Radon-Nikodym property

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Abstract. For a Banach space E and a probability space $(X, \mathcal{A}, \lambda)$, a new proof is given that a measure $\mu : \mathcal{A} \to E$, with $\mu \ll \lambda$, has RN derivative with respect to λ iff there is a compact or a weakly compact $C \subset E$ such that $|\mu|_C : \mathcal{A} \to [0, \infty]$ is a finite valued countably additive measure. Here we define $|\mu|_C(A) = \sup\{\sum_k |\langle \mu(A_k), f_k \rangle|\}$ where $\{A_k\}$ is a finite disjoint collection of elements from $\mathcal{A}$, each contained in A, and $\{f_k\} \subset E'$ satisfies $\sup_k |f_k(C)| \leq 1$. Then the result is extended to the case when E is a Frechet space.

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1. Introduction and notations

In this paper K will always denote the field of real or complex numbers (we will call them scalars), $\mathbb{R}$ the real numbers and $\mathbb{N}$ the set of natural numbers. All locally convex space are assumed to be Hausdorff and are over K and notations and results of [8] will be used. Given a locally convex space E with E' its dual, for $x \in E$ and $f \in E'$, we will also write $\langle f, x \rangle = \langle x, f \rangle$ for $f(x)$; for an $A \subset E$, $\Gamma(A)$ will denote the absolute convex hull of A. Let $(X, \mathcal{A}, \lambda)$ be a complete probability space. By a measure we will always mean a countably additive measure. For a measure μ, $|\mu|$ will denote its total variation measure. For measures and vector measures we refer to [2]; see also [4], [5].

In [3], for a Banach space E, an interesting characterization is given for a vector measure $\mu : \mathcal{A} \to E$ of bounded variation, $\mu \ll \lambda$, to have a derivative. It is proved that μ has derivative iff there is a compact or weakly compact $C \subset E$ such that $|\mu|_C : \mathcal{A} \to [0, \infty]$ is a finite valued countably additive measure. Here we define $|\mu|_C(A) = \sup\{\sum_k |\langle \mu(A_k), f_k \rangle|\}$ where $\{A_k\}$ is a finite disjoint collection of elements from $\mathcal{A}$, each contained in A, and $\{f_k\} \subset E'$ satisfies $\sup_k |f_k(C)| \leq 1$. First we give a new proof of this result and then extend this to Frechet spaces.

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2. Main results

Theorem 1. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, E a Frechet space and $\mu : \mathcal{A} \to E$ countably additive measure with $\mu \ll \lambda$. Suppose μ has finite variation with respect to every continuous semi-norm on E. Then μ has RN derivative relative to λ iff there is a compact or weakly compact $C \subset E$ such that $|\mu|_C : \mathcal{A} \to [0, \infty]$ is a finite-valued measure. Here we define $|\mu|_C(A) = \sup\{\sum_k |\langle \mu(A_k), f_k\rangle|\}$ where $\{A_k\}$ is a finite disjoint collection of elements from $\mathcal{A}$, each contained in A, and $\{f_k\} \subset E'$ satisfies $\sup_k |f_k(C)| \leq 1$.*

Proof: First we consider E to be a Banach and give an entirely different proof than the one given in [3]; we will reduce it to a reflexive Banach subspace of E with a finer topology. Take an absolutely convex weakly compact $C \subset E$ with a countably additive measure $\nu = |\mu|_C \leq 1$. This implies $\mu(\mathcal{A}) \subset C$ ([3, Theorem 2.1(1)]). By [1], there is a reflexive Banach space $E_0 \subset E$ such that $C \subset E_0$, C is weakly compact in E_0, and the identity mapping $E_0 \to E$ is continuous. Take an $f_0 \in E_0'$ with norm ≤ 1 and fix $c > 0$. If we consider C as a subset of E, $(f_0)_{|C}$ is an affine continuous function on C. It is proved in [7, Proposition 3.5, p. 31] that $(E'_{|C} + K)$ is uniformly dense in the space of all continuous affine functions on C (this is proved when $K = \mathbb{R}$ but easily extends to general K). Thus there is an $f \in E'$ and $r \in K$ such that $\sup |(f + r - f_0)(C)| \leq c$. Since C is absolutely convex (that implies $0 \in C$), we get $\sup |(f - f_0)(C)| \leq 2c$. Take a decreasing sequence $\{A_n\} \subset \mathcal{A}$ such that $A_n \downarrow \emptyset$. Now $f \circ \mu(A_n) \to 0$ and since $\sup |(f - f_0)(C)| \leq 2c$ and c is arbitrarily small, we get $f_0 \circ \mu(A_n) \to 0$ and so $\mu : \mathcal{A} \to E_0$ is countably additive. Now we will prove that $\mu : \mathcal{A} \to E_0$ is of bounded variation. Take $p > 0$ with $pC \subset B$ (the unit ball of E_0), a finite collection $\{f_i\}$ in the closed unit ball of E_0', and disjoint elements $\{A_i\} \subset \mathcal{A}$. We have $|pf_i(C)| \leq 1\ \forall i$. As explained above, take $\{f_i'\} \subset E'$ with $\sup |(pf_i - f_i')(C)| \leq \frac{1}{2^i}\ \forall i$. We get

$$\sum |\langle f_i, \mu(A_i)\rangle| \leq \frac{1}{p} \sum |(pf_i - f_i')\mu(A_i)| + \frac{1}{p} \sum |(f_i')\mu(A_i)| \leq \frac{1}{p} + \frac{1}{p}|\mu|_C(X) \leq \frac{2}{p}.$$

This proves $\mu : \mathcal{A} \to E_0$ is of bounded variation. Since E_0 is reflexive, there is an $h \in L_1(X, E_0)$ with $\mu = h\lambda$. From this it easily follows that $h \in L_1(X, E)$. The converse is same as for Frechet space which we will consider now.

Now we consider the case when E is a Frechet space. Suppose μ has RN derivative $\frac{d\mu}{d\lambda} = g \in L^1(\lambda, E) = L^1(\lambda)\hat{\otimes} E$ (the completion in projective tensor product). Thus $g = \sum_i \alpha_i g_i x_i$, $\{g_i\}$, $\{x_i\}$ being null sequences in $L^1(\lambda)$ and E respectively and $\{\alpha_i\} \in \ell_1$ ([8, Theorem 6.4, p. 94]); we can assume that $\int |g_i| d\lambda \leq 1\ \forall i$. Let C be the closed, absolutely convex hull of $\{x_i\}$; C is compact. Take a finite, disjoint family $\{A_k\}$ of elements of $\mathcal{A}$ and $\{f_k\}$ elements of E' with $\sup |f_k(C)| \leq 1\ \forall k$. We have $\sum_k |\langle |\mu(A_k), f_k|\rangle| \leq \sum_k \sum_i \int_{A_k} |\alpha_i|\, |g_i|\, |f_k(x_i)| d\lambda \leq \alpha$ where $\alpha = \sum_i |\alpha_i|$. Thus $|\mu|_C$ is finite-valued. It is a routine verification that $|\mu|_C$ is countably additive ([2, p. 4], [3, Theorem 2.1(5), p. 142]).

Conversely suppose for an absolutely convex weakly compact $C \subset E$, $\nu = |\mu|_C$ is finite-valued. We have $\mu \ll |\mu|_C$. Also it follows from the definition of $|\mu|_C$ that, for an $f \in E'$ with $\sup|f(C)| \leq 1$, we have $|\mu|_C \geq |f \circ \mu|$. Denoting the completion of $|\mu|_C$ by $|\mu|_C$ again, we fix a lifting ρ_0 for this measure ([9]) and take the lifting topology $\mathcal{T}_0$ on X which has $\{\rho_0(A) : A \in \mathcal{A}\}$ as the base of open sets; we can assume this topology to be Hausdorff and denote by $C_b(X)$ all scalar-valued bounded continuous functions on X. For each $f \in E'$ there is a $\phi_f \in L_1(|\mu|_C)$ such that $f \circ \mu = \phi_f |\mu|_C$. Put $\sup|f(C)| = p$; we claim $|\phi_f| \leq p$ *a.e.* $[|\mu|_C]$. Suppose this is not true. Then there is a $c > 0$ such that $|\mu|_C(A) > 0$ where $A = \{x \in X : |\phi_f(x)| \geq p + c\}$. Now, since $|\frac{1}{p}f(C)| \leq 1$, we have $|\mu|_C(A) \geq \frac{1}{p}|f \circ \mu|(A) = \frac{1}{p}\int_A |\phi_f| d|\mu|_C \geq \frac{1}{p}(p+c)|\mu|_C(A)$ which is a contradiction. Thus there is a unique function in $C_b(X)$ which is equal to ϕ_f *a.e.* $[|\mu|_C]$; we denote this function also by ϕ_f.

Define $\phi : X \to K^{E'}$, $(\phi(x))_f = \phi_f(x)$. It is a simple verification that $\phi_{f_1+f_2} = \phi_{f_1} + \phi_{f_2}$ and $\phi_{rf} = r\phi_f$ for any $f_1, f_2, f \in E'$ and any $r \in K$. Also E, with weak topology, can be considered as a subspace of $K^{E'}$ with product topology. We claim that $\phi(X) \subset C$. If this is not true, by separation theorem ([8, 9.2, p. 65]), $\exists x_0 \in X$ and $f \in E'$ such that $p = \sup|f(C)| < Rl(\phi_f(x_0)) = p + 3\eta$ for some $\eta > 0$ (note C is absolutely convex and so $\sup(Rl(f(C))) = \sup|f(C)|$). Now $\nu(A) > 0$ where $A = \{x \in X : |\phi_f(x)| > p + 2\eta\}$. Since $|\phi_f| \leq p$, we have $p\nu(A) \geq \int_A |\phi_f| d\nu \geq (p + 2\eta)\nu(A)$, a contradiction. By [6, Theorem 2, p. 389], ϕ is weakly equivalent to a function ϕ_0 such that $\phi_0(X)$ is contained in a separable weakly compact subset of E; thus ϕ_0 is bounded. Now it is well-known that if a weakly measurable function has a separable range in E then it is strongly measurable ([2, Theorem 2, p. 42]; it is proved for a Banach space but thus easily extends to a Frechet space). Now being bounded and strongly measurable, $\phi_0 \in L^1(\nu, E)$. Since $|\mu| \ll \nu$ and $\nu \ll \lambda$, the result follows. $\square$

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