)$, $\gamma'(t)\le\Gamma'(t),\forall t\in[0,+\infty)$ and $\gamma''(+\infty)\le\Gamma''(+\infty)$. Consider the set

$$E=\left\{(t,x(t),y(t),z(t))\in\mathbb{R}_0^+\times\mathbb{R}^3:\begin{array}{c}\gamma(t)\le x(t)\le\Gamma(t),\\ \gamma'(t)\le y(t)\le\Gamma'(t),\\ \gamma''(+\infty)\le z(+\infty)\le\Gamma''(+\infty)\end{array}\right\}.$$

The following Nagumo condition allows some *a priori* bounds on the second derivative of the solution:

Definition 4. A function $f:E\to\mathbb{R}$ is said to satisfy a Nagumo-type growth condition in E if, for some positive continuous functions ψ, h and some $\nu>1$, such that

$$\sup\psi(t)(1+t)^\nu<+\infty,\ \int_0^{+\infty}\frac{s}{h(s)}\,ds=+\infty,\tag{6}$$

it verifies

$$|f(t,x,y,z)|\le\psi(t)h(|z|),\forall(t,x,y,z)\in E.\tag{7}$$

Lemma 5. *Let $f:\mathbb{R}_0^+\times\mathbb{R}^3\to\mathbb{R}$ be a L^1-Carathéodory function satisfying* (6) *and* (7) *in E. Then there exists $R>0$* (*not depending on u*) *such that every u solution of* (1) *satisfying, for $t\ge0$,*

$$\begin{gathered}\gamma(t)\le u(t)\le\Gamma(t),\\ \gamma'(t)\le u'(t)\le\Gamma'(t),\\ \gamma''(+\infty)\le u''(+\infty)\le\Gamma''(+\infty),\end{gathered}\tag{8}$$

verifies $\|u''\|_2<R$.

Proof: Let u be a solution of (1) verifying (8). Consider $r>0$ such that

$$r>\max\{|\gamma''(+\infty)|,|\Gamma''(+\infty)|\}.\tag{9}$$

By the previous inequality we cannot have $|u''(t)|>r,\forall t\in[0,+\infty)$, because

$$|u''(+\infty)|<r.$$

If $|u''(t)|\le r,\forall t\in[0,+\infty)$, taking $R>r$ the proof is complete as

$$\|u''\|_2=\sup_{0\le t<+\infty}|u''(t)|\le r<R.$$

In the following it will be proved that even when there exists $t \in [0,+\infty)$ such that $|u''(t)| > r$, the norm $\|u''\|_2$ remains bounded.

Suppose there exists $t_0 \in \mathbb{R}^+$ such that $|u''(t_0)| > r$, that is $u''(t_0) > r$ or $u''(t_0) < -r$.

In the first case, by (6), we can take $R > r$ such that

$$\int_r^R \frac{s}{h(s)}\,ds > M \max\left\{ M_1 + \sup_{0\le t<+\infty} \frac{\Gamma'(t)}{1+t}\frac{\nu}{\nu-1}, M_1 - \inf_{0\le t<+\infty} \frac{\gamma'(t)}{1+t}\frac{\nu}{\nu-1} \right\}$$

with

$$M := \sup_{0\le t<+\infty} \psi(t)(1+t)^{\nu} \quad\text{and}\quad M_1 := \sup_{0\le t<+\infty} \frac{\Gamma'(t)}{(1+t)^{\nu}} - \inf_{0\le t<+\infty} \frac{\gamma'(t)}{(1+t)^{\nu}}.$$

If condition (7) holds, then by (9) there are $t_*, t_+ \in [0,+\infty)$ such that $t_* < t_+, u''(t_*) = r$ and $u''(t) > r$, $\forall t \in (t_*, t_+]$. Therefore

$$\begin{aligned}
\int_{u''(t_*)}^{u''(t_+)} \frac{s}{h(s)}\,ds &= \int_{t_*}^{t_+} \frac{u''(s)}{h(u''(s))}u'''(s)\,ds \le \int_{t_*}^{t_+} \psi(s)u''(s)\,ds \\
&\le M\int_{t_*}^{t_+} \frac{u''(s)}{(1+s)^{\nu}}\,ds = M\int_{t_*}^{t_+} \left[\left(\frac{u'(s)}{(1+s)^{\nu}} \right)' + \frac{\nu u'(s)}{(1+s)^{1+\nu}} \right] ds \\
&\le \quad M \left(M_1 + \sup_{0\le t<+\infty} \frac{\Gamma'(t)}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^{\nu}}\,ds \right) < \int_r^R \frac{s}{h(s)}\,ds.
\end{aligned}$$

So $u''(t_+) < R$ and as t_* and t_+ are arbitrary in $[0,+\infty)$, we have that $u''(t) < R$, $\forall t \in [0,+\infty)$.

Similarly, it can be proved the case where there are $t_-, t_* \in [0,+\infty)$ such that $t_- < t_*$ and $u''(t_*) = -r, u''(t) < -r, \forall t \in [t_-, t_*)$.

Therefore $\|u''\|_2 < R, \forall t \in [0,+\infty)$. □

The lack of compactness of X is overcome by the following lemma which gives a general criterion for relative compactness, suggested in [1] or [5]:

Lemma 6. *A set $Z \subset X$ is relatively compact if the following conditions hold:*

(i) *all functions from Z are uniformly bounded;*

(ii) *all functions from Z are equicontinuous on any compact interval of $[0,+\infty[$;*

(iii) *all functions from Z are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $t_\varepsilon > 0$ such that*

$$\begin{aligned}
&\left| \tfrac{x(t)}{1+t^2} - \lim_{t\to+\infty} \tfrac{x(t)}{1+t^2} \right| < \varepsilon, \\
&\left| \tfrac{x'(t)}{1+t} - \lim_{t\to+\infty} \tfrac{x'(t)}{1+t} \right| < \varepsilon, \\
&\left| x''(t) - \lim_{t\to+\infty} x''(t) \right| < \varepsilon \;\; \textit{for all} \;\; t > t_\varepsilon,\; x \in Z.
\end{aligned}$$

The existence tool will be Schauder's fixed point theorem.

Theorem 7 ([19])**.** *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

The functions considered as lower and upper solutions for the initial problem are defined as it follows, with $W^{3,1}(\mathbb{R}_0^+)$ the usual Sobolev space:

Definition 8. A function $\alpha \in X \cap W^{3,1}(\mathbb{R}_0^+)$ is a lower solution of problem (1), (2) if

$$\begin{cases} \alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t)),\ t \in [0, +\infty), \\ L_0(\alpha, \alpha(0)) \geq 0, \\ L_1(\alpha, \alpha'(0)) \geq 0, \\ L_2(\alpha, \alpha''(+\infty)) > 0. \end{cases}$$

A function β is an upper solution if it satisfies the reverse inequalities.

Remark 9. If $\alpha'(t) \leq \beta'(t)$ and $\alpha(0) \leq \beta(0)$, by integration on $[0, t]$ we have $\alpha(t) \leq \beta(t)$, $\forall t \geq 0$.

The following lemma, suggested by [18], and ensuring the existence and convergence of the derivative of some truncature-function, will be used:

Lemma 10. *For $y_1, y_2 \in C^1(\mathbb{R}_0^+)$ such that $y_1(t) \leq y_2(t)$, $\forall t \geq 0$, define*

$$p(t, v) = \begin{cases} y_2(t), & v > y_2(t) \\ v, & y_1(t) \leq v \leq y_2(t) \\ y_1(t), & v < y_1(t). \end{cases}$$

Then, for each $v \in C^1(\mathbb{R}_0^+)$ the next two properties hold:

(i) $\frac{d}{dt} p(t, v(t))$ *exists for a.e. $t \in [0, +\infty)$;*

(ii) *if $v, v_m \in C^1(\mathbb{R}_0^+)$ and $v_m \to v$ in $C^1(\mathbb{R}_0^+)$ then*

$$\frac{d}{dt} p(t, v_m(t)) \to \frac{d}{dt} p(t, v(t)) \text{ for a.e. } t \in [0, +\infty).$$

3. Existence and localization results

In this section we prove the existence and the localization of at least one solution for the problem (1), (2). The following assumptions are needed.

(H_1) There are α, β lower and upper solutions of (1), (2), respectively, with $\alpha'(t) \leq \beta'(t)$, $\alpha(0) \leq \beta(0)$ and $\alpha''(+\infty) \leq \beta''(+\infty)$.

(H_2) f satisfies the Nagumo condition on

$$E_* := \left\{ (t, x(t), y(t), z(t)) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \begin{array}{c} \alpha(t) \leq x(t) \leq \beta(t), \\ \alpha'(t) \leq y(t) \leq \beta'(t), \\ \alpha''(+\infty) \leq z(+\infty) \leq \beta''(+\infty) \end{array} \right\}.$$

(H_3) $f(t,x,y,z)$ verifies the growth condition

$$\begin{array}{rcl} f(t,\alpha(t),\alpha'(t),\alpha''(t)) & \geq & f(t,x,\alpha'(t),\alpha''(t)), \\ f(t,\beta(t),\beta'(t),\beta''(t)) & \leq & f(t,x,\beta'(t),\beta''(t)), \end{array}$$

for $t \geq 0$ fixed and $\alpha(t) \leq x \leq \beta(t)$.

(H_4) The continuous functions $L_i : C(\mathbb{R}_0^+) \times \mathbb{R} \to \mathbb{R}$, $i = 0,1,2$, are such that

$$\begin{cases} L_i(\alpha,\alpha^{(i)}(0)) \leq L_i(v,\alpha^{(i)}(0)) \text{ and } L_i(\beta,\beta^{(i)}(0)) \geq L_i(v,\beta^{(i)}(0)), \\ \text{ for } i = 0,1 \text{ and } \ \alpha \leq v \leq \beta; \\ L_2(\alpha,\alpha''(+\infty)) \leq L_2(v,\alpha''(+\infty)) \text{ and } L_2(\beta,\beta''(+\infty)) \geq L_2(v,\beta''(+\infty)), \\ \text{ for } \ \alpha \leq v \leq \beta, \\ \lim_{t\to+\infty} L_2(v,w) \in \mathbb{R}, \text{ for } \alpha \leq v \leq \beta, \text{ and } \alpha''(+\infty) \leq w \leq \beta''(+\infty). \end{cases}$$

Theorem 11. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1-Carathéodory function. If hypotheses (H_1)–(H_4) are verified, then problem* (1), (2) *has at least one solution $u \in X \cap W^{3,1}(\mathbb{R}_0^+)$ and there exists $R > 0$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t),\ \alpha'(t) \leq u'(t) \leq \beta'(t),\ -R \leq u''(t) \leq R,\ t \in [0,+\infty),$$

and

$$\alpha''(+\infty) \leq u''(+\infty) \leq \beta''(+\infty).$$

Proof: Let $\alpha, \beta \in X \cap W^{3,1}(\mathbb{R}_0^+)$ verifying (H_1).

Consider the modified and perturbed problem composed by the third order differential equation

$$\begin{aligned} u'''(t) = f\left(t, \delta_0(t,u(t)), \delta_1(t,u'(t)), \frac{d}{dt}\left(\delta_1(t,u'(t))\right)\right) \\ + \frac{1}{1+t^4}\frac{u'(t)-\delta_1(t,u'(t))}{1+|u'(t)-\delta_1(t,u'(t))|},\ t \in [0,+\infty), \end{aligned} \tag{10}$$

and the functional boundary equations

$$\begin{cases} u(0) = \delta_0\left(0, u(0) + L_0\left(\delta_F(u), u(0)\right)\right) \\ u'(0) = \delta_1\left(0, u'(0) + L_1\left(\delta_F(u), u'(0)\right)\right) \\ u''(+\infty) = \delta_\infty(u''(+\infty)) + L_2\left(\delta_F(u), \delta_\infty\left(u''(+\infty)\right)\right), \end{cases} \tag{11}$$

where functions $\delta_i : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ are given by

$$\delta_i(t,x) = \begin{cases} \beta^{(i)}(t), & x > \beta^{(i)}(t) \\ x, & \alpha^{(i)}(t) \le x \le \beta^{(i)}(t) \\ \alpha^{(i)}(t), & x < \alpha^{(i)}(t) \end{cases}, \ i = 0, 1,$$

$$\delta_\infty(x) = \begin{cases} \beta''(+\infty), & x > \beta''(+\infty) \\ x, & \alpha''(+\infty) \le x \le \beta''(+\infty) \\ \alpha''(+\infty), & x < \alpha''(+\infty) \end{cases}$$

$$\delta_F(v) = \begin{cases} \beta, & v > \beta \\ v, & \alpha \le v \le \beta \\ \alpha, & v < \alpha \end{cases}.$$

For clearness, the proof follows several steps.

STEP 1: *If u is a solution of* (10), (11), *then $\alpha'(t) \le u'(t) \le \beta'(t)$, $\alpha(t) \le u(t) \le \beta(t)$, $-R \le u''(t) \le R$, $\forall t \in [0, +\infty)$ and $\alpha''(+\infty) \le u''(+\infty) \le \beta''(+\infty)$.*

Let u be a solution of the modified problem (10), (11) and suppose, by contradiction, that there exists $t \ge 0$ such that $\alpha'(t) > u'(t)$. Therefore,

$$\inf_{0 \le t < +\infty} (u'(t) - \alpha'(t)) < 0.$$

- If the infimum is attained at $t = 0$, then

$$\min_{0 \le t < +\infty} (u'(t) - \alpha'(t)) = u'(0) - \alpha'(0) < 0,$$

therefore we have the contradiction

$$\begin{aligned} 0 &> u'(0) - \alpha'(0) = \delta_1\left(0, u'(0) + L_1\left(\delta_F(u), u'(0)\right)\right) - \alpha'(0) \\ &\ge \alpha'(0) - \alpha'(0) = 0. \end{aligned}$$

- If the infimum occurs at $t = +\infty$, then

$$\inf_{0 \le t < +\infty} (u'(t) - \alpha'(t)) = u'(+\infty) - \alpha'(+\infty) < 0.$$

Therefore $u''(+\infty) - \alpha''(+\infty) \le 0$ and by (H_4) and Definition 8 the contradiction holds

$$\begin{aligned} &0 \ge u''(+\infty) - \alpha''(+\infty) = \delta_\infty(u''(+\infty)) + L_2\left(\delta_F(u), \delta_\infty(u''(+\infty))\right) \\ &\quad \ge L_2(\delta_F(u), \alpha''(+\infty)) \ge L_2\left(\alpha, \alpha''(+\infty)\right) > 0. \end{aligned} \tag{12}$$

- If there is an interior point $t_* \in (0, +\infty)$ such that

$$\min_{0 \le t < +\infty} (u'(t) - \alpha'(t)) := u'(t_*) - \alpha'(t_*) < 0,$$

then there exists $0 \le t_1 < t_*$ where

$$\begin{array}{rcl} u'(t)-\alpha'(t) & < & 0,\ u''(t)-\alpha''(t)\le 0,\ \forall t\in[t_1,t_*], \\ u'''(t)-\alpha'''(t) & \ge & 0,\ \text{a.e. } t\in[t_1,t_*]. \end{array}$$

Therefore, for $t\in[t_1,t_*]$ by (H_3) and Definition 8 we get the contradiction

$$\begin{array}{rcl} 0 & \le & \displaystyle\int_{t_1}^{t} [u'''(s)-\alpha'''(s)]\,ds \\ & = & \displaystyle\int_{t_1}^{t} \left[f\left((s,\delta_0(s,u(s)),\delta_1(s,u'(s)),\frac{d}{ds}(\delta_1(s,u'(s))) \right)\right. \\ & & \displaystyle\left. +\frac{1}{1+s^4}\frac{u'(s)-\delta_1(s,u'(s))}{1+|u'(s)-\delta_1(s,u'(s))|}-\alpha'''(s)\right] ds \\ & \le & \displaystyle\int_{t_1}^{t}\left[f(s,\alpha(s),\alpha'(s),\alpha''(s))+\frac{u'(s)-\alpha'(s)}{1+|u'(s)-\alpha'(s)|}-\alpha'''(s)\right] ds \\ & \le & \displaystyle\int_{t_1}^{t}\left[\frac{u'(s)-\alpha'(s)}{1+|u'(s)-\alpha'(s)|}\right] ds<0. \end{array}$$

So $u'(t)\ge\alpha'(t)$ for $t>0$.

In a similar way it can be proved that $u'(t)\le\beta'(t)$, and, therefore,

$$\alpha'(t)\le u'(t)\le\beta'(t),\ \forall t\in[0,+\infty). \tag{13}$$

Remark that $\alpha(0)\le u(0)$, otherwise, by (H_4) and Definition 8, it will happen the contradiction

$$\begin{array}{rcl} 0 & > & u(0)-\alpha(0)=\delta_0(0,u(0)+L_0\left(\delta_F(u),u(0)\right))-\alpha(0) \\ & \ge & L_0\left(\delta_F(u),u(0)\right))\ge L_0\left(\alpha,\alpha(0)\right))\ge 0. \end{array}$$

Analogously, it can be proved that $u(0)\le\beta(0)$. So, integrating (13) in $[0,t]$, it is easily obtained that $\alpha(t)\le u(t)\le\beta(t)$, $\forall t\in[0,+\infty)$.

Arguing like in (12) we can prove that $u''(+\infty)\ge\alpha''(+\infty)$ and, similarly, that $u''(+\infty)\le\beta''(+\infty)$.

Therefore, $(t,u(t),u'(t),u''(t))\in E_*$ and the inequality $-R\le u''(t)\le R$ is a direct consequence of Lemma 5.

STEP 2: *The problem* (10), (11) *has at least one solution.*

Define the operator $T:X\to X$

$$Tu(t)=\Delta+\Gamma t+\frac{\Psi t^2}{2}+\int_0^{+\infty} G(t,s)F_u(s)\,ds,$$

where

$$\Delta:=\delta_0\left(0,u(0)+L_0\delta_F(u),u(0)\right)\right),$$

$$\Gamma := \delta_1\left(0, u'(0) + L_0\left(\delta_F(u), u'(0)\right)\right),$$

$$\Psi := \delta_\infty(u''(+\infty)) + L_2\left(\delta_F(u), \delta_\infty(u''(+\infty))\right),$$

$G(t,s)$ is the Green function given by (5) associated with the problem

$$\begin{cases} u'''(t) = F_u(t),\ t \in [0,+\infty), \\ u(0) = \Delta, \\ u'(0) = \Gamma, \\ u''(+\infty) = \Psi, \end{cases} \tag{14}$$

and

$$F_u(t) := f\left(t, \delta_0(t,u(t)), \delta_1(t,u'(t)), \frac{d}{dt}(\delta_1(t,u'(t)))\right) + \frac{1}{1+t^4}\frac{u'(t)-\delta_1(t,u'(t))}{1+|u'(t)-\delta_1(t,u'(t))|}.$$

By Lemma 2 the fixed points of T are solutions of (14) and, therefore, of problem (10), (11).

So it is enough to prove that T has a fixed point.

STEP 2.1: *T is well defined and, for a compact $D \subset X$, $TD \subset D$.*

As f is a L^1-Carathéodory function, $Tu \in C^2(\mathbb{R}_0^+)$ and for any $u \in X$ with

$$\rho > \max\left\{\|u\|_X, \|\alpha\|_X, \|\beta\|_X, R\right\}$$

there exists a positive function $\phi_\rho(t)$ such that $t^2\phi_\rho(t), t\phi_\rho(t), \phi_\rho(t) \in L^1\left(\mathbb{R}_0^+\right)$ and

$$\begin{aligned} \int_0^{+\infty} |F_u(s)|\, ds &\leq \int_0^{+\infty}\left(\phi_\rho(s) + \frac{1}{1+s^4}\right) ds < +\infty, \\ \int_0^{+\infty} |sF_u(s)|\, ds &\leq \int_0^{+\infty}\left(s\phi_\rho(s) + \frac{s}{1+s^4}\right) ds < +\infty, \\ \int_0^{+\infty} \left|s^2F_u(s)\right|\, ds &\leq \int_0^{+\infty}\left(s^2\phi_\rho(s) + \frac{s^2}{1+s^4}\right) ds < +\infty. \end{aligned}$$

That is $F_u, tF_u, t^2F_u \in L^1(\mathbb{R}_0^+)$.

By Lebesgue Dominated Convergence Theorem, Lemma 5 and (H_4), setting

$$L := \lim_{t\to\infty} L_2\left(\delta_F(u), \delta_\infty(u''(+\infty))\right),$$

$$M_\infty := \max\left\{|\alpha''(+\infty)| + |L|, |\beta''(+\infty)| + |L|\right\},$$

$$M(s) := \max\left\{\sup_{0\leq t<+\infty}\frac{|G(t,s)|}{1+t^2}, \sup_{0\leq t<+\infty}\frac{|G_1(t,s)|}{1+t}, 1\right\},$$

we have

$$\begin{aligned}
\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^2} &= \lim_{t\to+\infty}\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}+\int_0^{+\infty}\lim_{t\to+\infty}\frac{G(t,s)}{1+t^2}F_u(s)\,ds\\
&\leq \frac{M_\infty}{2}+M(s)\int_0^{+\infty}\left(\phi_\rho(s)+\frac{1}{1+s^4}\right)ds<+\infty,\\
\lim_{t\to+\infty}\frac{(Tu)'(t)}{1+t} &= \lim_{t\to+\infty}\frac{\Gamma+\Psi t}{1+t}+\int_0^{+\infty}\lim_{t\to+\infty}\frac{G_1(t,s)}{1+t}F_u(s)\,ds\\
&\leq M_\infty+M(s)\int_0^{+\infty}\phi_\rho(s)+\frac{1}{1+s^4}\,ds<+\infty,\\
\lim_{t\to+\infty}(Tu)''(t) &= M_\infty+\lim_{t\to+\infty}\int_t^{+\infty}F_u(s)\,ds<+\infty.
\end{aligned}$$

Therefore $Tu\in X$.

Consider now the subset $D\subset X$ given by $D:=\{x\in X:\|u\|_X<\rho_0\}$, with $\rho_0>0$ such that

$$\begin{aligned}
\rho_0 \quad > \quad & \max\{|\alpha(0)|,|\beta(0)|\}+\max\{|\alpha'(0)|,|\beta'(0)|\}+|k_0|\\
&+\int_0^{+\infty}M(s)\left(\phi_\rho(s)+\frac{1}{1+s^4}\right)ds,
\end{aligned}$$

where

$$k_0:=\max\{|\alpha''(+\infty)|,|\beta''(+\infty)|\}+\sup_{0\leq t<+\infty}L_2(v,w),$$

for $\alpha\leq v\leq\beta$, and $\alpha''(+\infty)\leq w\leq\beta''(+\infty)$.

So, for $t\in[0,+\infty)$,

$$\begin{aligned}
\|Tu\|_0 \quad &= \sup_{0\leq t<+\infty}\frac{|Tu(t)|}{1+t^2}\leq\sup_{0\leq t<+\infty}\left(\frac{\left|\Delta+\Gamma t+\frac{\Psi t^2}{2}\right|}{1+t^2}\right)\\
&\quad+\sup_{0\leq t<+\infty}\left(\int_0^{+\infty}\frac{|G(t,s)|}{1+t^2}\left|F_u(s)\right|\,ds\right)\\
&\leq |\Delta|+|\Gamma|+\frac{|\Psi|}{2}+\int_0^{+\infty}M(s)\left(\phi_{\rho_0}(s)+\frac{1}{1+s^4}\right)ds<\rho_0.
\end{aligned}$$

$$\begin{aligned}
\|(Tu)'\|_1=\sup_{0\leq t<+\infty}\frac{|(Tu)'|}{1+t}&\leq\sup_{0\leq t<+\infty}\left(\frac{|\Gamma+\Psi t|}{1+t}+\int_0^{+\infty}\frac{|G_1(t,s)|}{1+t}\left|F_u(s)\right|\,ds\right)\\
&\leq |\Gamma|+|\Psi|+\int_0^{+\infty}M(s)\left(\phi_{r_1}(s)+\frac{1}{1+s^4}\right)ds<\rho_0,
\end{aligned}$$

and

$$\begin{aligned}\|(Tu)''\|_2 &= \sup_{0\leq t<+\infty}|(Tu)''| \leq \sup_{0\leq t<+\infty}\left(|\Psi| + \int_t^{+\infty}|F_u(s)|\,ds\right)\\ &\leq \sup_{0\leq t<+\infty}\left(|\Psi| + \int_t^{+\infty}\phi_{r_1}(s) + \frac{1}{1+s^4}\,ds\right) < \rho_0.\end{aligned}$$

So, $TD \subset D$.

STEP 2.2: *T is continuous.*

Consider a convergent sequence $u_n \to u$ in X, there exists $\rho_1 > 0$ such that $\max\{\sup_n \|u_n\|_X, \|\alpha\|_X, \|\beta\|_X, R\} < \rho_1$. By Lemma 10 we have

$$\begin{aligned}\|Tu_n - Tu\|_X &= \max\left\{\|Tu_n - Tu\|_0, \|(Tu_n)' - (Tu)'\|_1, \|(Tu_n)'' - (Tu)''\|_2\right\}\\ &\leq \int_0^{+\infty} M(s)\,|F_{u_n}(s) - F_u(s)|\,ds \longrightarrow 0, \text{ as } n \to +\infty\end{aligned}$$

STEP 2.3: *T is compact.*

Let $B \subset X$ be any bounded subset. Therefore there is $r > 0$ such that $\|u\|_X < r$, $\forall u \in B$.

For each $u \in B$, and for $\max\{r, R, \|\alpha\|_X, \|\beta\|_X\} < r_1$, we can apply similar arguments to Step 2.1 and prove that $\|Tu\|_0$, $\|(Tu)'\|_1$ and $\|(Tu)''\|_2$ are finite.

So $\|Tu\|_X = \max\{\|Tu\|_0, \|(Tu)'\|_1, \|(Tu)''\|_2\} < +\infty$, that is, TB is uniformly bounded in X.

TB is equicontinuous, because, for $L > 0$ and $t_1, t_2 \in [0, L]$, we have, as $t_1 \to t_2$,

$$\begin{aligned}\left|\frac{Tu(t_1)}{1+t_1^2} - \frac{Tu(t_2)}{1+t_2^2}\right| &\leq \left|\frac{\Delta + \Gamma t_1 + \frac{\Psi t_1}{2}}{1+t_1^2} - \frac{\Delta + \Gamma t_2 + \frac{\Psi t_2}{2}}{1+t_2^2}\right|\\ &\quad + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right| |F(u(s))|\,ds\\ &\leq \left|\frac{\Delta + \Gamma t_1 + \frac{\Psi t_1}{2}}{1+t_1^2} - \frac{\Delta + \Gamma t_2 + \frac{\Psi t_2}{2}}{1+t_2^2}\right|\\ &\quad + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right|\left(\phi_{r_1}(s) + \frac{1}{1+s^4}\right) ds \longrightarrow 0,\end{aligned}$$

$$\begin{aligned}\left|\frac{(Tu)'(t_1)}{1+t_1} - \frac{(Tu)'(t_2)}{1+t_2}\right| &\leq \left|\frac{\Gamma + \Psi t_1}{1+t_1} - \frac{\Gamma + \Psi t_2}{1+t_2}\right|\\ &\quad + \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1} - \frac{G_1(t_2,s)}{1+t_2}\right| |F(u(s))|\,ds\\ &\leq \left|\frac{\Gamma + \Psi t_1}{1+t_1} - \frac{\Gamma + \Psi t_2}{1+t_2}\right|\end{aligned}$$

$$+\int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1}-\frac{G_1(t_2,s)}{1+t_2}\right|\left(\phi_{r_1}(s)+\frac{1}{1+s^4}\right)ds\longrightarrow 0,$$

$$\begin{aligned}|(Tu)''(t_1)-(Tu)''(t_2)| &= \left|\int_{t_1}^{+\infty}F_u(s)\,ds-\int_{t_2}^{+\infty}F_u(s)\,ds\right| \\ &\le \int_{t_1}^{t_2}|F_u(s)|\,ds\le\int_{t_1}^{t_2}\phi_{r_1}(s)+\frac{1}{1+s^4}\,ds\longrightarrow 0.\end{aligned}$$

Moreover TB is equiconvergent at infinity, because, as $t\to+\infty$,

$$\left|\frac{Tu(t)}{1+t^2}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t^2}\right|\le\left|\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}-\frac{\Psi}{2}\right|+\int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2}+\frac{1}{2}\right||F_u(s)|\,ds$$

$$\le\left|\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}-\frac{\Psi}{2}\right|+\int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2}+\frac{1}{2}\right|\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\to 0,$$

$$\begin{aligned}\left|\frac{(Tu)'(t)}{1+t}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t}\right| &\le \left|\frac{\Gamma+\Psi t}{1+t}-\Psi\right|+\int_0^{+\infty}\left|\frac{G_1(t,s)}{1+t}+1\right||F_u(s)|\,ds \\ \le\left|\frac{\Gamma+\Psi t}{1+t}-\Psi\right| &+ \int_0^{+\infty}\left|\frac{G_1(t,s)}{1+t}+1\right|\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\to 0,\end{aligned}$$

and

$$\begin{aligned}\left|(Tu)''(t)-\lim_{t\to+\infty}(Tu)''(t)\right| &= \int_t^{+\infty}|F_u(s)|\,ds \\ &\le \int_t^{+\infty}\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\longrightarrow 0.\end{aligned}$$

So, by Lemma 6, TB is relatively compact.

Then by Schauder's Fixed Point Theorem, T has at least one fixed point $u_1\in X$.

STEP 3: *u_1 is a solution of* (1), (2).

Suppose, by contradiction, that

$$\alpha(0)>u_1(0)+L_0(\delta_F,u_1(0)).$$

Then, by (11), $u_1(0)=\alpha(0)$ and, by (H_4) and Definition 8, the following contradiction holds

$$\begin{aligned}u_1(0)+L_0(\delta_F(u_1),u_1(0)) &= \alpha(0)+L_0(\delta_F(u_1),\alpha(0)) \\ &\ge \alpha(0)+L_0(\alpha,\alpha(0))\ge\alpha(0).\end{aligned}$$

So $\alpha(0) \leq u_1(0) + L_0(\delta_F, u_1(0))$ and in a similar way we can prove that $u_1(0) + L_0(\delta_F(u_1), u_1(0)) \leq \beta(0)$.

Assuming, by contradiction, that $\alpha'(0) > u_1'(0) + L_1(\delta_F(u_1), u_1'(0))$, then $u_1'(0) = \alpha'(0)$ and, by (H_4) and Definition 8, this contradiction is achieved:

$$\begin{aligned} u_1'(0) + L_1(\delta_F(u_1), u_1'(0)) &= \alpha'(0) + L_1(\delta_F(u_1), \alpha'(0)) \\ &\geq \alpha'(0) + L_1(\alpha, \alpha'(0)) \geq \alpha'(0). \end{aligned}$$

So $\alpha'(0) \leq u_1'(0) + L_1(\delta_F(u_1), u_1'(0))$. By similar arguments it can be proved that $u_1'(0) + L_1(\delta_F(u_1), u_1'(0)) \leq \beta'(0)$.

By Step 1 we have that $\alpha(0) \leq u_1(0) \leq \beta(0)$, $\alpha'(0) \leq u_1'(0) \leq \beta'(0)$ and $-R \leq u_1''(+\infty) \leq R$ therefore, $u_1(t)$ verifies the differential equation (1) and boundary conditions (2), that is, u_1 is a solution of (1), (2). $\square$

4. Application

A classical third-order differential equation, known as the Falkner-Skan equation, is at the form

$$u'''(t) + au(t)u''(t) + b(1 - (u'(t))^2) = 0, \; t \in [0, +\infty), \tag{15}$$

where a, b are real numbers.

This general equation is obtained from partial differential equations, by some transformation technique (see [20]).

When $b = 0$, (15) is known as the Blasius equation, and it models the behavior of a viscous flow over a flat plate.

Two-dimensional flow over a fixed impenetrable surface creates a boundary layer, as particles move more slowly near the surface than near the free stream. Thus we can subject this equation to the following boundary conditions on the half line

$$u(0) = 0, u'(0) = 0, u'(+\infty) = 1. \tag{16}$$

As far as we know, in the literature, only numerical techniques are applied to deal with this type of problems (15), (16), with general a, b (see, for instance, [21]).

To illustrate our result we consider a boundary value problem of this family, with a more generalized differential equation, where the constant coefficients are replaced by functions with an adequate asymptotic behavior, that is, composed by the third order fully differential equation

$$u'''(t) = \frac{(u'(t))^2 - 1}{1 + t^6} - \frac{u(t)\,|u''(t)|}{e^{3t}} + \frac{u''(t)}{1 + t^4}, \; t \in [0, +\infty), \tag{17}$$

and the functional boundary conditions on the half-line:

$$
\begin{aligned}
\int_0^{+\infty} \frac{|u(t)|}{(t^2+t+1)(t^2+1)}\,dt - 2u(0) &= \quad 0,\\
u'(0) &= \quad 1, \qquad (18)\\
\inf_{0\le t<+\infty} \frac{u(t)}{1+t^2} - u''(+\infty) &= \; -0.5.
\end{aligned}
$$

Remark that the above problem is a particular case of (1), (2) with

$$f(t,x,y,z) = \frac{y^2-1}{1+t^6} - \frac{x\,|z|}{e^{3t}} + \frac{z}{1+t^4},$$

and $L_i : C(\mathbb{R}_0^+) \times \mathbb{R} \to \mathbb{R}$, $i = 0,1,2$, given by

$$
\begin{aligned}
L_0(w,k_0) &= \int_0^{+\infty} \frac{|w(t)|}{(t^2+t+1)(t^2+1)}\,dt - 2k_0\\
L_1(w,k_1) &= k_1 - 1 \qquad (19)\\
L_2(w,k_2) &= \inf_{0\le t<+\infty} \frac{w(t)}{1+t^2} - k_2 + 0.5.
\end{aligned}
$$

The functions $\beta(t) = t^2+t+1$ and $\alpha(t) = t$ are, respectively, upper and lower solutions of the problem (17), (18) verifying (H_1).

The nonlinear function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ verifies the assumptions of Theorem 11. In fact:

- f is a L^1-Carathéodory function as for $|x| < \rho(1+t^2)$, $|y| < \rho(1+t)$ and $|z| < \rho$, we have

  $$|f(t,x,y,z)| \le \frac{\rho^2(1+t)^2+1}{1+t^6} + \frac{\rho^2(1+t^2)}{e^{3t}} + \frac{\rho}{1+t^4} := \phi_\rho(t),$$

  with $\phi_\rho, t\phi_\rho, t^2\phi_\rho \in L^1(\mathbb{R}_0^+)$;
- f verifies the Nagumo condition on the set

  $$E_* = \left\{ (t,x(t),y(t),z(t)) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \begin{array}{c} t \le x(t) \le t^2+t+1, \\ 1 \le y(t) \le 2t+1, \\ 0 \le z(+\infty) \le 2 \end{array} \right\},$$

  with $\psi(t) = \frac{k}{1+t^4}$ and $h = 1$, where $k > 0$ is a real constant;
- $f(t,x,y,z)$ is nonincreasing in x, therefore it satisfies (H_3).

The functions L_i, $i = 0,1,2$, given by (19), verify (H_4), then, by Theorem 11, there is at least one solution u of (17), (18) such that

$$t \le u(t) \le t^2+t+1,\; 1 \le u'(t) \le 2t+1,\; 0 \le u''(t) \le 2, \text{ for } t \in [0,+\infty[.$$

This localization part shows that this solution is unbounded, nonnegative, increasing and convex.

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