Variations of uniform completeness related to realcompactness

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Abstract. Various characterizations of realcompactness are transferred to uniform spaces giving non-equivalent concepts. Their properties, relations and characterizations are described in this paper. A Shirota-like characterization of certain uniform realcompactness proved by Garrido and Meroño for metrizable spaces is generalized to uniform spaces. The paper may be considered as a unifying survey of known results with some new results added.

Keywords: realcompactness; realcompleteness; uniform space

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1. Introduction

Various classes of uniform spaces close to the class of realcompact spaces appeared about 40–60 years ago in works by J.R. Isbell, M.D. Rice, G.D. Reynolds, A.W. Hager, L. Nachbin, O. Njastad, J. Pelant and others. Recently, similar classes appeared again, e.g., in works by A.A. Chekeev, M.I. Garrido and A.S. Meroño, M. Hušek and A. Pulgarín. Our approach is to unify all those concepts, to generalize some results and to give possibly new looks at them.

There are several possibilities how to transfer realcompactness into uniform spaces according to what characterization of realcompactness is used. We want those spaces to be epireflective in uniform spaces and, in some sense, to be compatible with realcompact spaces. Various terms were used for those realcompact-like uniform spaces. We suggest to use terms coming from their modification of completeness.

Basic references for topological and uniform concepts are [3], [19]. We shall repeat some of the concepts used more often in this paper.

All the topological spaces are assumed to be Tikhonov (i.e., Hausdorff and completely regular) and, thus, all uniform spaces are separated. A topological property used for a uniform space is the property of the induced topological space. Under our conditions, epireflective classes of uniform spaces coincide with productive and closed hereditary subclasses.

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In the next, $\mathfrak{m}$ denotes the Ulam measurable cardinal, if it exists. If it does not exist, then the inequality $\kappa \leq \mathfrak{m}$ for a cardinal κ is regarded as always fulfilled. For an infinite cardinal κ, a filter $\mathcal{F}$ is said to have κ-intersection property if $\bigcap \mathcal{F}' \neq \emptyset$ for any $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < \kappa$. Another term is κ-complete filter. We also say that a filter has cip instead of saying that it has ω_1-intersection property. Not every filter with κ-intersection property is contained in an ultrafilter with κ-intersection property.

For a uniform space X we denote by $\operatorname{cov} X$ the covering character of X, i.e., the least cardinal κ, such that every uniform cover has a uniform subcover of cardinality less than κ. In other words, every uniformly discrete subset is of cardinality less than κ. The cardinal function $w_u X$ (uniform weight) is the smallest infinite cardinality of a base of a uniform space X. Point character pcX of a uniform space X is less than an infinite cardinal κ if X has a base of uniform covers such that for every $x \in X$ and every cover $\mathcal{U}$ from the base one has $|\mathcal{V}| < \kappa$ whenever $\mathcal{V} \subset \mathcal{U}, \bigcap \mathcal{V} \ni x$. Spaces X with $pcX < \omega$ are called point-finite. If X is a uniform space, by $U(X)$ we denote the set of all uniformly continuous real-valued functions on X, by $U^*(X)$ its subset of bounded functions.

For a uniform space X, pX is the totally bounded modification of X (finite uniform covers of X form a base of pX), eX is the uniformly separable modification (countable uniform covers of X form a base of eX). For any infinite cardinal κ there is a modification $p_\kappa X$ of a uniform space X that is the finest space Y with $\operatorname{cov} Y \leq \kappa$ coarser than X. Thus $pX = p_\omega X, eX = p_{\omega_1} X$. The uniform space weakly generated by $U(X)$ is denoted as cX (preimages of uniform covers of $\mathbb{R}$ by all $f \in U(X)$ form a subbase of cX). By γX we denote a completion of X, so that γpX is the Samuel compactification sX of X.

We shall also use the (topologically) fine coreflection $t_f X$, the finest uniform space inducing the same topology as X does.

By $H(\kappa), \kappa$ is an infinite cardinal, we denote the metric hedgehog with κ-many spines, i.e. the disjoint sum of κ-many intervals $[0,1]$ sewed together at the point 0 and endowed with the standard metric.

We shall need uniform zero sets and their properties: $\mathcal{Z}_u(X)$ is the collection of zero sets $f^{-1}(0)$ of $f \in U(X)$ (i.e., of $U^*(X)$ so that $\mathcal{Z}_u(X)$ is the same for spaces proximally equivalent to X). The collection $\mathcal{Z}(X)$ consists of zero sets of $f \in C(X)$, thus equals to $\mathcal{Z}_u(t_f X)$. As in topological spaces, $\mathcal{Z}_u(X)$ is a σ-ring of sets with respect to $(\cup, \cap)$.

By z_u-filter in a uniform space X we mean a filter $\mathcal{F}$ in X such that $\mathcal{F} \cap \mathcal{Z}_u(X)$ is a base of $\mathcal{F}$. A z_u-filter $\mathcal{F}$ is said to be z_u-ultrafilter if $\mathcal{F} \cap \mathcal{Z}_u(X)$ is a maximal filter in $\mathcal{Z}_u(X)$, i.e., $Z \in \mathcal{Z}_u(X)$ belongs to $\mathcal{F}$ provided $Z \cap P \neq \emptyset$ for all $P \in \mathcal{F}$. If the index u is omitted it concerns $\mathcal{Z}(X)$ instead of $\mathcal{Z}_u(X)$. It follows from Kuratowski-Zorn lemma that every z_u-filter is contained in a z_u-ultrafilter.

The following simple property of z_u-filters was known to Z. Frolík, M.D. Rice and others in 1970's.

Lemma 1.1. *Let $\kappa > \omega$ and a uniform space Y be finer than a uniform space X. If $\mathcal{F}$ is a z_u-ultrafilter with κ-intersection property on Y then the filter $\mathcal{F}^*$ having the base $\mathcal{F} \cap \mathcal{Z}_u(X)$ is a z_u-ultrafilter on X (clearly, with κ-intersection property).*

Proof: It suffices to show that if $Z' \in \mathcal{Z}_u(X)$ and $Z' \cap Z \neq \emptyset$ for every $Z \in \mathcal{F} \cap \mathcal{Z}_u(X)$ then $Z' \cap A \neq \emptyset$ for every $A \in \mathcal{F}$. Assume $Z' \cap A = \emptyset$ for some $A \in \mathcal{F}$. We have $Z' = \bigcap C_n$ for some uniform cozero sets C_n in X and thus, $A \subset \bigcup (X \setminus C_n)$. Since $\mathcal{F}$ has cip, there is some n with $X \setminus C_n \in \mathcal{F}$. The last set belongs to $\mathcal{F} \cap \mathcal{Z}_u(X)$ and is disjoint with Z' — a contradiction. $\square$

It should be recalled that the converse does not hold. There are z_u-ultrafilters with cip that cannot be extended to ultrafilters with cip. For instance, take $X = \omega_1$ with the usual order topology and Y the same set with the uniformly discrete uniformity. There is a unique uniformity on X with exactly one free z_u-ultrafilter $\mathcal{F}$. It has for its base the intervals $(\alpha, \omega_1), \alpha \in \omega_1$ and, therefore, it has cip. Since every ultrafilter with cip on Y is fixed, $\mathcal{F}$ cannot be extended to an ultrafilter with cip.

So, not every z_u-filter with cip extends to a z_u-ultrafilter with cip. There is a big class of z_u-filters with cip that always extend to z_u-ultrafilters with cip:

Lemma 1.2. *Every z_u-filter in a uniform space X containing a z_u-filter with κ-intersection property converging in sX has κ-intersection property.*

Proof: We may assume $\kappa > \omega$. Let $\mathcal{F}$ be a z_u-filter in X with κ-intersection property converging to $\xi \in sX$ and $\mathcal{H}$ be a z_u-filter in X containing $\mathcal{F}$. Assume there is $\lambda < \kappa$ and $H_\alpha \in \mathcal{H} \cap \mathcal{Z}_u(X), \alpha \in \lambda$, with $\bigcap_\lambda H_\alpha = \emptyset$. Take some $f_\alpha \in U(X, [0,1])$ with $f_\alpha^{-1}(0) = H_\alpha$. Every $Z_\alpha^n = f_\alpha^{-1}[0, 1/n]$ is a trace of a neighborhood of ξ on X and, thus, belongs to $\mathcal{F}$. That gives a contradiction with κ-intersection property of $\mathcal{F}$ since $\bigcap_{\alpha \in \lambda, n \in \mathbb{N}} Z_\alpha^n = \bigcap_\lambda H_\alpha = \emptyset$. $\square$

Corollary 1.3. *Every z_u-ultrafilter in a uniform space X containing a Cauchy z_u-filter with κ-intersection property has κ-intersection property.*

Lemma 1.1 implies that if Y is finer than X and $t_f Y = t_f X$ then every z_u-ultrafilter in Y with κ-intersection property converges provided every z_u-ultrafilter in X with κ-intersection property converges. Lemma 1.2 implies that every Cauchy filter with κ-intersection property converges in X iff every Cauchy z_u-ultrafilter with κ-intersection property converges in X (for $\kappa > \omega, \operatorname{cov} X \leq \mathfrak{m}$ iff every z_u-ultrafilter with κ-intersection property converges in X).

The next assertion comes from [9, Theorem 15.20]. It is a consequence of the well-known Stone theorem (every uniform cover of a metric space can be refined by an open cover expressed as a union of countably many uniformly discrete open collections).

Lemma 1.4. *Every z_u-ultrafilter with cip on a uniform space X is Cauchy provided $\operatorname{cov} X \leq \mathfrak{m}$.*

2. Realcompact spaces

We shall now give a list of several characterizations of realcompact topological spaces. That concept has more terms, e.g., Q-spaces (Hewitt), saturated spaces (Nachbin), Hewitt-Nachbin spaces (Weir), functionally closed spaces, $\mathbb{R}$-compact spaces. The term *realcompact spaces* (originally real-compact), used most often now, was suggested and used by L. Gillman in [8]. J.R. Isbell in [18, p. 117] recalls other terms from that time, like e-complete spaces and supports a proposal of M. Jerison, namely real-complete spaces (I thank to M.D. Rice for that information).

The Hewitt's definition uses algebraic structures of $C(X)$ and is further developed by Gillman and Jerison in [9] and by M.D. Weir in [30]. In this paper we shall not investigate characterizations of realcompactness using algebraic structures of some function spaces although it may give interesting properties for uniform spaces.

Theorem 2.1. *For Tikhonov spaces X any of the next properties is equivalent to realcompactness.*

1. *X is homeomorphic to a closed subspace of a power of reals.*
2. *$c(t_f X)$ is complete.*
3. *$e(t_f X)$ is complete.*
4. *$t_f X$ is complete and no closed discrete subset has Ulam measurable cardinality.*
5. *Every zero-ultrafilter with cip in X converges in X.*
6. *X is G_δ-closed in βX.*
7. *X is the intersection of all cozero sets in βX containing X.*
8. *For each Y containing X as a dense subspace and for each $y \in Y \setminus X$ there exists $f \in C(X)$ that cannot be extended continuously to y (into $\mathbb{R}$).*
9. *For each $\xi \in \beta X \setminus X$ one can find $f \in C(X)$ that cannot be continuously extended to ξ.*
10. *For each $\xi \in \beta X \setminus X$ one can find $f \in C^*(X), f > 0$, that continuously extends to ξ with the value 0.*

Hewitt showed that every realcompact space is homeomorphic to a closed subspace of a power of $\mathbb{R}$ and asked whether the converse is true. The converse was proved by T. Shirota in [29]. The second property was used by L. Nachbin in [23] for his saturated spaces. An equivalence of both properties can be found, e.g., in [9]. The third and the fourth property was found by T. Shirota in [28] and [29]. The fifth characterization comes from the original E. Hewitt's paper [13], the sixth was proved by S. Mrówka in [21] and the next one was shown by Z. Frolík in [4]. There are more characterizations like those in 6 and 7 using either other compactifications than βX or using other subsets between X and βX than cozero sets (e.g., F_σ-sets or σ-compact sets). From the last three characterizations using extensions of functions, the first two were proved by M. Katětov in [20] and the last one by S. Mrówka in [22].

3. Realcompactness in uniform spaces

We shall now transfer the previous characterization of realcompactness to uniform spaces. Some of the next concepts are known under different names.

3.1 $\mathbb{R}$-complete spaces.

Definition 1. A uniform space is said to be $\mathbb{R}$-*complete* if it is uniformly homeomorphic to a closed subspace of a power of $\mathbb{R}$.

We are using a modification of Mrówka's terminology for E-compact spaces. We prefer to use the term E-complete spaces to distinguish the uniform case from the topological one.

By J. Isbell ([17]), a cover of X is linear if it can be indexed by integers, as $\{U_n\}$, so that U_n, U_m meet only if $|n-m| \leq 1$.

Theorem 3.1.

1. *A uniform space X is $\mathbb{R}$-complete iff it is complete and has a subbase of linear covers.*
2. *A uniform space X is $\mathbb{R}$-complete if $X = \gamma cX$.*
3. *The class of all $\mathbb{R}$-complete spaces is epireflective in all uniform spaces. The epireflection of X is γcX.*
4. *Every $\mathbb{R}$-complete space is realcompact.*
5. *A topological space is realcompact iff its topology is induced by an $\mathbb{R}$-complete uniformity.*
6. *Covering character of $\mathbb{R}$-complete spaces is at most ω_1 (thus the fine modification of an $\mathbb{R}$-complete space need not be $\mathbb{R}$-complete).*
7. *A precompact space is $\mathbb{R}$-complete iff it is compact.*
8. *A uniformly zero-dimensional space X is $\mathbb{R}$-complete iff it is complete and* $\operatorname{cov} X \leq \omega_1$. *Thus, a uniformly discrete space is $\mathbb{R}$-complete iff it is at most countable.*
9. *The hedgehog $H(\omega_1)$ is a complete metric space with* $\operatorname{cov} H(\omega_1) = \omega_1$ *and it is not $\mathbb{R}$-complete.*

Proof: It was proved by Isbell in [17] that a uniform space can be embedded into a power of $\mathbb{R}$ iff it has a subbase of linear covers. Now the assertion 1 follows. The items 2–7 are very easy to show. The assertion 8 follows from the fact that a uniformly zero-dimensional space X has $\operatorname{cov} X \leq \omega_1$ iff it can be embedded into a power of $\mathbb{N}$. The hedgehog $H(\omega_1)$ is not precompact but $U(H(\omega_1)) = U^*(H(\omega_1))$ so that $c(H(\omega_1)) = p(H(\omega_1))$. □

3.2 r-complete spaces. In the next, r stays for an upper modification of uniform spaces preserving proximity, i.e., X is finer than rX that is finer than pX, and the identity map $X \to rX$ is a reflection into spaces coinciding with their r–modification. For us the main such modifications are c, p_κ.

Definition 2. A uniform space X is said to be *r-complete* if its upper modification rX is complete (i.e., $\gamma rX = rX$).

At first we shall show some general properties of r-complete spaces and then their special properties for special choices of r. We shall denote by $X(r)$ the uniform space having for its underlying set γrX and for its uniformity the finest one coarser than $t_f(\gamma rX)$ and $X+(\gamma rX\setminus X)_D$, where Y_D is the uniformly discrete space with the underlying set Y. The space $X(r)$ is the finest one inducing the topology of γrX and making the identity map $X \to \gamma rX$ uniformly continuous into $X(r)$. For $r = p$ we have $X(p) = sX$ so that X need not be a uniform subspace of $X(r)$. Clearly, X is a topological subspace of $X(r)$ that is a topological subspace of sX.

We want γrX to be r-complete, so we shall assume $r\gamma r = \gamma r$ in the next (the main applications $r = c, p_\kappa$ satisfy that equality). A comparison $r \geq t$ of two modifications means rX is coarser than tX for all X.

Theorem 3.2. 1. *Every r-complete space is complete.*
2. *The spaces γrX are r-complete.*
3. *If X is r-complete then every uniform space finer than rX and inducing the same topology as X is r-complete.*
4. *The class of all r-complete spaces is epireflective in all uniform spaces. An epireflection of X is $X(r)$.*
5. *A precompact space is r-complete iff it is compact.*
6. *If $t \leq r$ then every r-complete space is t-complete.*
7. *If $r \geq e$ then every r-complete space is realcompact (for spaces X with $\operatorname{cov} t_f X \leq \mathfrak{m}$ the condition $r \geq e$ can be omitted). If $r \leq c$ then $t_f X$ is r-complete provided X is realcompact (then a uniformly discrete space X is r-complete iff $|X| < \mathfrak{m}$).*

Proof: The items 1, 2, 5 and 6 are easy. To show 3 it suffices to realize that if Y is finer than rX and coarser than $t_f X$, then rY is finer than rX and is complete provided rX is complete.

We shall now prove the assertion 4. Since we are in Hausdorff spaces, it suffices to show that the class of r-complete spaces is closed hereditary and productive. So, let X be r-complete and Y be its closed subspace. Then the inclusion map $Y \to X$ maps rY into rX and, consequently, rY is complete provided rX is complete. Let X_i, $i \in I$, be a family of r-complete spaces. The projections $\prod_I X_i \to X_i$ are uniformly continuous also as maps $r\prod_I X_i \to rX_i$ that generate a uniformly continuous identity map $r\prod_I X_i \to \prod_I rX_i$. Since both $r\prod_I X_i, \prod_I rX_i$ induce the same topology and the second space is complete, also the first space is complete. Thus $\prod_I X_i$ is r-complete.

An alternative proof shows also that $X(r)$ is a reflection of X in the class of r-complete spaces. At first one must prove that $X(r)$ is r-complete. That follows from the fact that $r\gamma rX = \gamma rX$ (by our assumption) and, thus $rX(r)$ is finer than γrX with the same topology — consequently, $rX(r)$ is complete. To show that $X(r)$ is a reflection, take a uniformly continuous map $f : X \to Y, Y r$-complete. Then f is uniformly continuous as a map $rX \to rY$ and can be extended to a uniformly continuous map $f' : \gamma rX \to \gamma rY = rY$. Now, $f' : t_f\gamma rX \to t_f rY \to Y$

is uniformly continuous and $f' : X + (\gamma r X \setminus X)_D \to Y$ is uniformly continuous. Consequently, f' is uniformly continuous on $X(r)$.

It remains to prove 7. If $r \geq e$ then r-completeness implies completeness of $et_f X$, thus realcompactness of X. If $\operatorname{cov} t_f X \leq \mathfrak{m}$ completeness of $t_f X$ is sufficient for realcompactness of X. If $r \leq c$ and X is realcompact, then $ct_f X$ is complete, thus $rt_f X$ is complete. □

The extreme choices $r = p$ or r equal to identity are trivial. In the first case we get exactly compact spaces and in the latter case exactly complete spaces for r-complete spaces. It is easy to see that the first assertion in 7 does not hold for r equal to identity and the last one does not hold for $r = p$.

Probably, the most interesting cases are $r = c$ and $r = e$. Completeness of eX and cX was studied in many papers published 40-50 years ago (some authors: J.R. Isbell, A.W. Hager, O. Njastad, M.D. Rice, G.D. Reynolds, J. Pelant).

3.2.1 c-complete spaces. Taking a special choice $r = c$ one can add more assertions to those for general r in Theorem 3.2.

Theorem 3.3.
1. *A uniform space X is c-complete iff there exists a uniformly continuous homeomorphism of X onto a closed uniform subspace of a power of $\mathbb{R}$.*
2. *A uniformly zero-dimensional space is c-complete iff it is complete and its covering character is not bigger than $\mathfrak{m}$. Thus a uniformly discrete space is c-complete iff its cardinality is Ulam non-measurable.*
3. *Every $\mathbb{R}$-complete space is c-complete. The uniformly discrete space of cardinality ω_1 is c-complete but not $\mathbb{R}$-complete.*
4. *The hedgehog $H(\omega)$ is not c-complete.*

Proof: 1. Let Y be a closed uniform subspace of a power of $\mathbb{R}$ that is homeomorphic to X and the homeomorphism $X \to Y$ is uniformly continuous. We can assume that the homeomorphism is identity. Then $\gamma c Y = Y$ induces the same topology as X does and X is finer than Y. Consequently, cX is finer than Y and is complete, thus X is c-complete (use Theorem 3.2.3).

Every uniformly zero-dimensional space X can be embedded into a product of uniformly discrete spaces of cardinalities less than $\operatorname{cov} X$. By 1, a uniformly discrete space is c-complete iff it is realcompact, i.e., its cardinality is Ulam non-measurable. Those two assertions imply 2.

The assertion in 3 follows from 1 and Theorem 3.2.7 and 4 follows from the equality $c(H(\omega)) = p(H(\omega))$. □

For metrizable spaces, an interesting characterization of c-complete spaces was announced by M.I. Garrido, A.S. Meroño at Prague Toposym, June 2016 (see [6]). We shall prove their result in the setting of all uniform spaces (Theorem 3.7).[1]

[1]Shortly before submitting this paper, the author received their contribution to Toposym Proceedings [7], where the original result for metric spaces was also generalized to general uniform spaces.

Their basic idea is to use decomposition of uniform spaces using iterations that was used by J. Hejcman in his definition and study of uniform boundedness (see [10], [11]). For any symmetric uniform neighborhood U of diagonal in a uniform space X the relation of points $x \sim_U y$ if there exists $n \in \mathbb{N}$ such that $y \in U^n[x]$, is equivalence. The equivalence classes are the sets $U^\infty[x] = \bigcup_{\mathbb{N}} U^n[x]$ and they form a uniform partition of X. We should keep in mind that (unlike in topological spaces) a uniform partition $\{X_a\}_A$ of X does not mean that X is a coproduct of the subspaces $\{X_a\}_A$ (products of an infinite uniformly discrete space and of a convenient uniform space (e.g., of $\mathbb{R}$) show that).

Our main task is to characterize Cauchy filters on cX by means of properties of X. At first a general easy assertion. In the next assertion (probably known as folklore), we use a bireflective full subcategory $\mathcal{R}$ of uniform spaces containing all compact spaces. We denote by r the corresponding reflection preserving underlying sets. Then r preserves proximities.

Lemma 3.4. *Let $\{X_a\}_A$ be a uniform partition of a uniform space X and the uniformly discrete space A belongs to a bireflective subcategory $\mathcal{R}$ of uniform spaces. Then $\gamma rX \subset \bigcup_A \overline{X_a}^{sX}$.*

PROOF: Let $f : X \to A$ be a map assigning a to X_a. The map f is uniformly continuous. Since $A \in \mathcal{R}, f$ can be extended to a uniformly continuous map $\widetilde{f} : \gamma rX \to A$, which implies the requested inclusion. □

The next result belongs to Isbell since the cover $\{A_n \cup A_{n+1}\}_{\mathbb{N}}$ is a linear uniform cover of X (see the beginning of the proof of Theorem 3.1). We shall give a short proof here.

Lemma 3.5. *Let $\{A_n\}_{\mathbb{N}}$ be a partition of a uniform space X such that, for some uniform neighborhood U of Δ_X, $U[A_n] \cap A_k \neq \emptyset$ only if $|k-n| \leq 1$. Then the sequence $\{A_n \cup A_{n+1}\}_{\mathbb{N}}$ is a uniform cover in cX.*

PROOF: Let a continuous pseudometric $d \leq 1$ be subordinated to U (i.e., $d(x,y) < 1$ implies $(x,y) \in U$). Define $f(z) = 0$ for $z \in A_1$ and $f(z) = n-1+d(z,A_{n-1})$ if $z \in A_n, n > 1$. If $\delta < 1$ and $d(z,y) < \delta$ then z,y belong either to a same set A_n or to two neighboring sets A_n, A_{n+1} and, thus, the δ-cover refines the cover $\{A_n \cup A_{n+1}\}_{\mathbb{N}}$. It remains to show that f is uniformly continuous. Take again some $\delta > 0$ and z,y as before. We have

$$|f(z)-f(y)| = \begin{cases} |d(z,A_{n-1}) - d(y,A_{n-1})| \leq d(z,y)\,, \quad z,y \in A_n \\ |1+d(y,A_n) - d(z,A_{n-1})| = |d(y,A_{n-1}) + d(y,A_n) - d(z,A_{n-1})| \\ \qquad \leq d(y,z)+d(y,A_n) \leq 2d(z,y)\,, \quad z \in A_n, y \in A_{n+1} \end{cases}$$

Consequently, f is uniformly continuous on (X,d) and, thus, on X. The rest of the proof is clear. □

We shall now apply the previous assertion to c-complete spaces.

Proposition 3.6. *Let every uniform partition of X have cardinality smaller than $\mathfrak{m}$. Then an ultrafilter $\mathcal{F}$ on X is Cauchy in cX iff for every uniform neighborhood U of Δ_X in X there exists $x \in X$ and $n \in \mathbb{N}$ such that $U^n[x] \in \mathcal{F}$.*

Proof: Let $\mathcal{F}$ be a Cauchy filter in cX and U be a uniform neighborhood of Δ_X in X. The uniform partition $\{U^\infty[x]; x \in X\}$ has cardinality less than $\mathfrak{m}$. Since every uniformly discrete space of Ulam non-measurable cardinality is c-complete, it follows from Lemma 3.4 that there exists some $x \in X$ with $U^\infty[x] \in \mathcal{F}$. By Lemma 3.5, the sequence $\{U^n[x]\}_{\mathbb{N}}$ is a uniform cover of $cU^\infty[x]$, so that one of its members belongs to $\mathcal{F}$ (take $A_n = U^{n+1}[x] \setminus U^n[x]$ in Lemma 3.5).

Let, conversely, an ultrafilter $\mathcal{F}$ have the property from the proposition. To prove that $\mathcal{F}$ is Cauchy in cX, it suffices to show that $f(\mathcal{F})$ is Cauchy in $\mathbb{R}$ for any $f \in U(X)$. The property of $\mathcal{F}$ implies existence of $F \in \mathcal{F}$ with bounded $f(F)$. Thus, $f(\mathcal{F})$ is an ultrafilter on a compact set $\overline{f(F)}$, it converges and, thus, is Cauchy in $\mathbb{R}$. □

In the first part of the proof it was sufficient to assume $\mathcal{F}$ to be a filter and, moreover, it follows that one of the sets $U^{n+1}[x] \setminus U^n[x]$ belongs to $\mathcal{F}$.

We came to a characterization of Cauchy ultrafilters in cX by means of uniformity structure of X. Filters having the property from the previous proposition were defined and investigated for metrizable spaces in [5] under the name Bourbaki Cauchy filters. Since they are defined by means of iteration of uniform neighborhoods of diagonal, we shall call them iteratively Cauchy filters.

Definition 3. A filter $\mathcal{F}$ in a uniform space X is said to be *iteratively Cauchy* (briefly i-Cauchy) if for any uniform neighborhood U of Δ_X there exist $x \in X$, $n \in \mathbb{N}$ such that $U^n[x] \in \mathcal{F}$.

The space X is said to be *iteratively complete* if every i-Cauchy ultrafilter in X converges in X.

If we take filters instead of ultrafilters in the definition of iterative completeness, we must use accumulation points instead of limit points. For instance, every filter in $\mathbb{R}$ with nonempty bounded intersection is i-Cauchy. It would be possible to use limit points even for filters if we add a condition in the definition, e.g., that for any uniform neighborhoods U, V of $\Delta(X)$ with $V \circ V \subset U$ one has either $U[x] \in \mathcal{F}$ or $X \setminus V[x] \in \mathcal{F}$ for some $x \in X$ (i.e., $\mathcal{F}$ is Cauchy in pX).

The space $\mathbb{R}$ is iteratively complete since every its i-Cauchy filter contains a compact subset of $\mathbb{R}$. It is easy to see that i-Cauchy filters are preserved by uniformly continuous maps. Since every Cauchy filter is i-Cauchy, every iteratively complete space is complete.

As a corollary of Proposition 3.6 we get a generalization of the result by Garrido and Meroño from [6].

Theorem 3.7. *A uniform space X is c-complete iff it is iteratively complete and no uniform partition of X is of Ulam measurable cardinality.*

Corollary 3.8. *A topological space is realcompact iff it has an iteratively complete uniformity with no uniform partition of Ulam measurable cardinality.*

Another consequence of Theorem 3.7 is the following interesting result proved by G.D. Reynolds and M.D. Rice in [26].

Corollary 3.9. *Every complete space with covering character not bigger than* $\mathfrak{m}$ *and having a base of star-finite covers is c-complete.*

PROOF: If X is star-finite then every $U^n[x]$ is a union of a finite number of some $U[y]$. Thus, every i-Cauchy ultrafilter in X is Cauchy. □

There are two more properties that can be substituted instead of star-finiteness in the last corollary, namely local finiteness (see [19, Theorem VII.18]) and inversion property (see [27]). It seems it is easier to prove those results directly than to use Theorem 3.7.

We add a historical remark. O. Njastad in [24] defined a realcompact-like notion for proximity spaces X: if $\xi \in sX \setminus X$ then there exists a proximally continuous map $X \to \mathbb{R}$ that cannot be continuously extended to ξ. Since for proximally fine spaces proximal continuity and uniform continuity coincide, for such spaces (e.g., metrizable spaces or, more general, products of spaces having linearly ordered bases) the definition gives c-completeness. So, for metrizable uniform spaces, some characterizations of c-completeness and constructions of c-realcompactifications can be found in [24]. For instance, a metrizable space X is c-complete iff every maximal regular filter on X is equi-uniform.

3.2.2 e-complete spaces. The choice $r = e$ gives similar results as for the choice $r = c$. Perhaps, the term e-completeness is not quite natural since it has practically nothing common with reals.

Theorem 3.10.
1. *A uniform space X is e-complete iff there exists a uniformly continuous homeomorphism of X onto a closed subspace of products of complete metrizable separable spaces.*
2. *Every c-complete space is e-complete. The hedgehog $H(\omega)$ is e-complete but not c-complete.*
3. *A uniformly zero-dimensional space is e-complete iff it is c-complete* (*i.e., iff it is complete and its covering character is not bigger than* $\mathfrak{m}$). *Thus, a uniformly discrete space is e-complete iff its cardinality is Ulam non-measurable.*

PROOF: 1. For every separable metric space M one has $eM = M$ and every Y with $eY = Y$ can be uniformly embedded into a product of separable metric spaces $\prod_I M_i$. So, if Y is complete, it is embedded onto a closed subspace and for X with $eX = Y$ the embedding map $X \to \prod_I M_i$ is uniformly continuous and remains a homeomorphism.

The first part of the item 2 follows from Theorem 3.2.6. Since hedgehog $H(\omega)$ is complete and separable, it is e-complete; it is not c-complete ($cH(\omega) = pH(\omega)$).

The remaining item follows from the fact that for uniformly zero-dimensional spaces X one has $eX = cX$. $\square$

Every e-complete space is complete and no uniformly discrete space of cardinality at least $\mathfrak{m}$ is e-complete. Does there exist a complete space of smaller cardinality than $\mathfrak{m}$ that is not e-complete? It is not difficult to realize that if such an example exists, then $\ell_\infty(\kappa)$ (for some $\kappa < \mathfrak{m}$) is an example, too. The first such an example was constructed by J. Pelant in [25].

There are several positive answers to the question for some special classes of uniform spaces. M.D. Rice and G.D. Reynolds proved in [26] that eX is complete provided X is complete, $\operatorname{cov} X \leq \mathfrak{m}$ and it has a base of uniform covers composed of point-finite covers. By now it is a largest class of nice spaces, where completeness implies e-completeness. Point-finite uniform spaces coincide with spaces that can be embedded into powers of some $c_0(\kappa)$. Thus, it suffices to show that $c_0(\kappa)$ is e-complete provided $\kappa < \mathfrak{m}$ - that is not easy to show. Much easier is the special case of the so called distal spaces, i.e., spaces having a base of finitely dimensional covers. It uses the fact that every finite-dimensional uniform cover is refined by a uniform cover that is a union of finitely many uniformly discrete collections (see [19, IV.25]). That proof can be easily generalized for such unions of countably many uniformly discrete collections.

A uniform space is said to be uniformly σ-discrete if it has a base of uniformly σ-discrete uniform covers, i.e. of covers that are unions of countably many uniformly discrete collections.

Theorem 3.11. *A uniformly σ-discrete space is e-complete iff it is complete and $\operatorname{cov} X \leq \mathfrak{m}$.*

Proof: Necessity is clear. So, let X be uniformly σ-discrete, complete and $\operatorname{cov} X \leq \mathfrak{m}$. Take any Cauchy filter $\mathcal{F}$ on eX. To prove it converges it suffices to show it is a Cauchy filter in X. Take any uniformly σ-discrete uniform cover $\mathcal{U} = \bigcup_{\mathbb{N}} \mathcal{U}_n$, where $\mathcal{U}_n$ are uniformly discrete collections. Since $\{\bigcup\{U; U \in \mathcal{U}_n\}\}_{\mathbb{N}}$ is a countable uniform cover of X, there is some $n \in \mathbb{N}$ with $\bigcup\{U; U \in \mathcal{U}_n\} \in \mathcal{F}$. Since $\mathcal{U}_n$ is uniformly discrete and $|\mathcal{U}_n| < \mathfrak{m}$ there must exist some $U \in \mathcal{U}_n$ belonging to $\mathcal{F}$. $\square$

There may appear a question whether, as by c-completeness, one may require that uniform partitions of X have cardinalities smaller than $\mathfrak{m}$ instead of $\operatorname{cov} X \leq \mathfrak{m}$. The answer is in the negative. The hedgehog is always uniformly connected and 1-dimensional but $H(\mathfrak{m})$ is not e-realcompact.

If we use $r = p_\kappa$ for $\kappa > \omega_1$, the situation is similar as for $\kappa = \omega_1$, at least for existence of a complete space X such that $p_\kappa X$ is not complete. That was shown also by J. Pelant in [25] using sufficiently large spaces $\ell_\infty(\lambda)$ ($\lambda \geq \kappa^+$ suffices). In fact, later on J. Pelant improved that result for any non-identical upper modification r.

Clearly, if $\kappa \leq \lambda$ and X is p_κ-complete, it is p_λ-complete. Consequently, all complete point-finite spaces are p_κ-complete for any $\kappa \geq \omega_1$.

3.3 Realcompleteness by means of filters. The condition in Theorem 2.1.4 may have several variations in uniform spaces. A direct modification gives: a uniform space is complete and no its uniformly discrete subset has Ulam measurable cardinality (i.e., $\operatorname{cov} X \leq \mathfrak{m}$). One possibility to get other classes is not to use convergence of all Cauchy filters but of some only. We shall use cip for those filters and formulate definitions and results using κ-intersection property.

In [12] a hierarchy starting with compact and realcompact spaces was prolonged: a topological space X is said to be κ-compact if every its zero-ultrafilter with κ-intersection property converges in X. In [14] the definition was extended to totally bounded uniformities and proved that those classes of spaces are simple (generated by a single space P_κ like compactness is generated by [0,1] and realcompactness by $\mathbb{R}$). Moreover, P_κ is not generated by P_λ for $\lambda < \kappa$. We shall extend the definitions to all uniform spaces. The spaces P_κ are described as follows: $P_{\kappa^+} = [0,1]^\kappa - \{p\}$, where p is any point of $[0,1]^\kappa$, $P_\kappa = \prod\{P_{\lambda^+}; \lambda < \kappa\}$ for non-successor κ. We shall usually use $p = \{0\}$, i.e., p is the point having all its coordinates equal to zero.

There are several possibilities for transferring κ-compactness to uniform spaces. One may use combinations of various Cauchy filters and z_u-filters. We should have in mind that in a uniform space every Cauchy filter with κ-intersection property converges iff every Cauchy z_u-filter with κ-intersection property converges and that every minimal Cauchy filter is z_u-filter.

Definition 4. A uniform space X is said to be κ-*complete* if one of the following equivalent conditions holds.

1. Every minimal Cauchy filter in X with κ-intersection property converges.
2. Every Cauchy z_u-filter in X with κ-intersection property converges.
3. Every Cauchy z_u-ultrafilter in X with κ-intersection property converges.

It follows from Lemma 1.4 that for $\kappa > \omega$ and spaces X with $\operatorname{cov} X \leq \mathfrak{m}$ the word "Cauchy" can be omitted in the condition 3. In that case, κ-completeness is a property of proximity spaces, i.e., a uniform space X is κ-complete iff pX has that property. That is not the case for $\kappa = \omega$.

The above properties for $\kappa = \omega_1$ appeared elsewhere under different names. For instance, in [1] the author calls the spaces satisfying the property 2 as weakly complete. In the same paper the property 3 without assuming z_u-ultrafilter to be Cauchy is investigated under the name Wallman realcompactness, in [2] as $\mathbb{R} - z_u$-completeness. Those concepts are studied there on uniformities finest among uniformities having the same collection of uniform zero sets. One can find there references to related earlier notions (e.g. to papers by S. Mrówka or by A.K. Steiner and E.F. Steiner).

At first we look at basic properties of κ-complete spaces.

Proposition 3.12. 1. *X is complete iff it is ω-complete.*

2. *Every uniform space X is $w_u(X)^+$-complete (thus, every metrizable uniform space is ω_1-complete).*

3. *If $\kappa < \lambda$ then every κ-complete space is λ-complete and there exists a uniformly zero-dimensional uniform space that is λ-complete and not κ-complete.*
4. *If X is κ-complete then any Y finer than X and inducing the same topology is κ-complete.*
5. *A topological space X is realcompact iff it is induced by an ω_1-complete uniform space with covering character not larger than $\mathfrak{m}$.*
6. *The class of κ-complete spaces is epireflective.*
7. *Every e-complete space is ω_1-complete. There is an ω_1-complete space that is not complete, thus not e-complete.*

Proof: The assertions 1 and 4 are easy. To prove 2, take a Cauchy filter $\mathcal{F}$ in X with κ intersection property, where $\kappa > w_u(X)$. There is a base $\mathcal{U}_\alpha, \alpha \leq w_u(X)$, of uniform covers of X. For every α there is $U_\alpha \in \mathcal{F} \cap \mathcal{U}_\alpha$. Then $\bigcap U_\alpha \neq \emptyset$, which implies that $\mathcal{F}$ converges to a point $x \in \bigcap U_\alpha$. Indeed, the neighborhood base $\{\mathrm{st}_{\mathcal{U}_\alpha} x\}_\alpha$ of $x \in \bigcap U_\alpha$ belongs to $\mathcal{F}$.

The first part of 3 is trivial. The totally bounded spaces P_κ described above are κ-complete and not λ-complete (see [14]). The same procedure can be used if one uses $2^\kappa \subset \{0\}$ instead of $[0,1]^\kappa \setminus \{0\}$.

In 5, realcompactness of a topological space X implies $\mathrm{cov}(X,u) \leq \mathfrak{m}$ for any uniformity u on X and a convergence of every z-ultrafilter with cip, which together entails ω_1-completeness of $t_f X$. Conversely, if a uniform space X is ω_1-complete and $\mathrm{cov}\, X \leq \mathfrak{m}$ then for every z-ultrafilter $\mathcal{F}$ with cip on X the filter with the base $\mathcal{F} \cap \mathcal{Z}_u(X)$ is a z_u-ultrafilter with cip on X (Lemma 1.1) and it is Cauchy by Lemma 1.4. Consequently, it converges and X is realcompact.

To show 6, we must prove that the classes under consideration are closed hereditary and productive. The closed hereditary property is proved in a standard way. To show productivity, it suffices to realize that projections preserve Cauchy filters with κ-intersection property.

If X is e-realcompact, then it is complete and, thus, ω_1-complete. Every metrizable non-complete space witnesses the last assertion. □

It was proved by M.D. Rice in [27] that each z_u ultrafilter with cip in a uniform space X converges iff meX is complete (m is the metric-fine coreflection). For X with $\mathrm{cov}\, X \leq \mathfrak{m}$ one has the following result.

Theorem 3.13 (M.D. Rice). *If $\mathrm{cov}\, X \leq \mathfrak{m}$ then X is ω_1-complete iff mX is complete.*

We see a difference between r-completeness defined by means of an upper modification and between ω_1-completeness characterized by means of completeness of a lower modification, at least for spaces with not huge covering character. A question is whether Theorem 3.13 may be modified for uncountable cardinals κ, i.e., for κ-complete spaces, to get a generalization of the previous theorem. We can show that it is not possible.

Theorem 3.14. *For $\kappa > \omega_1$ there are no upper and lower modifications in uniform spaces, both preserving topology, such that X is κ-complete iff some combination of those modifications applied to X is complete.*

PROOF: The space $X = [0,1]^{\omega_1} \setminus \{0\}$ is κ-complete for any $\kappa > \omega_1$. Since $\beta X = [0,1]^{\omega_1}$ (see, e.g., [14]), there is a unique uniformity on X inducing its topology. The space X is not complete. □

3.4 Realcompleteness using positions of X in sX. We now look at the conditions 6 and 7 from Theorem 2.1. The following transfer to uniform spaces gives equivalent properties (equivalent to ω_1-completeness if $\operatorname{cov} X \le \mathfrak{m}$) — see the proof of Proposition 3.15 or [1], where references to some implications from around 1970 are given.

1. X is G_δ-closed in sX;
2. X is the intersection of all cozero sets in sX containing X.

It is not difficult to modify the properties to higher cardinals such that they remain equivalent. We need to define or recall some concepts. A G_κ-closure of A in Y consists of those points $x \in Y$ such that every intersection of less than κ neighborhoods of x meets A. A κ-zero set in Y is an intersection of less than κ many zero sets in Y, κ-cozero set is a complement of a κ-zero set. Clearly, $Z \subset Y$ is κ-zero iff there exists a continuous map into some $[0,1]^\lambda, \lambda < \kappa$, with $Z = f^{-1}(\{0\})$.

Proposition 3.15. *Let X be a uniform space and κ be an infinite cardinal. Then the following subsets of sX coincide:*

$$\begin{aligned} X_z &= \{\textstyle\bigcap \overline{Z}^{sX}; Z \in \mathcal{F}, \mathcal{F} \text{ is a } z_u\text{-ultrafilter in } X \text{ with } \kappa\text{-intersection property}\} \\ X_c &= \textstyle\bigcap \{C; C \text{ is a } \kappa\text{-cozero set in } sX \text{ containing } X\} \\ X_u &= {}^{\kappa}\overline{X}^{sX} \text{ (i.e., } G_\kappa \text{ closure of } X \text{ in } sX). \end{aligned}$$

PROOF: Let $\xi \in X_z \setminus X$. Then ξ is a limit point of a z_u-ultrafilter $\mathcal{F}$ with κ-intersection property. For any open neighborhood U of ξ in sX one has $U \cap X \in \mathcal{F}$ and, consequently, intersection of less than κ many neighborhoods of ξ with X is non-empty. Thus $\xi \in X_u$ and $X_z \subset X_u$.

Now, let $\xi \in X_u$ and assume there is a κ-cozero set C in sX containing X with $\xi \notin C$. There is some $g \in U(sX,[0,1]^\lambda), \lambda < \kappa$, with $g(\xi) = \{0\}$, $\{0\} \notin g(C)$. Denote by g_α the composition of g with the α-th projection of $[0,1]^\lambda$ onto $[0,1]$ and take $U_\alpha^n = g_\alpha^{-1}[0,1/n]$. Then $C \cap \bigcap_{\alpha,n} U_\alpha^n = \emptyset$, which implies $X \cap \bigcap_{\alpha,n} U_\alpha^n = \emptyset$ and that is contradiction with $\xi \in X_u$. Consequently, $X_u \subset X_c$.

Take $\xi \in X_c$ and let ξ be a limit of a z_u-ultrafilter $\mathcal{F}$ from X that has not κ-intersection property. That means $\bigcap Z_\alpha = \emptyset$ for some $Z_\alpha \in \mathcal{F} \cap \mathcal{Z}_u(X), \alpha < \lambda < \kappa$. For every α there is a zero set S_α in sX with $S_\alpha \cap X = Z_\alpha$. The intersection $S = \bigcap S_\alpha$ is a nonvoid κ-zero set in sX disjoint with X. Since $\xi \in S$, we have $\xi \notin sX \setminus S$ and the last set is a κ-cozero set in sX containing X. Thus $\xi \notin X_c$

and that contradiction proves $X_c \subset X_z$. The proof of equalities $X_z = X_c = X_u$ is finished. $\square$

The next assertion can also be deduced from the results on Wallman realcompactifications in [1], [2] formulated for $\kappa = \omega_1$.

Corollary 3.16. *The first two of the following properties for a uniform space X are equivalent and are equivalent to the third one if* $\operatorname{cov} X \leq \mathfrak{m}$.

1. *X is G_κ-closed in sX.*
2. *$sX \setminus X$ is κ-zero in sX.*
3. *X is κ-complete.*

It follows from the previous corollary that no new concept need be defined using a position of X in sX.

3.5 Realcompleteness using extension of maps. The remaining properties from Theorem 2.1 have the following corresponding formulations in uniform spaces. All those uniform modifications are equivalent to properties defined earlier.

Proposition 3.17. *Consider the following properties for a uniform space X.*

1. *For each Y containing X as a dense uniform subspace and for each $y \in Y \setminus X$ there exists $f \in U(X)$ that cannot be extended continuously to y (into $\mathbb{R}$).*
2. *For each $\xi \in sX \setminus X$ one can find $f \in U(X)$ that cannot be continuously extended to ξ.*
3. *For each $\xi \in sX \setminus X$ one can find $f \in U^*(X), f > 0$, that continuously extends to ξ with the value* 0.

Then the first property is equivalent to completeness, the second one to c-realcompactness and the last one to ω_1-completeness provided $\operatorname{cov} X \leq \mathfrak{m}$.

Proof: That completeness is equivalent to 1 follows immediately from the fact that complete spaces are absolutely closed.

Clearly, cX is complete iff the property 2 holds.

To prove the last assertion, assume first that X is ω_1-complete, $\operatorname{cov} X \leq \mathfrak{m}$ and take any $\xi \in sX \setminus X$. Then the trace $\mathcal{F}$ of the neighborhood filter of ξ in sX to X has not cip. We can find neighborhoods U_n of ξ in sX with $X \cap \bigcap U_n = \emptyset$. We may assume that $U_n \supset U_{n+1}$. There is $f_n \in U(sX, [0,1])$ with $f_n(\xi) = 0, f_n(x) = 1$ for $x \in sX \setminus U_n$. Then the restriction f of the function $g = \sum f_n/2^n$ to X belongs to $U(X, (0,1])$ and $g(\xi) = 0$. Conversely, assume the property 3 is fulfilled and X is not ω_1-complete. Then there exists $\xi \in sX \setminus X$ such that the trace $\mathcal{F}$ of the neighborhood filter of ξ in sX to X has cip. Take any $f \in U(X, (0,1])$ and suppose $\widetilde{f}(\xi) = 0$. The preimages $U_n = f^{-1}([0, 1/n])$ belong to $\mathcal{F}$, therefore, $\bigcap U_n \neq \emptyset$. Consequently, there is some $x \in X$ with $f(x) = 0$, which is a contradiction. $\square$

It is possible to reformulate the previous property 3 to get a characterization of κ-completeness:

Proposition 3.18. *Every z_u-ultrafilter with κ-intersection property in a uniform space X converges iff for each $\xi \in sX \setminus X$ one can find $f : X \to P_{\lambda^+}$ for some $\lambda < \kappa$, that continuously extends to ξ with the value $\{0\}$.*

Proof: We may assume $\kappa > \omega_1$. Suppose first that every z_u-ultrafilter with κ-intersection property in X converges and let $\xi \in sX \setminus X$. The point ξ is a limit point of a z_u-ultrafilter $\mathcal{F}$ on X not having κ-intersection property. Thus there is $\lambda < \kappa$ and $Z_\alpha \in \mathcal{F}, \alpha \in \lambda$, with $\bigcap_\lambda Z_\alpha = \emptyset$. For each $\alpha < \lambda$ we find $f_\alpha \in U(X, [0,1])$ with $Z_\alpha = f_\alpha^{-1}(0)$. The reduced product f of all f_α is a uniformly continuous map of X into $[0,1]^\lambda \setminus \{0\}$. Clearly, the continuous extension $sX \to [0,1[^\lambda$ of f has the value $\{0\}$ at ξ.

Conversely, assume there is a non-converging z_u-ultrafilter on X with κ-intersection property and let $\xi \in sX$ be its limit. Take $\lambda < \kappa$, any $f \in U(X, P_{\lambda^+})$ extending continuously to ξ with the value $\{0\}$ and denote by f_α the composition of f with the α-projection $[0,1]^\lambda$ onto its α-th coordinate space. Then all the sets $Z_\alpha^n = f_\alpha^{-1}([0,1/n])$ belong to $\mathcal{F}$ and, thus, $\bigcap Z_\alpha^n \neq \emptyset$, which implies that $f(x) = \{0\}$ for some $x \in \bigcap Z_\alpha^n \subset X$ and that is not possible. Consequently, any $f \in U(X, P_{\lambda^+})$ extends continuously to ξ with a value different from $\{0\}$. $\square$

Corollary 3.19. *A uniform space X with* $\operatorname{cov} X \leq \mathfrak{m}$ *is κ-complete iff for each $\xi \in sX \setminus X$ one can find $f : X \to P_{\lambda^+}$ for some $\lambda < \kappa$, that continuously extends to ξ with the value $\{0\}$.*

3.6 Closing remarks. We have the following implications concerning epireflective classes:

$$\mathbb{R}\text{-complete} \to \text{c-complete} \to \text{e-complete} \to \text{complete} \to \omega_1\text{-complete}$$

As it follows from the preceding text, no arrow can be converted.

The above classes contained in complete spaces are characterized by completeness of bireflections, that one containing all complete spaces by completeness of a coreflection (for not huge covering characters). That suggests possibility to define other classes of uniform spaces by means of completeness of some reflections or coreflections or their combinations.

The classes are productive and closed-hereditary. They are also closed under some coproducts: if a uniformly discrete space D belongs to a class $\mathcal{C}$ above, then any coproduct of $|D|$ many spaces X_d from $\mathcal{C}$ also belongs to $\mathcal{C}$ (since such a coproduct is finer than a closed subspace of $D \times \prod_D X_d$ and is its topological subspace). None of the classes is closed under quotients. To show that we need the fact that every uniform space X is a quotient of a complete uniformly zero-dimensional space X_q of the same cardinality (see, e.g., [19, p. 52]). Not every countable uniform space is complete but every countable complete uniformly zero-dimensional space is $\mathbb{R}$-complete. So, a quotient of some $\mathbb{R}$-complete space need not be complete. That procedure does not work for ω_1-complete spaces since every countable uniform space is ω_1-complete. The ordered space X of countable ordinals with its unique uniformity is not ω_1-complete but X_q is c-complete. Thus quotients of c-complete spaces need not be ω_1-complete.

It is seen from the previous consideration that the only property that should be recognized as a uniform counterpart of realcompactness is c-completeness. It could be called realcompleteness or, better, uniform realcompleteness.

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