# Statistical convergence of sequences of functions with values in semi-uniform spaces

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*Abstract.* We study several kinds of statistical convergence of sequences of functions with values in semi-uniform spaces. Particularly, we generalize to statistical convergence the classical results of C. Arzelà, Dini and P. S. Alexandroff, as well as their statistical versions studied in [Caserta A., Di Maio G., Kočinac L. D. R., *Statistical convergence in function spaces*,. Abstr. Appl. Anal. 2011, Art. ID 420419, 11 pp.] and [Caserta A., Kočinac L. D. R., *On statistical exhaustiveness*, Appl. Math. Lett. **25** (2012), no. 10, 1447–1451].

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## 1. Introduction

In [17] Morita defines a generalization for uniform structures using the covering concept of Tukey [22]. Subsequently many researchers have dealt with this issue. On the other hand, classical results about sequences and nets of functions have been extended from metric to uniform and generalized uniform spaces (see, for example, [4], [10], [13], [15]).

The concept of convergence of a sequence has been extended to statistical convergence by Fast [11], Fridy [12], Šalát [19], Schoenberg [20], Steinhaus [21], and Zygmund [23]. This convergence has many applications in mathematical analysis (see, for instance, [14]). In recent years, a lot of papers have been written on sequences of real functions and functions between metric spaces by using the idea of statistical convergence (see [3], [7], [8], [16]).

In this paper, we present and investigate the quasi uniform, Alexandroff, almost uniform and Dini statistical convergence for a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of an arbitrary topological space $X$ into a semi-uniform space $Y$. Particularly, the continuity of the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ is studied. Since each uniform space is a semi-uniform space, the results of the paper remain valid in the case that $Y$ is a uniform space.

The rest of this paper is organized as follows. Sections 2 and 3 contain preliminaries and basic concepts, respectively. In Section 4 we give the quasi uniform

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and Alexandroff statistical convergence for sequences of functions with values in semi-uniform spaces and we present modifications of the results of Arzelà [2] (see also [6]) and Alexandroff [1] (a survey on these results can be found in [5]). In Section 5 we present the almost uniform statistical convergence and define the notion of st-equicontinuous family of functions. Finally, the concept of Dini statistical convergence of a sequence of functions with values in a semi-uniform space is investigated in Section 6.

## 2. Preliminaries

First, we recall some of the basic concepts related to the uniform spaces. There are several ways to approach the theory of uniform spaces. Here we use Tukey description of a uniform space in terms of covers [22]. For more details we refer the reader to [9], [13], [18].

The *power set* of a set $Y$ is denoted by $\mathcal{P}(Y)$. Let $Y$ be a set and let $y \in Y$, $B \subseteq Y$, and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(Y)$. We use the following terminology and notations.

(1) The family $\mathcal{A}$ is called a *cover* of $Y$ if $\bigcup\{A \subseteq Y \colon A \in \mathcal{A}\} = Y$.
(2) The family $\mathcal{A}$ is called a *refinement* of $\mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$. In this case we write $A \prec \mathcal{B}$.
(3) $\mathcal{A} \vee \mathcal{B} = \{A \cup B \colon A \in \mathcal{A}, B \in \mathcal{B}\}$.
(4) $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \colon A \in \mathcal{A}, B \in \mathcal{B}\}$.
(5) $\mathrm{St}(B, \mathcal{A}) = \bigcup\{A \in \mathcal{A} \colon A \cap B \neq \emptyset\}$.
(6) $\mathrm{St}(y, \mathcal{A}) = \mathrm{St}(\{y\}, \mathcal{A}) = \bigcup\{A \in \mathcal{A} \colon y \in A\}$.
(7) $\mathrm{St}^{n+1}(B, \mathcal{A}) = \mathrm{St}^{n}(\mathrm{St}(B, \mathcal{A}), \mathcal{A})$, $n = 1, 2, \ldots$.

Let $\Phi$ be a nonempty family of covers of a set $Y$ and $\mathcal{A}, \mathcal{B} \in \Phi$.

(1) The family $\mathcal{B}$ is called a *star-refinement of* $\mathcal{A}$ if $\{\mathrm{St}(B, \mathcal{B}) \colon B \in \mathcal{B}\} \prec \mathcal{A}$.
(2) The family $\mathcal{B}$ is called a *local star-refinement of* $\mathcal{A}$ *in* $\Phi$ if for each $B \in \mathcal{B}$ there exist $\mathcal{A}_B \in \Phi$ and $A \in \mathcal{A}$ such that $\mathrm{St}(B, \mathcal{A}_B) \subseteq A$.

**Definition 2.1** ([18]). A *uniformity* on a set $Y$ is a nonempty family $\Phi$ of covers of $Y$ satisfying the following properties.

($\Phi_1$) If $\mathcal{A}_1, \mathcal{A}_2 \in \Phi$, then there exists $\mathcal{B} \in \Phi$ such that $\mathcal{B} \prec \mathcal{A}_1 \wedge \mathcal{A}_2$.
($\Phi_2$) If $\mathcal{A} \in \Phi$ and $\mathcal{B}$ is a cover of $Y$ such that $\mathcal{A} \prec \mathcal{B}$, then $\mathcal{B} \in \Phi$.
($\Phi_3$) For each $\mathcal{A} \in \Phi$ there exists $\mathcal{B} \in \Phi$ which is a star-refinement of $\mathcal{A}$.

**Definition 2.2** ([18]). A *semi-uniformity* on a set $Y$ is a nonempty family $\Phi$ of covers of $Y$ satisfying conditions ($\Phi_1$) and ($\Phi_2$) from Definition 2.1 and the following:

($\Phi_4$) For each $\mathcal{A} \in \Phi$ there exists $\mathcal{B} \in \Phi$ which is a local star-refinement of $\mathcal{A}$ in $\Phi$.

A *semi-uniform space* is a pair $(Y, \Phi)$ consisting of a set $Y$ and a semi-uniformity $\Phi$ on the set $Y$.

For every semi-uniform space $(Y, \Phi)$ the semi-uniform topology $\tau_\Phi$ on $Y$ is the family of all subsets $O$ of $Y$ such that for each $y \in O$ there is $\mathcal{A} \in \Phi$ such that $\mathrm{St}(y, \mathcal{A}) \subseteq O$.

A mapping $f$ of a topological space $X$ into a semi-uniform space $(Y, \Phi)$ is called *continuous at* $x_0 \in X$ if for each $\mathcal{A} \in \Phi$ there exists an open neighbourhood $O_{x_0}$ of $x_0$ such that $f(O_{x_0}) \subseteq \mathrm{St}(f(x_0), \mathcal{A})$. The mapping $f$ is called *continuous* if it is continuous at every point of $X$.

Now, we recall the notion of asymptotic density (see, for instance, [14]). By $\mathbb{N}$ we denote the set of positive integers.

Let $K \subseteq \mathbb{N}$. For every $n \in \mathbb{N}$ we set $\delta_n(K) = \{k \in K : k \leq n\}$. The *asymptotic density* $\delta(K)$ of $K$ is equal to

$$\lim_{n\to\infty} \frac{|\delta_n(K)|}{n}$$

whenever this limit exists. A set $K \subseteq \mathbb{N}$ is said to be *statistically dense* if $\delta(K) = 1$. The family $\mathcal{F} = \{K \subseteq \mathbb{N} : \delta(K) = 1\}$ is a proper filter on $\mathbb{N}$, that is the following conditions hold.

(1) $\mathcal{F} \neq \emptyset$.
(2) $\emptyset \notin \mathcal{F}$.
(3) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
(4) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

The following is a summary of some properties of asymptotic density.

(1) $0 \leq \delta(K) \leq 1$.
(2) $\delta(\mathbb{N}) = 1$.
(3) For every $x \in [0, 1]$ there exists a subset $K_x$ of $\mathbb{N}$ such that $\delta(K_x) = x$.
(4) If $K$ is a finite subset of $\mathbb{N}$, then $\delta(K) = 0$.
(5) If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$, provided that both densities exist.
(6) If $\delta(K)$ exists, then $\delta(\mathbb{N} \setminus K) = 1 - \delta(K)$.
(7) The even integers have asymptotic density 1/2, as do the odd integers.
(8) The prime numbers have asymptotic density 0.

## 3. Basic concepts

The concept of statistical convergence for sequences of functions between metric spaces was investigated in [7] and [8]. Here we deal with sequences of functions with values in semi-uniform spaces. In what follows we consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$.

**Definition 3.1.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge pointwise to* $f$ *on* $X$ if for every $x \in X$ and for every $\mathcal{A} \in \Phi$ there exists a statistically dense set $K \subseteq \mathbb{N}$ such that for every $n \in K$ we have $f_n(x) \in \mathrm{St}(f(x), \mathcal{A})$. In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$. We shall say that the sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges pointwise on* $X$ if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$.

**Definition 3.2.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge uniformly to $f$ on $X$* if for every $\mathcal{A} \in \Phi$ there exists a statistically dense set $K \subseteq \mathbb{N}$ such that for every $x \in X$ and for every $n \in K$ we have $f_n(x) \in \mathrm{St}(f(x), \mathcal{A})$. In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-u}} f$. We shall say that the sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges uniformly on $X$* if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-u}} f$.

In what follows we will use frequently the following facts:

**Fact 3.3.** Let $(Y, \Phi)$ be a semi-uniform space and $\mathcal{A} \in \Phi$. The following statements are true.

(1) If $Y_1 \subseteq Y$, then $Y_1 \subseteq \mathrm{St}(Y_1, \mathcal{A})$.
(2) If $Y_1 \subseteq Y_2 \subseteq Y$, then $\mathrm{St}(Y_1, \mathcal{A}) \subseteq \mathrm{St}(Y_2, \mathcal{A})$.
(3) If $y \in \mathrm{St}(x, \mathcal{A})$, then $\mathrm{St}(y, \mathcal{A}) \subseteq \mathrm{St}^2(x, \mathcal{A})$.
(4) If $y \in Y$ and $\mathcal{B} \prec \mathcal{A}$, then $\mathrm{St}(y, \mathcal{B}) \subseteq \mathrm{St}(y, \mathcal{A})$.

Proof: (1) Let $y \in Y_1$. Since $\mathcal{A}$ is a cover of $Y$, there exists $A \in \mathcal{A}$ such that $y \in A$. Hence, $Y_1 \cap A \neq \emptyset$ and, therefore, $y \in A \subseteq \mathrm{St}(Y_1, \mathcal{A})$.

(2) Let $y \in \mathrm{St}(Y_1, \mathcal{A})$. Then, there exists $A \in \mathcal{A}$ such that $y \in A$ and $Y_1 \cap A \neq \emptyset$. Since $Y_1 \subseteq Y_2$, we have $Y_2 \cap A \neq \emptyset$. Therefore, $y \in \mathrm{St}(Y_2, \mathcal{A})$.

(3) This follows by statement (2) for $Y_1 = \{y\}$ and $Y_2 = \mathrm{St}(x, \mathcal{A})$.

(4) Let $z \in \mathrm{St}(y, \mathcal{B})$. Then, there exists $B \in \mathcal{B}$ such that $y, z \in B$. Since $\mathcal{B} \prec \mathcal{A}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$. Therefore, $y, z \in A$ and, hence $z \in \mathrm{St}(y, \mathcal{A})$. □

**Lemma 3.4.** *Let $(Y, \Phi)$ be a semi-uniform space, $\mathcal{A} \in \Phi$ and $y_0 \in Y$. Then, there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}^3(y_0, \mathcal{B}) \subseteq \mathrm{St}(y_0, \mathcal{A})$.*

Proof: Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be local star-refinements of $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ in $\Phi$, respectively. Since $\mathcal{A}_3$ is a cover of $Y$, there exists $A_3 \in \mathcal{A}_3$ such that $y_0 \in A_3$. We have successively:

(i) There exist $\mathcal{B}_3 \in \Phi$ and $A_2 \in \mathcal{A}_2$ such that $\mathrm{St}(A_3, \mathcal{B}_3) \subseteq A_2$. Hence, $y_0 \in A_2$.
(ii) There exist $\mathcal{B}_2 \in \Phi$ and $A_1 \in \mathcal{A}_1$ such that $\mathrm{St}(A_2, \mathcal{B}_2) \subseteq A_1$. Hence, $y_0 \in A_1$.
(iii) There exist $\mathcal{B}_1 \in \Phi$ and $A \in \mathcal{A}$ such that $\mathrm{St}(A_1, \mathcal{B}_1) \subseteq A$. Hence, $y_0 \in A$.

Let $\mathcal{B}_4 \in \Phi$ such that $\mathcal{B}_4 \prec \mathcal{B}_1 \wedge \mathcal{B}_2$ and let $\mathcal{B} \in \Phi$ such that $\mathcal{B} \prec \mathcal{B}_4 \wedge \mathcal{B}_3$. Therefore, $\mathcal{B} \prec \mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \mathcal{B}_3$. We prove that $\mathrm{St}^3(y_0, \mathcal{B}) \subseteq \mathrm{St}(y_0, \mathcal{A})$.

First, we prove that

$$\mathrm{St}^3(y_0, \mathcal{B}) \subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(A_3, \mathcal{B}_3), \mathcal{B}_2), \mathcal{B}_1),$$

where

$$\begin{aligned}\mathrm{St}(\mathrm{St}(\mathrm{St}(A_3, \mathcal{B}_3), \mathcal{B}_2), \mathcal{B}_1) &= \bigcup\{B_1 \in \mathcal{B}_1 \colon B_1 \cap \mathrm{St}(\mathrm{St}(A_3, \mathcal{B}_3), \mathcal{B}_2) \neq \emptyset\} \\ &= \bigcup\{B_1 \in \mathcal{B}_1 \colon B_1 \cap B_2 \neq \emptyset \text{ for some } B_2 \in \mathcal{B}_2 \text{ with } B_2 \cap \mathrm{St}(A_3, \mathcal{B}_3) \neq \emptyset\}.\end{aligned}$$

Let us have

$$\begin{aligned} y \in \mathrm{St}^3(y_0,\mathcal{B}) &= \mathrm{St}(\mathrm{St}(\mathrm{St}(y_0,\mathcal{B}),\mathcal{B}),\mathcal{B}) = \bigcup\{B \in \mathcal{B}\colon B \cap \mathrm{St}(\mathrm{St}(y_0,\mathcal{B}),\mathcal{B}) \neq \emptyset\} \\ &= \bigcup\{B \in \mathcal{B}\colon B \cap B' \neq \emptyset \text{ for some } B' \in \mathcal{B} \text{ with } B' \cap \mathrm{St}(y_0,\mathcal{B}) \neq \emptyset\}. \end{aligned}$$

Then, there exists $B \in \mathcal{B}$ such that $y \in B$ and $B \cap B' \neq \emptyset$ for some $B' \in \mathcal{B}$ with $B' \cap \mathrm{St}(y_0,\mathcal{B}) \neq \emptyset$. Therefore, there exists $B'' \in \mathcal{B}$ such that $y_0 \in B''$ and $B' \cap B'' \neq \emptyset$. Since $\mathcal{B} \prec \mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \mathcal{B}_3$, there exist $B_1, B_1', B_1'' \in \mathcal{B}_1$, $B_2, B_2', B_2'' \in \mathcal{B}_2$, $B_3, B_3', B_3'' \in \mathcal{B}_3$ such that

$$B \subseteq B_1 \cap B_2 \cap B_3, \quad B' \subseteq B_1' \cap B_2' \cap B_3', \quad B'' \subseteq B_1'' \cap B_2'' \cap B_3''.$$

Since $y \in B$, we have $y \in B_1$. Moreover, $B_1 \cap B_2' \neq \emptyset$. It suffices to prove that $B_2' \cap \mathrm{St}(A_3,\mathcal{B}_3) \neq \emptyset$. Indeed, we have $y_0 \in B_3''$ and $y_0 \in A_3$. Hence, $B_3'' \subseteq \mathrm{St}(A_3,\mathcal{B}_3)$. Since $B_2' \cap B_3'' \neq \emptyset$, we have $B_2' \cap \mathrm{St}(A_3,\mathcal{B}_3) \neq \emptyset$. Thus, $y \in \mathrm{St}(\mathrm{St}(\mathrm{St}(A_3,\mathcal{B}_3),\mathcal{B}_2),\mathcal{B}_1)$.

Now, we prove that $\mathrm{St}^3(y_0,\mathcal{B}) \subseteq \mathrm{St}(y_0,\mathcal{A})$. Indeed, we have

$$\mathrm{St}^3(y_0,\mathcal{B}) \subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(A_3,\mathcal{B}_3),\mathcal{B}_2),\mathcal{B}_1) \subseteq \mathrm{St}(\mathrm{St}(A_2,\mathcal{B}_2),\mathcal{B}_1) \subseteq \mathrm{St}(A_1,\mathcal{B}_1) \subseteq A.$$

Since $y_0 \in A$, we have $A \subseteq \mathrm{St}(y_0,\mathcal{A})$. Therefore, $\mathrm{St}^3(y_0,\mathcal{B}) \subseteq \mathrm{St}(y_0,\mathcal{A})$. □

**Proposition 3.5.** *If $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-u}} f$ and the functions $f_n$, $n \in \mathbb{N}$ are continuous, then the function $f$ is continuous.*

Proof: Suppose that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-u}} f$ and let $x_0 \in X$. We prove that $f$ is continuous at $x_0$. Let $\mathcal{A} \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that

$$\mathrm{St}^3(f(x_0),\mathcal{B}) \subseteq \mathrm{St}(f(x_0),\mathcal{A}).$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-u}} f$, there exists a statistically dense set $K \subseteq \mathbb{N}$ such that for every $x \in X$ and for every $n \in K$ we have $f_n(x) \in \mathrm{St}(f(x),\mathcal{B})$. Let $n_0 \in K$. Then,

$$f_{n_0}(x_0) \in \mathrm{St}(f(x_0),\mathcal{B}). \tag{1}$$

Since $f_{n_0}$ is continuous at $x_0$, there exists an open neighbourhood $O_{x_0}$ of $x_0$ such that $f_{n_0}(x) \in \mathrm{St}(f_{n_0}(x_0),\mathcal{B})$, for every $x \in O_{x_0}$. Let $x \in O_{x_0}$. Then,

$$f_{n_0}(x) \in \mathrm{St}(f_{n_0}(x_0),\mathcal{B}) \tag{2}$$

and

$$f_{n_0}(x) \in \mathrm{St}(f(x),\mathcal{B}). \tag{3}$$

Therefore, using successively the relations (1), (2), and (3), we have

$$\begin{aligned} f(x) \in \mathrm{St}(f_{n_0}(x), \mathcal{B}) &\subseteq \mathrm{St}(\mathrm{St}(f_{n_0}(x_0), \mathcal{B}), \mathcal{B}) \\ &\subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(f(x_0), \mathcal{B}), \mathcal{B}), \mathcal{B}) \\ &= \mathrm{St}^3(f(x_0), \mathcal{B}) \end{aligned}$$

and the continuity of $f$ is proved. □

## 4. Quasi uniform and Alexandroff statistical convergence

In this section we introduce the notions of quasi uniform and Alexandroff statistical convergence for sequences of functions with values in semi-uniform spaces and then we generalize the classical theorems of C. Arzelà [2] (see also [6]) and P. S. Alexandroff [1].

In [7], a statistical version of the quasi uniform convergence of sequences of functions between metric spaces was defined. An analogous definition can be given for sequences of functions with values in semi-uniform spaces.

**Definition 4.1.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge quasi uniformly to $f$ on $X$* if $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st}} f$ and for every $\mathcal{A} \in \Phi$ and for every statistically dense set $K \subseteq \mathbb{N}$, there exists a finite subset $\{n_1, \dots, n_r\}$ of $K$ such that for each $x \in X$ at least one of the following relations holds:

$$f_{n_i}(x) \in \mathrm{St}(f(x), \mathcal{A}), \quad i = 1, \dots, r.$$

In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-qu}} f$. We shall say that the sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges quasi uniformly on $X$* if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-qu}} f$.

**Lemma 4.2** ([18, Proposition 1.11])**.** *Let $(Y, \Phi)$ be a semi-uniform space, $\mathcal{A} \in \Phi$, and $y_0 \in Y$. Then, there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}^2(y_0, \mathcal{B}) \subseteq \mathrm{St}(y_0, \mathcal{A})$.*

Proof: It is similar to the proof of Lemma 3.4. □

**Lemma 4.3.** *Let $(Y, \Phi)$ be a semi-uniform space and $\mathcal{A} \in \Phi$. Then, there exists an open cover $\mathcal{O}$ of $Y$ in the semi-uniform topology $\tau_\Phi$ such that $\mathcal{O} \in \Phi$ and $\mathcal{O} \prec \mathcal{A}$.*

Proof: We set $\mathcal{O} = \{\mathrm{Int}_\Phi(A) \colon A \in \mathcal{A}\}$, where

$$\mathrm{Int}_\Phi(A) = \{y \in Y \colon \text{there exists } \mathcal{B} \in \Phi \text{ such that } \mathrm{St}(y, \mathcal{B}) \subseteq A\}.$$

First, we prove that $\mathcal{O} \in \Phi$. Let $\mathcal{B} \in \Phi$ be a local star-refinement of $\mathcal{A}$ in $\Phi$. Then, for every $B \in \mathcal{B}$ there exist $\mathcal{E} \in \Phi$ and $A \in \mathcal{A}$ such that $\mathrm{St}(B, \mathcal{E}) \subseteq A$. This shows that $B \subseteq \mathrm{Int}_\Phi(A)$. Hence, $\mathcal{B} \prec \mathcal{O}$ and, therefore, $\mathcal{O} \in \Phi$.

Now, we prove that $\mathcal{O}$ is an open cover of $Y$. Let $A \in \mathcal{A}$. We prove that $\mathrm{Int}_\Phi(A)$ is an open subset of $Y$. Let $y \in \mathrm{Int}_\Phi(A)$. It suffices to prove that there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}(y, \mathcal{B}) \subseteq \mathrm{Int}_\Phi(A)$. Since $y \in \mathrm{Int}_\Phi(A)$, there exists

$\mathcal{B}_1 \in \Phi$ such that $\mathrm{St}(y, \mathcal{B}_1) \subseteq A$. By Lemma 4.2 there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}^2(y, \mathcal{B}) \subseteq \mathrm{St}(y, \mathcal{B}_1)$. Let $z \in \mathrm{St}(y, \mathcal{B})$. Then, we have

$$\mathrm{St}(z, \mathcal{B}) \subseteq \mathrm{St}^2(y, \mathcal{B}) \subseteq \mathrm{St}(y, \mathcal{B}_1) \subseteq A.$$

Hence, $z \in \mathrm{Int}_\Phi(A)$ and, therefore, $\mathrm{St}(y, \mathcal{B}) \subseteq \mathrm{Int}_\Phi(A)$. □

**Lemma 4.4.** *Let $f$ and $g$ be two continuous functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$. The following statements are true.*

(1) *The function $m\colon X \to (Y \times Y, \tau_\Phi \times \tau_\Phi)$ defined by $m(x) = (f(x), g(x))$, for every $x \in X$ is continuous.*
(2) *If $\mathcal{O} \in \Phi$ is an open cover of $Y$ in the semi-uniform topology $\tau_\Phi$, then the set $M = \{x \in X\colon f(x) \in \mathrm{St}(g(x), \mathcal{O})\}$ is open.*

Proof: (1) Let $x \in X$ and let $\mathrm{St}(f(x), \mathcal{A}) \times \mathrm{St}(g(x), \mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \Phi$, be an open neighbourhood of $m(x)$. Since $f$ is continuous at $x$, there exists an open neighbourhood $O_x$ of $x$ such that

$$f(O_x) \subseteq \mathrm{St}(f(x), \mathcal{A}).$$

Since $g$ is continuous at $x$, there exists an open neighbourhood $O'_x$ of $x$ such that

$$g(O'_x) \subseteq \mathrm{St}(g(x), \mathcal{B}).$$

We consider the open neighbourhood of $x$:

$$O''_x = O_x \cap O'_x.$$

Then, we have

$$m(O''_x) \subseteq f(O_x) \times g(O'_x) \subseteq \mathrm{St}(f(x), \mathcal{A}) \times \mathrm{St}(g(x), \mathcal{B}).$$

(2) Let $\mathcal{O} \in \Phi$ be an open cover of $Y$ in the semi-uniform topology $\tau_\Phi$. It suffices to prove that

$$M = m^{-1}\Bigg(\bigcup_{O \in \mathcal{O}} (O \times O)\Bigg).$$

Let $x \in M$. Then, $f(x) \in \mathrm{St}(g(x), \mathcal{O})$. Therefore, there exits $O_x \in \mathcal{O}$ such that $f(x), g(x) \in O_x$. Hence,

$$m(x) = (f(x), g(x)) \in O_x \times O_x \subseteq \bigcup_{O \in \mathcal{O}} (O \times O).$$

Thus, $M \subseteq m^{-1}\left(\bigcup_{O \in \mathcal{O}} (O \times O)\right)$.

Conversely, let $x \in m^{-1}\left(\bigcup_{O \in \mathcal{O}} (O \times O)\right)$. Then,

$$m(x) = (f(x), g(x)) \in \bigcup_{O \in \mathcal{O}} (O \times O).$$

Hence, there exits $O_x \in \mathcal{O}$ such that $f(x), g(x) \in O_x$ which means $f(x) \in \mathrm{St}(g(x), \mathcal{O})$. Thus, $x \in M$ and $m^{-1}\left(\bigcup_{O\in\mathcal{O}}(O \times O)\right) \subseteq M$. □

**Lemma 4.5.** *Let $f$ be a continuous function of a topological space $X$ into a semi-uniform space $(Y, \Phi)$ and $x_0 \in X$. The following statements are true.*

(1) *The function $m\colon X \to (Y \times Y, \tau_\Phi \times \tau_\Phi)$ defined by $m(x) = (f(x), f(x_0))$, for every $x \in X$ is continuous.*
(2) *If $\mathcal{O} \in \Phi$ is an open cover of $Y$ in the semi-uniform topology $\tau_\Phi$, then the set $M = \{x \in X\colon f(x) \in \mathrm{St}(f(x_0), \mathcal{O})\}$ is open.*

Proof: It is similar to the proof of Lemma 4.4. □

In [7] (see Theorem 3.3) and [8] (see Theorem 4.8) some characterizations of the continuity of statistical pointwise limit for sequences of functions between metric spaces were given. Similar results are true for sequences of functions with values in semi-uniform spaces.

**Theorem 4.6.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$. If the sequence $(f_n)_{n\in\mathbb{N}}$ statistically converges pointwise to a continuous limit, then the statistical convergence is quasi uniform on every compact subset of $X$. Conversely, if the sequence $(f_n)_{n\in\mathbb{N}}$ statistically converges quasi uniformly on a subset of $X$, then the limit is continuous on this subset.*

Proof: Suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges pointwise to a continuous function $f$. Let $C$ be compact subset of $X$, $\mathcal{A} \in \Phi$, and let $K \subseteq \mathbb{N}$ be a statistically dense set. By Lemma 4.3 there exists an open cover $\mathcal{O}$ of $Y$ in the semi-uniform topology $\tau_\Phi$ such that $\mathcal{O} \in \Phi$ and $\mathcal{O} \prec \mathcal{A}$. Let $c \in C$. Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{st} f$, there exists a statistically dense set $K_c \subseteq \mathbb{N}$ such that for every $n \in K_c$,

$$f_n(c) \in \mathrm{St}(f(c), \mathcal{O}).$$

Choose $n_c \in K_c \cap K$ and set

$$O_c = \{x \in X\colon f_{n_c}(x) \in \mathrm{St}(f(x), \mathcal{O})\}.$$

Since $f_{n_c}$ and $f$ are continuous, by Lemma 4.4, $O_c$ is an open set containing $c$. Thus, the family

$$\{O_c \cap C\colon c \in C\}$$

is an open cover of $C$. By compactness of $C$, there are $c_1, \ldots, c_r \in C$ such that

$$C = \bigcup_{i=1}^{r} O_{c_i} \cap C.$$

The set $\{n_{c_1}, \ldots, n_{c_r}\}$ is a finite subset of $K$ such that for each $x \in C$ at least one of the following relations holds:

$$f_{n_{c_i}}(x) \in \mathrm{St}(f(x), \mathcal{O}), \quad i = 1, \ldots, r.$$

Since $\mathcal{O} \prec \mathcal{A}$, for each $x \in C$ it holds that $f_{n_{c_i}}(x) \in \mathrm{St}(f(x), \mathcal{A})$ for at least one $i = 1, \ldots, r$. Thus, $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-qu}} f$ on $C$.

Conversely, suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges quasi uniformly to $f$ on a subset $X'$ of $X$. Let $x_0 \in X'$. We prove that $f$ is continuous at $x_0$. Let $\mathcal{A} \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}^3(f(x_0), \mathcal{B}) \subseteq \mathrm{St}(f(x_0), \mathcal{A})$. By Lemma 4.3 there exists an open cover $\mathcal{O}$ of $Y$ in the semi-uniform topology $\tau_\Phi$ such that $\mathcal{O} \in \Phi$ and $\mathcal{O} \prec \mathcal{B}$. Let

$$K_0 = \{n \in \mathbb{N} \colon f_n(x_0) \in \mathrm{St}(f(x_0), \mathcal{O})\}.$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st}} f$, there exists a statistically dense set $K \subseteq \mathbb{N}$ such that for every $n \in K$ we have

$$f_n(x_0) \in \mathrm{St}(f(x_0), \mathcal{O}).$$

Hence, $K \subseteq K_0$ and, therefore $K_0$ is a statistically dense set. By assumption, there exists a finite subset $\{n_1, \ldots, n_r\}$ of $K_0$ such that for each $x \in X'$ at least one of the following relations holds:

$$f_{n_i}(x) \in \mathrm{St}(f(x), \mathcal{O}), \quad i = 1, \ldots, r. \tag{4}$$

Since $\{n_1, \ldots, n_r\} \subseteq K_0$, by the definition of $K_0$, we have

$$f_{n_i}(x_0) \in \mathrm{St}(f(x_0), \mathcal{O}), \quad i = 1, \ldots, r. \tag{5}$$

Let

$$O_i = \{x \in X \colon f_{n_i}(x) \in \mathrm{St}(f_{n_i}(x_0), \mathcal{O})\}, \quad i = 1, \ldots, r. \tag{6}$$

Since the functions $f_{n_i}$, $i = 1, \ldots, r$ are continuous, by Lemma 4.5, the sets $O_i$, $i = 1, \ldots, r$ are open in $X$ and contain $x_0$. We consider the set

$$O_{x_0} = X' \cap \bigcap_{i=1}^{r} O_i.$$

Then, $O_{x_0}$ is open in $X'$ and contains the point $x_0$. Let $x \in O_{x_0}$. Using relations (4), (5), and (6), for proper choice of $i$, we obtain

$$f(x) \in \mathrm{St}^3(f(x_0), \mathcal{O}) \subseteq \mathrm{St}^3(f(x_0), \mathcal{B}) \subseteq \mathrm{St}(f(x_0), \mathcal{A}).$$

Thus, $f(O_{x_0}) \subseteq \mathrm{St}(f(x_0), \mathcal{A})$. We conclude that $f$ is continuous at $x_0$ completing the proof of the theorem. □

In [8] (see also [7, Definition 3.1]), a statistical version of the Alexandroff convergence of sequences of functions between metric spaces was defined. An analogous definition can be given for sequences of functions with values in semi-uniform spaces.

**Definition 4.7.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge Alexandroff to $f$ on $X$* if $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st}} f$ and for every $\mathcal{A} \in \Phi$ and for every statistically dense set $K \subseteq \mathbb{N}$, there exist an open cover $\mathcal{U} = \{O_n : n \in \mathbb{N}\}$ of $X$ and an infinite set $M = \{k_1, k_2, \ldots, k_n, \ldots\} \subseteq K$ such that for every $n \in \mathbb{N}$ and for each $x \in O_n$ we have $f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{A})$. In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Al}} f$. We shall say that the sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges Alexandroff on $X$* if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Al}} f$.

The next theorem is a generalization of [8, Theorem 4.8] to sequences of functions with values in semi-uniform spaces.

**Theorem 4.8.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$ and suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges pointwise to $f$ on $X$. Then, the statistical convergence is Alexandroff if and only if $f$ is continuous.*

Proof: Suppose that $f$ is continuous. We prove that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Al}} f$. Let $\mathcal{A} \in \Phi$ and let $K \subseteq \mathbb{N}$ be a statistically dense set. By Lemma 4.3 there exists an open cover $\mathcal{O}$ of $Y$ in the semi-uniform topology $\tau_\Phi$ such that $\mathcal{O} \in \Phi$ and $\mathcal{O} \prec \mathcal{A}$. Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st}} f$, for every $x \in X$ there exists a statistically dense set $N_x \subseteq \mathbb{N}$ such that for every $n \in N_x$ we have $f_n(x) \in \mathrm{St}(f(x), \mathcal{O})$. Let $N = \bigcup_{x\in X} N_x$. We consider the set

$$M = K \cap N = \{k_1, k_2, \ldots, k_n, \ldots\} \subseteq K.$$

Moreover, for each $n \in \mathbb{N}$ we set

$$O_n = \{x \in X : f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{O})\}.$$

Since $f_{k_n}$ and $f$ are continuous, by Lemma 4.4, $O_n$ is an open set. We prove that the family $\mathcal{U} = \{O_n : n \in \mathbb{N}\}$ is an open cover of $X$. Indeed, let $x \in X$. Then, there exists $n \in \mathbb{N}$ such that $k_n \in K \cap N_x$. Hence, $f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{O})$ and, therefore, $x \in O_n$. For every $n \in \mathbb{N}$ and for each $x \in O_n$ we have

$$f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{O}) \subseteq \mathrm{St}(f(x), \mathcal{A}).$$

Conversely, suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges Alexandroff to $f$ on $X$. Let $x_0 \in X$. We prove that $f$ is continuous at $x_0$. Let $A \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that $\mathrm{St}^3(f(x_0), \mathcal{B}) \subseteq \mathrm{St}(f(x_0), \mathcal{A})$. Let $K$ be an arbitrary statistically dense subset of $\mathbb{N}$. By assumption, there exist an open cover

$$\mathcal{U} = \{O_n : n \in \mathbb{N}\}$$

of $X$ and an infinite set

$$M = \{k_1, k_2, \ldots, k_n, \ldots\} \subseteq K$$

such that for every $n \in \mathbb{N}$ and for each $x \in O_n$ we have $f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{B})$. Let $n_0 \in \mathbb{N}$ such that $x_0 \in O_{n_0}$. Since the function $f_{k_{n_0}}$ is continuous at $x_0$, there exists an open neighbourhood $O_{x_0}$ of $x_0$ such that

$$f_{k_{n_0}}(x) \in \mathrm{St}(f_{k_{n_0}}(x_0), \mathcal{B}) \quad \text{for every } \ x \in O_{x_0}.$$

We set $H_{x_0} = O_{n_0} \cap O_{x_0}$. Then, the set $H_{x_0}$ is an open neighbourhood of $x_0$. We prove that $f(H_{x_0}) \subseteq \mathrm{St}(f(x_0), \mathcal{A})$. For every $x \in H_{x_0}$ we have

$$f_{k_{n_0}}(x_0) \in \mathrm{St}(f(x_0), \mathcal{B}), \quad f_{k_{n_0}}(x) \in \mathrm{St}(f(x), \mathcal{B}), \quad f_{k_{n_0}}(x) \in \mathrm{St}(f_{k_{n_0}}(x_0), \mathcal{B})$$

and, therefore,

$$f(x) \in \mathrm{St}^3(f(x_0), \mathcal{B}) \subseteq \mathrm{St}(f(x_0), \mathcal{A}).$$

We conclude that the function $f$ is continuous at $x_0$ completing the proof of the theorem. □

## 5. Almost uniform statistical convergence

The following definition is the statistical version of the almost uniform convergence of sequences of functions with values in semi-uniform spaces (see [4], [10]).

**Definition 5.1.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge almost uniformly to* $f$ *on* $X$ if for every $x \in X$ and for every $\mathcal{A} \in \Phi$ there exist a statistically dense set $K \subseteq \mathbb{N}$ and an open neighbourhood $O_x$ of $x$ such that for every $n \in K$ and for every $t \in O_x$ we have $f_n(t) \in \mathrm{St}(f(t), \mathcal{A})$. In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-au}} f$. We shall say that the sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges almost uniformly on* $X$ if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-au}} f$.

**Theorem 5.2.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$. If $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-au}} f$, then the function $f$ is continuous.*

Proof: Suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges almost uniformly to a function $f$. We prove that $f$ is continuous. Let $x \in X$ and $\mathcal{A} \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that

$$\mathrm{St}^3(f(x), \mathcal{B}) \subseteq \mathrm{St}(f(x), \mathcal{A}).$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-au}} f$, there exist a statistically dense set $K \subseteq \mathbb{N}$ and an open neighbourhood $O_x$ of $x$ such that for every $n \in K$ and for every $t \in O_x$ we have $f_n(t) \in \mathrm{St}(f(t), \mathcal{B})$. Let $n_0 \in K$. Then,

$$f_{n_0}(x) \in \mathrm{St}(f(x), \mathcal{B}).$$

Since the function $f_{n_0}$ is continuous at $x$, there exists an open neighbourhood $O'_x$ of $x$ such that $f_{n_0}(t) \in \mathrm{St}(f_{n_0}(x), \mathcal{B})$, for all $t \in O'_x$. We consider the set

$$H_x = O_x \cap O'_x.$$

Then, $H_x$ is an open neighbourhood of $x$. For every $t \in H_x$ we have

$$f_{n_0}(t) \in \mathrm{St}(f(t), \mathcal{B}).$$

Therefore,

$$\begin{aligned} f(t) \in \mathrm{St}(f_{n_0}(t), \mathcal{B}) &\subseteq \mathrm{St}(\mathrm{St}(f_{n_0}(x), \mathcal{B}), \mathcal{B}) \\ &\subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(f(x), \mathcal{B}), \mathcal{B}), \mathcal{B}) \\ &= \mathrm{St}^3(f(x), \mathcal{B}) \end{aligned}$$

and the continuity of $f$ is proved. □

**Definition 5.3.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$. The family $\{f_n : n \in \mathbb{N}\}$ is called *st-equicontinuous at a point $x_0$ of $X$* if for every $\mathcal{A} \in \Phi$ there exists a statistically dense set $K \subseteq \mathbb{N}$ and an open neighbourhood $O_{x_0}$ of $x_0$ such that

$$f_n(x) \in \mathrm{St}(f_n(x_0), \mathcal{A}) \quad \text{for all } \ n \in K \ \text{ and for all } \ x \in O_{x_0}.$$

The family $\{f_n : n \in \mathbb{N}\}$ is called *st-equicontinuous* if it is equicontinuous at each point of $X$.

**Theorem 5.4.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions of a topological space $X$ with values in a semi-uniform space $(Y, \Phi)$ such that the family $\{f_n : n \in \mathbb{N}\}$ is st-equicontinuous. If $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$, where $f$ is a continuous function, then the statistical convergence is almost uniform.*

Proof: Suppose that $(f_n)_{n\in\mathbb{N}}$ statistically converges pointwise to a continuous function $f$. Let $x \in X$ and $\mathcal{A} \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that

$$\mathrm{St}^3(f(x), \mathcal{B}) \subseteq \mathrm{St}(f(x), \mathcal{A}).$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$, there exists a statistically dense set $K_x \subseteq \mathbb{N}$ such that

$$f_n(x) \in \mathrm{St}(f(x), \mathcal{B}) \quad \text{for every } \ n \in K_x.$$

By the st-equicontinuity of the family $\{f_n : n \in \mathbb{N}\}$ at the point $x$, there exist a statistically dense set $K'_x$ and an open neighbourhood $O_x$ of $x$ such that

$$f_n(t) \in \mathrm{St}(f_n(x), \mathcal{B}) \quad \text{for all } \ n \in K'_x \ \text{ and for all } \ t \in O_x.$$

Since the function $f$ is continuous at $x$, there exists an open neighbourhood $O'_x$ of $x$ such that

$$f(t) \in \mathrm{St}(f(x), \mathcal{B}) \quad \text{for all } \ t \in O'_x.$$

We consider the set

$$H_x = O_x \cap O'_x.$$

Then, $H_x$ is an open neighbourhood of $x$. We set

$$K = K_x \cap K'_x.$$

For every $n \in K$ and for every $t \in H_x$ we have

$$\begin{aligned} f_n(t) \in \mathrm{St}(f_n(x), \mathcal{B}) &\subseteq \mathrm{St}(\mathrm{St}(f(x), \mathcal{B}), \mathcal{B}) \\ &\subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(f(t), \mathcal{B}), \mathcal{B}), \mathcal{B}) \\ &= \mathrm{St}^3(f(t), \mathcal{B}). \end{aligned}$$

Thus, the sequence $(f_n)_{n\in\mathbb{N}}$ statistically converges almost uniformly to $f$. □

## 6. Dini statistical convergence

In [7], a statistical version of the Dini convergence of sequences of functions between metric spaces was defined. An analogous definition can be given for sequences of functions with values in semi-uniform spaces.

**Definition 6.1.** The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *statistically converge Dini to $f$ on $X$* if $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$ and for every $\mathcal{A} \in \Phi$ and for every statistically dense set $K \subseteq \mathbb{N}$, there exists an infinite set $M = \{k_1, k_2, \ldots\} \subseteq K$ such that for each $x \in X$ and each $n \in \mathbb{N}$ we have $f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{A})$. In this case we write $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Di}} f$. We shall say that sequence $(f_n)_{n\in\mathbb{N}}$ *statistically converges Dini on $X$* if there is a function $f$ such that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Di}} f$.

The next theorem is a generalization of [7, Theorem 3.5] to sequences of functions with values in semi-uniform spaces.

**Theorem 6.2.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions of a topological space $X$ into a semi-uniform space $(Y, \Phi)$. If $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Di}} f$ and the functions $f_n$, $n \in \mathbb{N}$ are continuous, then the function $f$ is continuous.*

Proof: Suppose that $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Di}} f$ and let $x_0 \in X$. We prove that $f$ is continuous at $x_0$. Let $\mathcal{A} \in \Phi$. By Lemma 3.4 there exists $\mathcal{B} \in \Phi$ such that

$$\mathrm{St}^3(f(x_0), \mathcal{B}) \subseteq \mathrm{St}(f(x_0), \mathcal{A}).$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{st}} f$, there exists a statistically dense set $K \subseteq \mathbb{N}$ such that for every $n \in K$ we have

$$f_n(x_0) \in \mathrm{St}(f(x_0), \mathcal{B}). \tag{7}$$

Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-Di}} f$, there exists an infinite set $K_{\mathcal{A}} = \{k_1, k_2, \ldots\} \subseteq K$ such that for each $x \in X$ and each $n \in \mathbb{N}$ we have $f_{k_n}(x) \in \mathrm{St}(f(x), \mathcal{B})$. Let $n_0 \in \mathbb{N}$. Since $k_{n_0} \in K$, by relation (7) we have

$$f_{k_{n_0}}(x_0) \in \mathrm{St}(f(x_0), \mathcal{B}). \tag{8}$$

Since $f_{k_{n_0}}$ is continuous at $x_0$, there exists an open neighbourhood $O_{x_0}$ of $x_0$ such that $f_{k_{n_0}}(x) \in \mathrm{St}(f_{k_{n_0}}(x_0), \mathcal{B})$, for every $x \in O_{x_0}$. Let $x \in O_{x_0}$. Then,

$$f_{k_{n_0}}(x) \in \mathrm{St}(f_{k_{n_0}}(x_0), \mathcal{B}) \tag{9}$$

and

$$f_{k_{n_0}}(x) \in \mathrm{St}(f(x), \mathcal{B}). \tag{10}$$

Therefore, using successively the relations (8), (9), and (10), we have

$$\begin{aligned} f(x) \in \mathrm{St}(f_{k_{n_0}}(x), \mathcal{B}) &\subseteq \mathrm{St}(\mathrm{St}(f_{k_{n_0}}(x_0), \mathcal{B}), \mathcal{B}) \\ &\subseteq \mathrm{St}(\mathrm{St}(\mathrm{St}(f(x_0), \mathcal{B}), \mathcal{B}), \mathcal{B}) \\ &= \mathrm{St}^3(f(x_0), \mathcal{B}) \end{aligned}$$

and the continuity of $f$ is proved. □

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