Resolvability in c.c.c. generic extensions

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Abstract. Every crowded space X is ω-resolvable in the c.c.c. generic extension $V^{\mathrm{Fn}(|X|,2)}$ of the ground model.

We investigate what we can say about λ-resolvability in c.c.c. generic extensions for $\lambda > \omega$.

A topological space is *monotonically* ω_1*-resolvable* if there is a function $f : X \to \omega_1$ such that

$$\{x \in X : f(x) \geq \alpha\} \subset^{dense} X$$

for each $\alpha < \omega_1$.

We show that given a T_1 space X the following statements are equivalent:

(1) X is ω_1-resolvable in some c.c.c. generic extension;

(2) X is monotonically ω_1-resolvable;

(3) X is ω_1-resolvable in the Cohen-generic extension $V^{\mathrm{Fn}(\omega_1,2)}$.

We investigate which spaces are monotonically ω_1-resolvable. We show that if a topological space X is c.c.c., and $\omega_1 \leq \Delta(X) \leq |X| < \omega_\omega$, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$, then X is monotonically ω_1-resolvable.

On the other hand, it is also consistent, modulo the existence of a measurable cardinal, that there is a space Y with $|Y| = \Delta(Y) = \aleph_\omega$ which is not monotonically ω_1-resolvable.

The characterization of ω_1-resolvability in c.c.c. generic extension raises the following question: is it true that crowded spaces from the ground model are ω-resolvable in $V^{\mathrm{Fn}(\omega,2)}$?

We show that (i) if $V = L$ then every crowded c.c.c. space X is ω-resolvable in $V^{\mathrm{Fn}(\omega,2)}$, (ii) if there are no weakly inaccessible cardinals, then every crowded space X is ω-resolvable in $V^{\mathrm{Fn}(\omega_1,2)}$.

Moreover, it is also consistent, modulo a measurable cardinal, that there is a crowded space X with $|X| = \Delta(X) = \omega_1$ such that X remains irresolvable after adding a single Cohen real.

Keywords: resolvable; monotonically ω_1-resolvable; measurable cardinal

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1. Introduction

Notion of resolvability was introduced and studied first by E. Hewitt [4], in 1943. A topological space X is κ*-resolvable* if it can be partitioned into κ many dense subspaces. X is *resolvable* iff it is 2-resolvable, and *irresolvable* otherwise.

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Irresolvable spaces with many interesting extra properties were constructed, but there are no "absolute" examples for crowded irresolvable spaces, because if X is a crowded space, then clearly

$$V^{\mathrm{Fn}(|X|,2)} \models X \text{ is } \omega\text{-resolvable.}$$

In this paper we investigate what we can say about λ-resolvability in c.c.c. generic extensions for $\lambda > \omega$.

To characterize spaces which are ω_1-resolvable in some c.c.c. generic extension we introduce the notion of monotone κ-resolvability.

Definition 1.1. Let κ be an infinite cardinal. A topological space X is *monotonically κ-resolvable*† if there is a function $f : X \to \kappa$ such that

$$\{x \in X : f(x) \geq \alpha\} \subset^{dense} X$$

for each $\alpha < \kappa$. We will say that f *witnesses* that X is monotonically κ-resolvable.

Clearly a space X is monotonically κ-resolvable iff X has a partition $\{X_\zeta : \zeta < \kappa\}$ of X such that

$$\operatorname{int}\left(\bigcup\{X_\zeta : \zeta < \xi\}\right) = \emptyset$$

for all $\xi < \kappa$.

Theorem 1.2. *Let X be a T_1 topological space. The following statements are equivalent:*

(1) *X is ω_1-resolvable in some c.c.c. generic extension,*
(2) *X is monotonically ω_1-resolvable,*
(3) *X is ω_1-resolvable in the Cohen generic extension $V^{\mathrm{Fn}(\omega_1,2)}$.*

Which spaces are monotonically ω_1-resolvable?

Theorem 1.3. *If a topological space X is c.c.c., and $\omega_1 \leq \Delta(X) \leq |X| < \omega_\omega$, then X is monotonically ω_1-resolvable.*

Theorem 1.4. *If κ is a measurable cardinal, then there is a space X with $|X| = \Delta(X) = \kappa$ which is not monotonically ω_1-resolvable.*

What about spaces of cardinality ω_ω?

Theorem 1.5. *It is consistent, modulo the existence of a measurable cardinal, that there is a space X with $|X| = \Delta(X) = \omega_\omega$ which is not monotonically ω_1-resolvable.*

Do we really need to add $|X|$-many Cohen reals to make X resolvable?

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†In [13] a "monotonically ω-resolvable" space is called "almost-ω-resolvable". However, in [12] a space X is *almost-κ-resolvable* if it contains a family of κ dense sets with pairwise nowhere dense intersections.

Theorem 1.6. (1) *It is consistent, modulo a measurable cardinal, that there is a crowded space X with $|X| = \Delta(X) = \omega_1$ (so X is monotonically ω_1-resolvable) such that*

$$V^{\mathrm{Fn}(\omega,2)} \models \text{“}X \text{ is irresolvable.”}$$

(2) *If $V = L$, then every crowded space with $|X| = \Delta(X) = \mathrm{cf}(|X|)$ is monotonically ω-resolvable, and so it is ω-resolvable in $V^{\mathrm{Fn}(\omega,2)}$.*
(3) *If the cardinality of a crowded c.c.c. space X is less than the first weakly inaccessible cardinal, then X is ω-resolvable in $V^{\mathrm{Fn}(\omega_1,2)\S}$.*

The almost resolvability of c.c.c. spaces was investigated by Pavlov in [11]: on page 53 Pavlov writes that mimicking Malykhin's method, by using Ulam matrices, he showed that every crowded c.c.c. space of cardinality ω_1 is almost resolvable. In [3, Theorem 2.22] a stronger result was proved: a crowded c.c.c. space is almost resolvable if its cardinality is less than the first weakly inaccessible cardinal. Theorem 1.6(2) is a further improvement of this result because monotone ω-resolvability implies almost resolvability.

In [1, 3.12 Problem (2)] the authors ask *if every space with countable cellularity and cardinality less than the first inaccessible non-countable cardinal is almost-ω-resolvable*. As we will see Theorem 1.6(3) gives a positive answer to a weakening of this question.

2. Characterization of ω_1-resolvability in c.c.c. extensions

Instead of Theorem 1.2 we prove the following stronger result. We say that a function $g : X \to \kappa$ *witnesses* that X is κ-resolvable if

$$\{x \in X : g(x) = \alpha\} \subset^{dense} X$$

for each $\alpha < \kappa$.

Theorem 2.1. *Assume that X is a crowded topological space and κ is an infinite cardinal. If $\kappa = \mathrm{cf}([\kappa]^\omega, \subset)$ then the following statements are equivalent:*
(1) *X is κ-resolvable in some c.c.c. generic extension;*
(2) *there is a function $h : X \to [\kappa]^\omega$ such that $\bigcup h''U = \kappa$ for each non-empty open $U \subset X$;*
(3) *X is κ-resolvable in the Cohen-generic extension $V^{\mathrm{Fn}(\kappa,2)}$.*

Proof: First we show that (1) $\to$ (2). Assume that $\mathbb{P}$ is a c.c.c. poset such that there is a function $g \in V^{\mathbb{P}}$ witnessing the κ-resolvability of X.

For each $x \in X$ define

$$h(x) = \{\alpha < \kappa : \exists p^x_\alpha \in \mathbb{P}(p^x_\alpha \Vdash \dot{g}(\check{x}) = \check{\alpha})\}.$$

Since the conditions $\{p^x_\alpha : \alpha \in h(x)\}$ are pairwise incomparable and $\mathbb{P}$ is c.c.c., the set $h(x)$ is countable.

§ω_1 is not a misprint here.

We now show that the function h defined above satisfies (2). Fix $\alpha < \kappa$ and U an open subset of X. We need to show that there exists $x \in U$ such that $\alpha \in h(x)$. Since

$$V^{\mathbb{P}} \models g^{-1}(\{\alpha\}) \subset^{dense} X$$

it follows that there is $x \in U$ such that

$$V^{\mathbb{P}} \models g(x) = \alpha.$$

Thus, there exists $p \in \mathbb{P}$ such that

$$p \Vdash \text{“}\dot{g}(\check{x}) = \check{\alpha}.\text{”}$$

Then $\alpha \in h(x)$.

Next we show that (2) $\to$ (3). Let $\mathcal{A}$ be a cofinal subset of $[\kappa]^\omega$ with $|\mathcal{A}| = \kappa$.

Let $\{A_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{A}$, and for each $x \in X$ pick

$$h^*(x) \in \mathcal{A} \text{ such that } h^*(x) \supset \bigcup_{\alpha \in h(x)} A_\alpha.$$

Then for all non-empty open U

$$(+) \qquad \{h^*(x) : x \in U\} \text{ is cofinal in } [\kappa]^\omega.$$

Next we note that forcing with $\mathrm{Fn}(\kappa, 2)$ is the same as forcing with $\mathrm{Fn}(\kappa, \omega)$. Further, $\mathrm{Fn}(\kappa, \omega)$ is isomorphic to

$$\mathbb{P} = \{p \in \mathrm{Fn}(\mathcal{A}, \kappa) \ : \ \forall A \in \mathrm{dom}(p)\ p(A) \in A\}.$$

Indeed, for each $A \in \mathcal{A}$ fix a bijection $\rho_A : \omega \to A$, and then for $q \in \mathrm{Fn}(\kappa, \omega)$ define $\varphi(q) \in \mathbb{P}$ as follows:

(i) $\mathrm{dom}(\varphi(q)) = \{A_\alpha : \alpha \in \mathrm{dom}(q)\}$, and
(ii) $\varphi(q)(A_\alpha) = \rho_{A_\alpha}(q(\alpha))$ for $A_\alpha \in \mathrm{dom}(\varphi(q))$.

Then φ is clearly an isomorphism between $\mathrm{Fn}(\kappa, \omega)$ and $\mathbb{P}$.

We will proceed using $\mathbb{P}$.

Let G be a $\mathbb{P}$-generic filter, and let $g = \bigcup G$. Then $g \in V^{\mathbb{P}}$ and $g : \mathcal{A} \to \kappa$ is such that $g(A) \in A$.

We claim that $f = g \circ h^*$ witnesses that X is κ-resolvable.

Fix $\alpha < \kappa$ and an open $U \subset X$.

Let $q \in \mathbb{P}$ be arbitrary. Then, by (+), there is $x \in U$ such that

$$\{\alpha\} \cup \bigcup \mathrm{dom}(q) \subsetneq h^*(x).$$

Then $h^*(x) \notin \mathrm{dom}(q)$, and $\alpha \in h^*(x)$, so

$$p = q \cup \{\langle h^*(x), \alpha\rangle\} \in \mathbb{P}_1,$$

and

$$p \Vdash (g \circ h^*)(\check{x}) = \check{\alpha}.$$

Thus, by genericity, there is $p \in G$ and $x \in U$ such that

$$p \Vdash (g \dot{\circ} h^*(\check{x}) = \check{\alpha}).$$

Hence

$$V^{\mathbb{P}} \models X \text{ is } \kappa\text{-resolvable.}$$

Finally (3) $\rightarrow$ (1) is trivial. $\square$

Problem 2.2. *Can we drop the assumption $\kappa = \mathrm{cf}([\kappa]^\omega, \subset)$ from Theorem 2.1?*

3. On monotone ω_1-resolvability of c.c.c. spaces

We start with an easy to prove observation.

Lemma 3.1. *Let X be a topological space and $\mathcal{B} \subset \mathcal{P}(X)$. If every $B \in \mathcal{B}$ is monotonically κ-resolvable, then so is $\overline{\cup \mathcal{B}}$. So every space contains a subspace which is the greatest monotonically κ-resolvable subspace (this subspace can be empty, of course).*

Corollary 3.2. *Let X be a topological space. Let Z be a dense subset of X. If Z is monotonically κ-resolvable, then X is also monotonically κ-resolvable.*

Before proving Theorem 1.3 we prove the following "stepping-down" theorem. The proof uses ideas from [8].

Theorem 3.3. *If X is a κ-c.c., monotonically κ^+-resolvable space, then X is monotonically κ-resolvable as well.*

Proof: Since an open subspace of a κ-c.c., monotonically κ^+-resolvable space is also κ-c.c. and monotonically κ^+-resolvable, by Lemma 3.1 it is enough to show that

(*) every κ-c.c., monotonically κ^+-resolvable space X has a monotonically κ-resolvable non-empty open subset.

Ulam [14] proved that there is a "matrix"

$$\left\langle M_{\alpha,\zeta} : \alpha < \kappa^+, \zeta < \kappa \right\rangle \subset \mathcal{P}(\kappa^+)$$

such that

(i) $M_{\alpha,\xi} \cap M_{\beta,\xi} = \emptyset$ for $\{\alpha, \beta\} \in [\kappa^+]^2$ and $\xi \in \kappa$,
(ii) $M_{\alpha,\xi} \cap M_{\alpha,\zeta} = \emptyset$ for $\alpha \in \kappa^+$ and $\{\xi, \zeta\} \in [\kappa]^2$, and
(iii) $|M^-_\alpha| \leq \kappa$, where $M^-_\alpha = \kappa^+ \setminus \bigcup_{\zeta<\kappa} M_{\alpha,\zeta}$ for $\alpha < \kappa^+$.

Fix a partition $\{Y_\eta : \eta < \kappa^+\}$ witnessing that X is monotonically κ^+-resolvable. Let

$$Z_{\alpha,\zeta} = \bigcup\{Y_\eta : \eta \in M_{\alpha,\zeta}\}$$

for $\alpha < \kappa^+$ and $\zeta < \kappa$, and let

$$Z_\alpha = \bigcup_{\zeta<\kappa} Z_{\alpha,\zeta}.$$

Since $Z_\alpha = \bigcup\{Y_\eta : \eta \in \kappa^+ \setminus M_\alpha^-\}$, assumption (iii) implies that every Z_α is dense in X.

Case 1. There is $\alpha < \kappa^+$ such that for all $\zeta < \kappa$

$$\bigcup_{\zeta\le\xi} Z_{\alpha,\xi} \subset^{dense} Z_\alpha.$$

Then $(Z_{\alpha,\zeta})_{\zeta<\kappa}$ witnesses that Z_α is monotonically κ-resolvable and so by Corollary 3.2, X is also monotonically κ-resolvable.

Case 2. For all $\alpha < \kappa^+$ there is $\zeta_\alpha < \kappa$ and there is a non-empty open set $U_\alpha \in \tau_X$ such that

$$(\dagger) \qquad \bigcup_{\zeta_\alpha\le\xi} Z_{\alpha,\xi} \cap U_\alpha = \emptyset.$$

Then there is a set $I \in [\kappa^+]^{\kappa^+}$ and there is an ordinal $\zeta < \kappa$ such that $\zeta_\alpha = \zeta$ for all $\alpha \in I$.

Fix an arbitrary $K \in [I]^\kappa$. By (iii) we can find $\rho < \kappa^+$ such that

$$\bigcup_{\alpha\in K} M_\alpha^- \subset \rho.$$

Let $Z = \bigcup_{\rho<\eta} Y_\eta$. Then $Z \subset^{dense} X$ and $Z \subset Z_\alpha$ for all $\alpha \in K$.

Claim. *If $L \in [K]^\kappa$ then*

$$\bigcap_{\alpha\in L} U_\alpha \cap Z = \emptyset.$$

Proof of the Claim: Assume on the contrary that $z \in \bigcap_{\alpha\in L} U_\alpha \cap Z$. Then $z \in Y_\eta$ for some $\rho < \eta$.

Let $\alpha \in L$. Then $\eta \in \kappa^+ \setminus \rho \subset \bigcup_{\xi<\kappa} M_{\alpha,\xi}$. Pick $\xi_\alpha < \kappa$ with $\eta \in M_{\alpha,\xi_\alpha}$. Then $Y_\eta \subset Z_{\alpha,\xi_\alpha}$, so $Z_{\alpha,\xi_\alpha} \cap U_\alpha \neq \emptyset$, so $\xi_\alpha < \zeta_\alpha = \zeta$ by ($\dagger$).

Since $\zeta < \kappa = |L|$, there are $\alpha \neq \beta \in [L]^2$ such that $\xi_\alpha = \xi_\beta$. Thus $\eta \in M_{\alpha,\xi_\alpha} \cap M_{\beta,\xi_\beta}$ which contradicts (i) because $\xi_\alpha = \xi_\beta$. □

Fix an enumeration $K = \{\chi_\xi : \xi < \kappa\}$, and let $V_\zeta = \bigcup_{\zeta<\xi} U_{\chi_\xi}$. Then the sequence $\langle V_\zeta : \zeta < \kappa\rangle$ is decreasing and

$$\bigcap_{\zeta<\kappa} V_\zeta \cap Z = \emptyset$$

by the Claim.

Since X is κ-c.c. there is $\xi < \kappa$ such that $\overline{V_\zeta} = \overline{V_\xi}$ for all $\xi < \zeta < \kappa$.

We can assume that $\xi = 0$. Let

$$T_\zeta = \begin{cases} V_0 \setminus Z & \text{if } \zeta = 0, \\ \\ ((\bigcap_{\xi<\zeta} V_\xi) \setminus V_\zeta) \cap Z & \text{if } \zeta > 0. \end{cases}$$

Then

$$\bigcup_{\xi<\zeta} T_\zeta \supset V_\xi \cap Z \subset^{dense} V.$$

Thus the partition $\{T_\zeta : \zeta < \kappa\}$ witnesses that V is monotonically κ-resolvable. □

Proof of Theorem 1.3: Let $\mathcal{Y} = \{Y \in \tau_X : |Y| = \Delta(Y)\}$.

Then $\bigcup \mathcal{Y}$ is dense in X, and every open subset of every $Y \in \mathcal{Y}$ is also in $\mathcal{Y}$. Thus by Lemma 3.1 it is enough to prove that a c.c.c. space Y with $\omega_1 \leq |Y| = \Delta(Y) < \omega_\omega$ is monotonically ω_1-resolvable.

Let $Y \in \mathcal{Y}$ such that $\omega_n = |Y|$. Clearly, Y is monotonically ω_n-resolvable because $|Y| = \Delta(Y) = \omega_n$. Since Y is c.c.c. then Y is ω_{n-1}-c.c. By Theorem 3.3, Y is monotonically ω_{n-1}-resolvable. By repeating the application of Theorem 3.3 $n-2$ times we conclude that Y is monotonically ω_1-resolvable. □

Problem 3.4. *Is it true that every crowded c.c.c. space with $\Delta(X) \geq \omega_1$ is monotonically ω_1-resolvable?*

4. Spaces which are not monotonically ω_1-resolvable

If X is a topological space, and $\mathcal{D} \subset \mathcal{P}(X)$, we write

$$\overline{\overline{\mathcal{D}}} = \{\overline{D} : D \in \mathcal{D}\}.$$

Lemma 4.1. *Let X be a topological space. Assume that $\overline{\overline{\mathcal{D}}}$ is point-countable for each point-countable family $\mathcal{D} \subset \mathcal{P}(X)$. Then X does not contain any monotonically ω_1-resolvable subspace Y.*

Proof: Assume that $\{Y_\zeta : \zeta < \omega_1\}$ is a partition of Y. Let $D_\xi = \bigcup\{Y_\zeta : \xi < \zeta\}$ for $\xi < \omega_1$. Then the family $\mathcal{D} = \{D_\xi : \xi < \omega_1\}$ is point-countable. Hence $\overline{\overline{\mathcal{D}}}$ is also point-countable. So D_ξ is not dense in Y for all but countably many ξ. Therefore the partition $\{Y_\zeta : \zeta < \omega_1\}$ does not witness that Y is monotonically ω_1-resolvable. □

To prove Theorems 1.4 and 1.5 we recall some definitions and results from [6] and [5].

Definition 4.2 ([6, Definition 3.1])**.** Let κ be an infinite cardinal, and let $\mathcal{F}$ be a filter on κ. Let T be the tree $\kappa^{<\omega}$. A topology $\tau_{\mathcal{F}}$ is defined on T by

$$\tau_{\mathcal{F}} = \big\{V \subset T : \forall t \in V \{\alpha \in \kappa : t^\frown \alpha \in V\} \in \mathcal{F}\big\},$$

and the space $\langle T, \tau_{\mathcal{F}}\rangle$ is denoted by $X(\mathcal{F})$.

Proof of Theorem 1.4: Let $\mathcal{U}$ be a κ-complete non-principal ultrafilter on κ.

The space $X = X(\mathcal{U})$ is monotonically normal by [6, Theorem 3.1].

An ultrafilter $\mathcal{U}$ is λ-*descendingly complete* if $\bigcap\{U_\zeta : \zeta < \lambda\} \neq \emptyset$ for each decreasing sequence $\{U_\zeta : \zeta < \lambda\} \subset \mathcal{U}$.

A σ-complete ultrafilter is clearly ω-descendingly-complete. In the proof of [6, Theorem 3.5] the authors prove Lemma 3.6 which claims that $\overline{\overline{\mathcal{D}}}$ is point-countable for each point-countable family $\mathcal{D} \subset \mathcal{P}(X(\mathcal{F}))$ provided that $\mathcal{F}$ is a ω-descendingly complete ultrafilter. So $\overline{\overline{\mathcal{D}}}$ is point-countable for each point-countable family $\mathcal{D} \subset \mathcal{P}(X)$, and so X is not monotonically ω_1-resolvable by Lemma 4.1. $\square$

Instead of Theorem 1.5 we prove the following theorem which is a slight improvement of [5, Theorem 5].

Theorem 4.3. *If it is consistent that there is a measurable cardinal, then it is also consistent that there is an ω-resolvable monotonically normal space X with $|X| = \Delta(X) = \omega_\omega$ such that if a family $\mathcal{D} \subset \mathcal{P}(X)$ is point-countable, then the family $\overline{\overline{\mathcal{D}}} = \{\overline{D} : D \in \mathcal{D}\}$ is also point countable. Hence X does not contain any monotonically ω_1-resolvable subspace.*

Proof: In [5, p. 665] the authors write that "*starting from one measurable, Woodin (*[15]*) constructed a model in which $\aleph_\omega$ carries an ω_1-descendingly complete uniform ultrafilter. Woodin's model V_1 can be embedded into a bigger ZFC model V_2 so that the pair of models (V_1, V_2) with $\kappa = \aleph_\omega$ satisfies the two models situation*", i.e.

(1) $\omega_1^{V_1} = \omega_1^{V_2}$,
(2) there is a countable subset A of ω_ω in V_2 such that no $B \in V_1$ of cardinality $< \omega_\omega$ covers A,
(3) for the filter $\mathcal{G}$ on ω_ω defined in V_2 by $B \in \mathcal{G}$ iff $A \setminus B$ is finite, we have $\mathcal{G} \cap V_1 \in V_1$.

(The "two model situation" is defined in [5, Theorem 4.5]).

Let $\mathcal{F} = \mathcal{G} \cap V_1$ and consider the space $X = X(\mathcal{F})$. As it was observed in [6], spaces obtained as $X(\mathcal{H})$ from some filter $\mathcal{H}$ are monotonically normal and ω-resolvable.

In [5, Theorem 4.1] Juhász and Magidor showed that the space $X(\mathcal{F})$ is actually hereditarily ω_1-irresolvable. They proved the following lemma:

Lemma 4.2 ([5])**.** *For any $D \subset X(F)$ and $t \in \overline{D}$ there is a finite sequence s of members of A such that $t^\frown s \in D$.*

Using this lemma we show that $\overline{\overline{\mathcal{D}}}$ is point-countable for each point-countable family $\mathcal{D} \subset \mathcal{P}(X)$, and so X is not monotonically ω_1-resolvable by Lemma 4.1.

Indeed, let $\mathcal{D} \subset \mathcal{P}(X)$ be an uncountable family such that $t \in \bigcap_{D\in\mathcal{D}} \overline{D}$. Then, by [5, Lemma 4.3], for each $D \in \mathcal{D}$ we can pick a finite sequence s_D of members of A such that $t^\frown s_D \in D$. Since there are only countable many finite sequences of elements of A there is s such that $s_D = s$ for uncountably many $D \in \mathcal{D}$. Then $t^\frown s$ is in uncountably many elements of $\mathcal{D}$, so $\mathcal{D}$ is not point-countable.

We have thus proved that no subspace of X is monotonically ω_1-resolvable. $\square$

5. ω-resolvability after adding a single Cohen real

Before proving Theorem 1.6 we need some preparation.

The notion of almost resolvability was introduced by Bolstein [2] in 1973: a topological space is *almost-resolvable* if it is a countable union of sets with empty interiors. The notion of monotone ω-resolvability was first considered in [13] under the name almost-ω-resolvability.

Clearly almost ω-resolvable (i.e. monotonically ω-resolvable) spaces are almost resolvable.

Lemma 5.1. *Let X be a crowded topological space.*
(1) *If X is monotonically ω-resolvable, then X is ω-resolvable in $V^{\mathrm{Fn}(\omega,2)}$.*
(2) *If X is resolvable in $V^{\mathrm{Fn}(\omega,2)}$, then X is almost-resolvable.*

Proof: (1) Assume that the function $f : X \to \omega$ witnesses the monotone ω-resolvability of X.

If $\mathcal{G}$ is the V-generic filter in $\mathrm{Fn}(\omega,\omega)$, and $g = \bigcup \mathcal{G}$, then the function $h = g \circ f$ witnesses that X is ω-resolvable.

We need to show that $\{y \in X : (g \circ f)(y) = n\}$ is dense in X.

Indeed, let $p \in \mathrm{Fn}(\omega,\omega)$ and $\emptyset \neq U \in \tau_X$. Since $f : X \to \omega$ witnesses the monotone ω-resolvability of X there is $y \in U$ such that

$$f(y) > \max \mathrm{dom}(p).$$

Let

$$q = p \cup \{\langle f(y), n\rangle\}.$$

Then $q \leq p$ and

$$g \Vdash (g \circ f)(y) = n.$$

So we proved that $g \circ f$ witnesses that X is ω-resolvable in the generic extension.

(2) Assume that

$$V^{\mathrm{Fn}(\omega,2)} \models \text{“}X \text{ has a partition } \{D_0, D_1\} \text{ into dense subsets.”}$$

For all $p \in \mathrm{Fn}(\omega,2)$ and $i < 2$ let

$$D_i^p = \{x \in X : p \Vdash x \in \dot{D}_i\}.$$

Then $X = \bigcup\{D_i^p : p \in \mathrm{Fn}(\omega,2), i < 2\}$, and we claim that $\operatorname{int} D_i^p = \emptyset$ for each $p \in \mathrm{Fn}(\omega,2)$, and $i < 2$.

Indeed, fix p and i and let U be an arbitrary non-empty open subset. Then $p \Vdash U \cap \dot{D}_{1-i} \neq \emptyset$, so there is $q \le p$ and $y \in U$ such that $q \Vdash y \in \dot{D}_{1-i}$. Then $q \Vdash y \notin \dot{D}_i$, hence $p \nVdash y \in \dot{D}_i$, and so $y \notin D_i^p$. Thus $U \not\subset D_i^p$. Since U was arbitrary, we proved $\operatorname{int} D_i^p = \emptyset$. □

After this preparation we can prove Theorem 1.6.

Proof of Theorem 1.6: (1) Kunen [7] proved that it is consistent, modulo a measurable cardinal, that there is a maximal independent family $\mathcal{A} \subset \mathcal{P}(\omega_1)$ which is also σ-independent.

In [9, Theorems 3.1 and 3.2] the authors proved that if there is a maximal independent family $\mathcal{A} \subset \mathcal{P}(\omega_1)$ which is also σ-independent, then there is a Baire space X with $|X| = \Delta(X) = \omega_1$ such that every open subspace of X is irresolvable, i.e. the space X is *OHI*.

It is well-known that a crowded OHI Baire space X is not almost resolvable: if $X = \bigcup_{n\in\omega} X_n$, then $\operatorname{int} X_n \neq \emptyset$ for some $n \in \omega$.

Indeed, if $\operatorname{int} X_n = \emptyset$, then $X \setminus X_n$ is dense, so $U_n = \operatorname{int}(X \setminus X_n)$ is dense in X because every open subset of X is irresolvable. Thus $\bigcap_{n\in\omega} U_n \neq \emptyset$ because X is Baire. However

$$\bigcap_{n\in\omega} U_n \subset \bigcap_{n\in\omega} (X \setminus X_n) = X \setminus \bigcup_{n\in\omega} X_n = \emptyset,$$

which is a contradiction.

Thus X is not almost resolvable, so it is not ω-resolvable in the model $V^{\mathrm{Fn}(\omega,2)}$ by Lemma 5.1(2).

(2) In [10] the authors proved that if $V = L$, then there are no crowded Baire irresolvable spaces. Hence, by [13], if $V = L$, then every crowded space X is almost-ω-resolvable (i.e. monotonically ω-resolvable).

So these spaces are ω-resolvable in the model $V^{\mathrm{Fn}(\omega,2)}$ by Lemma 5.1(1).

(3) Let X be a crowded c.c.c. space.

We can assume that $|X| = \Delta(X)$.

By induction we define a strictly decreasing sequence of cardinals:

$$\kappa_0, \kappa_1, \ldots, \kappa_n \ldots$$

as follows.

(i) $\kappa_0 = \Delta(X)$,
(ii) if κ_i is singular, then $\kappa_{i+1} = \mathrm{cf}(\kappa_i)$,
(iii) if $\kappa_i > \omega$ is regular, then $\kappa_i = \lambda^+$ (because $|X|$ is below the first weakly inaccessible cardinal,) and let $\kappa_{i+1} = \lambda$,
(iv) if $\kappa_i = \omega$ or $\kappa_i = \omega_1$, then we stop.

Assume that the construction stopped in the nth step.

Then we can prove, by finite induction, that X is monotonically κ_i-resolvable for all $i \leq n$ by Theorem 3.3. Thus X is monotonically ω-resolvable or monotonically ω_1-resolvable, and so either X is ω-resolvable in $V^{\mathrm{Fn}(\omega,2)}$ by Lemma 5.1(1), or X is ω_1-resolvable in $V^{\mathrm{Fn}(\omega_1,2)}$ by Theorem 2.1. $\square$

Problem 5.2 ([13, Questions 5.2.]). *Are almost resolvability and almost-ω-resolvability equivalent in the class of irresolvable spaces?*

Problem 5.3. *Is there, in ZFC, a crowded topological space X which is irresolvable in the Cohen generic extension $V^{\mathrm{Fn}(\omega,2)}$?*

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