Nil series from arbitrary functions in group theory

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Abstract. In an earlier paper distributors were defined as a measure of how close an arbitrary function between groups is to being a homomorphism. Distributors generalize commutators, hence we can use them to try to generalize anything defined in terms of commutators. In this paper we use this to define a generalization of nilpotent groups and explore its basic properties.

Keywords: finite group; nilpotent; arbitrary functions; nil-series; distributor

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1. Function conjugation

A notion of function conjugation was introduced by Hawthorn and Guo in [2]. If $f\colon G \to H$ is an arbitrary function between finite groups and $a \in G$, then we define a new function $f^a(x) = f(a)^{-1}f(ax)$ which we call *conjugate* of f by a. Clearly f is a group homomorphism if and only if $f^a = f$ for all $a \in G$. Consider for example the inverse function $(-1)\colon g \mapsto g^{-1}$. Then $(-1)^a(x) = ax^{-1}a^{-1} = [(-1)(x)]^a$, hence function conjugation generalizes the usual conjugate.

Note that $f^a(1) = 1$, hence conjugation maps the set of all functions onto the set of identity preserving ones. Furthermore function conjugation defines a group action of G on the set of identity preserving functions mapping from G to H since $f^1 = f$ and $(f^a)^b = f^{ab}$. Homomorphisms are precisely the functions invariant under this action. It follows that the number of function conjugates of a given identity preserving function f is the index of the stabilizer

$$\mathrm{Stab}_G(f) = \{x \in G\colon f(xg) = f(x)f(g)\ \forall\, g \in G\}.$$

Our function action was defined in terms of multiplication from the left. We could also have defined an action using multiplication from the right. We introduce temporary notation $f^{|x>}(g) = f^x(g)$ for the action from the left and define action from the right by $f^{<x|}(g) = f(gx^{-1})f(x^{-1})^{-1}$. Then $f^{<1|} = f$ and $(f^{<a|})^{<b|} = f^{<ab|}$ so this is indeed an action.

We might expect that the actions from the left and right are related. In fact they are equivalent. An intertwining map is given by $f^{(-1)}(x) = f(x^{-1})^{-1}$.

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Note that $(f^{(-1)})^{(-1)} = f$ hence $f \mapsto f^{(-1)}$ is of order two and in particular is a bijection. One can easily check that

$$(f^{<x|})^{(-1)} = (f^{(-1)})^{|x>},$$
$$(f^{|x>})^{(-1)} = (f^{(-1)})^{<x|},$$

so this defines an equivalence between the left and right actions.

Because the two actions are equivalent we will focus our attention on the left action and will revert to our initial less cumbersome notation. We can also use $((f^{(-1)})^x)^{(-1)}$ to refer to the right action if necessary.

Function conjugation is related to the notion of a planar function studied by P. Dembowski in [1] and the others in [3]. A function $f\colon G \to H$ is planar if the functions $f_u(x) = f(ux)f(x)^{-1}$ are bijections for all u which of course requires that G and H have the same order. Such a function generates a projective plane and is of interest for that reason.

In our notation $f_u(x) = f^{<x^{-1}|}(u)$. However whereas we are interested in this expression as a collection of functions of u, in planar function theory it is treated as collection of functions of x. The bijection condition in the definition of a planar function requires that all function conjugates of f be completely different in the sense that $f^a(x) = f^b(x)$ for some x if and only if $a = b$. Hence planar functions can be viewed as functions that are the opposite of homomorphisms since their conjugates are as different from each other as possible.

The stabilizer for the right action is

$$\mathrm{Stab}_G(f) = \{x \in G\colon f(gx) = f(g)f(x)\ \forall\, g \in G\}.$$

Our intertwining map tells us that the stabilizer under the right action of the function f is the same as the stabilizer under the left action of the function $f^{(-1)}$.

The function $(-1)\colon g \mapsto g^{-1}$ has the property that $(-1)^{(-1)} = (-1)$. Hence the stabilizers under the left and right actions of the function (-1) are the same and both are equal to the centre $Z(G)$. We can thus regard stabilizers of functions as generalizations of the centre. Since our aim is to generalize nilpotency these subgroups will be vitally important, you might even say central, to our discussion. Deciding how to name them is complicated by the fact that there are two of them and they need not be equal. For reasons that will become clear later we will call the stabilizer for the *right* action the *f-centre* $Z^f(G)$ of G so that

$$Z^f(G) = \{z \in G\colon f(gz) = f(g)f(z)\ \forall\, g \in G\}.$$

Hence the stabilizer $\mathrm{Stab}_G(f)$ for the more standard left action is $Z^{f^{(-1)}}$. The reasons for this apparently perverse choice will become clear later.

If $X \le G$ and $X \le Z^f(G)$ then we say that; X is an *f-central subgroup* of G; G *f-centralizes* X; and, f is an *X-centralizing function*; depending on which of X, G, or f is the focus of our attention.

We noted that since $(-1)^{(-1)} = (-1)$ the left and right centres for this function are equal. This will also be the case for any function f with the property that $f^{(-1)} = f$. We call such a function *inverse preserving* as it preserves the relationship of being inverse. This is an important and interesting class of functions. Inverse preserving functions are not difficult to directly construct. Furthermore any odd collection of identity preserving functions which is closed under the map $f \mapsto f^{(-1)}$ must contain an inverse preserving function.

If f is inverse preserving then in general f^a need not be inverse preserving. If all function conjugates of f are inverse preserving then we say that f is *strongly inverse preserving*. The inverse function is strongly inverse preserving as are all homomorphisms. This is an interesting class of functions worthy of further study.

2. Distributors

Distributors were also defined in [2]. They provide us with another measure of how closely an arbitrary function resembles a homomorphism.

If $f\colon G \to H$ is an arbitrary function between finite groups and $x, y \in G$ then we define the *f-distributor* $[x, y; f]$ of x and y to be

$$[x, y; f] = f(y)^{-1} f(x)^{-1} f(xy) = f(y)^{-1} f^x(y).$$

It follows that $f(xy) = f(x)f(y)[x, y; f]$.

Clearly $[x, y; f] = 1$ for all $x, y \in G$ if and only if f is a homomorphism and so the set of distributors provides a measure of how close f is to being a homomorphism. Distributors for the function $(-1)\colon g \mapsto g^{-1}$ are commutators

$$[x, y; (-1)] = yxy^{-1}x^{-1} = [y^{-1}, x^{-1}].$$

Hence distributors can be regarded as generalized commutators. They are also related to group cohomology. For example expanding $f(xyz)$ using distributors gives the *cocycle identity*

$$[y, z; f][x, yz; f] = [x, y; f]^{f(z)}[xy, z; f]. \tag{1}$$

Another useful elementary identity is the *product-conjugate identity*

$$[xy, z; f] = [x, z; f][y, z; f^x] \tag{2}$$

which can be easily checked by simply expanding both sides.

In the defining equation $f(xy) = f(x)f(y)[x, y; f]$ we chose to place the distributor on the right. We could equally well have chosen to place it on the left using the defining equation $f(xy) = [f; x, y]f(x)f(y)$ where the placement of the function on the left inside the brackets has been used to distinguish this from the standard definition. These other-handed distributors are related to the standard ones via the equations

$$[f; x, y] = [x, y; f]^{(f(x)f(y))},$$
$$[f; x, y]^{-1} = [y^{-1}, x^{-1}; f^{(-1)}].$$

If $X \leq G$ we define its *f-image* $f(X) \leq H$ by

$$f(X) = \langle f(x) \colon x \in X\rangle.$$

If $K \leq H$ and $K \leq f(X)$ then we will say that X is an *f-cover* of K. Note that the set of possible f-covers is generally not closed under intersection. The notions of f-image and f-cover relate subgroups of G and subgroups of H and are vitally important in what follows.

We also define the *closed f-image* $f^G(X) \leq H$ to be

$$f^G(X) = \langle f^g(x) \colon x \in X,\ g \in G\rangle.$$

Clearly $f^G(X) = \langle f^g(X) \colon g \in G\rangle$.

If $X, Y \leq G$ then we define their distributor $[X, Y; f] \leq H$ by

$$[X, Y; f] = \langle [x, y; f] \colon x \in X,\ y \in Y\rangle.$$

The f-center was defined earlier as $Z^f(G) = \{z \in G \colon f(gz) = f(g)f(z)\}$. We note that $[G, Z^f(G); f] = 1$ and $[G, X; f] = 1$ if and only if $X \leq Z^f(G)$. The other kind of center $Z^{f^{(-1)}}(G)$ can also be described in this way with $[Z^{f^{(-1)}}(G), G; f] = 1$ and $[X, G; f] = 1$ if and only if $X \leq Z^{f^{(-1)}}(G)$. These generalize a similar result for commutators and the ordinary center where $[X, G] = [G, X] = 1$ if and only if $X \leq Z(G)$.

In general anything which can be described in terms of commutators admits a generalization using distributors. For example consider normality. If $X \leq G$ we have $X \trianglelefteq G$ if and only if $[G, X] \leq X$. We can generalize this as follows. If $X \leq G$ we say X is *f-normal* in G if $[G, X; f] \leq f(X)$. Clearly the f-center is f-normal.

Also if X is f-normal and $Y \leq X$ with $[G, X; f] \leq f(Y)$ then clearly Y is also f-normal since $[G, Y; f] \leq [G, X; f] \leq f(Y)$. In particular every subgroup of the f-center is f-normal.

Similarly if $X, Y \leq G$ with $[Y, X; f] \leq f(X)$ we will say that X *is f-normalized by Y*.

In general there is no unique maximal subgroup Y which f-normalizes X, hence we cannot define something like an f-normalizer. To demonstrate this we give an example. Let $g \in G$, $g \neq 1$ and let $f \colon G \to \mathbb{Z}_2$ be the characteristic function to the cyclic group $\mathbb{Z}_2$ of order 2 defined (using additive notation) by

$$f(x) = \begin{cases} 0, & x \neq g, \\ 1, & x = g. \end{cases}$$

Then $[y,x;f]=0$ unless exactly one of the three elements y, x and yx is equal to g in which case $[y,x;f]=1$.

Now suppose $1\neq X\leq G$ and $g\notin X$. Then $f(X)=0$ and Y will f-normalize X if and only if $g\notin Y$ and $g\notin yX$. In general there is no unique maximal subgroup with these properties for a given $g\notin X$. Consider $G=A_5$, $g=(12)(34)$ and $X=\langle(123)\rangle$ then every Sylow 5-subgroup of A_5 f-normalizes X, but the group these generate is A_5 which clearly does not.

If $X,Y\leq G$ we say that Y *f-centralizes* X if $[Y,X;f]=1$. Once again while we can speak of one subgroup f-centralizing another, we cannot in general speak of the f-centralizer, a unique maximal subgroup which f-centralizes X. The previous example also suffices to show this.

In some special cases however a centralizer does exist. For example there is a unique maximal subgroup Y characterized by the property that $[Y,G;f]=1$, namely the group $Z^{f^{(-1)}}$. So G itself always has a f-centralizer. Furthermore for some functions (like the function (-1)) f-centralizers exist for every subgroup.

In [2] we proved

Proposition 2.1 ($[G,G;f]\trianglelefteq f(G)$)**.** *If $K\trianglelefteq f(G)$ and $\pi\colon f(G)\to f(G)/K$ is the natural projection map then πf is a homomorphism if and only if $[G,G,f]\leq K$.*

Note that πf is a homomorphism if and only if $\pi f(g)=\pi f^a(g)$ for all $a,g\in G$.
We need a generalization of this result. First a lemma.

Lemma 2.2. *If $f\colon G\to H$ is arbitrary, $X\leq G$ and $m\in G$, then*

$$[G,X;f]=[G,X;f^m].$$

Proof: The product-conjugate identity gives $[g,x;f^m]=[m,x;f]^{-1}[mg,x;f]$. Hence $[G,X;f^m]\subseteq[G,X;f]$ and the result follows by symmetry. $\square$

We can now state our generalization.

Proposition 2.3. *If $f\colon G\to H$ is arbitrary and $X\leq G$, then $[G,X;f]\trianglelefteq f^G(X)$. Furthermore if $K\trianglelefteq f^G(X)$ and $\pi\colon f^G(X)\to f^G(X)/K$ is the natural projection, then $\pi f(x)=\pi f^g(x)$ for all $x\in X$, $g\in G$ if and only if $[G,X;f]\leq K$.*

Proof: To show that $[G,X;f]\trianglelefteq f^G(X)$ it is enough to show that if $x,a\in X$ and $g,m\in G$ then $[g,x;f]^{f^m(a)}\in[G,X;f]$. The product-conjugate identity applied to $[m^{-1}g,x;f^m]$ gives $[g,x;f]=[m^{-1},x;f^m]^{-1}[m^{-1}g,x;f^m]$ and hence

$$[g,x;f]^{f^m(a)}=[m^{-1},x;f^m]^{-f^m(a)}[m^{-1}g,x;f^m]^{f^m(a)}.$$

This expression is generated by terms of the form $[k,x;f^m]^{f^m(a)}$ where $k\in G$ and it is therefore enough to show that each such term lies in $[G,X;f]$. But from the cocycle identity $[k,x;f^m]^{f^m(a)}=[x,a;f^m][k,xa;f^m][kx,a;f^m]^{-1}$ so these terms all lie in $[G,X;f^m]$, and $[G,X;f^m]=[G,X;f]$ by Lemma 2.2. Hence $[G,X;f]\trianglelefteq f^G(X)$ as claimed.

Let $K \trianglelefteq f^G(X)$ with natural projection $\pi\colon f^G(X) \to f^G(X)/K$. If $x \in X$ and $g \in G$ then $\pi f(x) = \pi f^g(x) \Leftrightarrow (\pi\left(f(x)\right)^{-1} f^g(x)\big) = 1 \Leftrightarrow [g, x; f] \in K$. So $\pi f(x) = \pi f^g(x)\ \forall\, x \in X, g \in G$ if and only if $[G, X; f] \leq K$ as claimed. $\square$

Corollary 2.4. *If $f\colon G \to H$ and $X \leq G$, then $f(X)[G, X; f] = f^G(X)$.*

PROOF: Since $[G, X; f] \trianglelefteq f^G(X)$ then $f(X)[G, X; f]$ is a semidirect product giving a well defined subgroup of $f^G(X)$. It is therefore enough to show all generating elements of $f^G(X)$ are in $f(X)[G, X; f]$. But $f^G(X)$ is generated by elements of the form $f^g(x)$ where $g \in G$ and $x \in X$. And $f^g(x) = f(x)[g, x; f] \in f(X)[G, X; f]$ so $f(X)[G, X; f] = f^G(X)$ as claimed. $\square$

Corollary 2.5. *If $X \leq G$ then X is f-normal if and only if $f^G(X) = f(X)$.*

PROOF: By definition X is f-normal if and only if $[G, X; f] \leq f(X)$. But $[G, X; f] \leq f(X) \Leftrightarrow f(X)[G, X; f] \leq f(X) \Leftrightarrow f^G(X) \leq f(X) \Leftrightarrow f^G(X) = f(X)$. $\square$

3. Nilmorphisms and f-nil groups

Consider an arbitrary $f\colon G \to H$. We say that f is a *nilmorphism* and also that G is an *f-nil group* if there is a chain of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$ with the property that $[G, G_i; f] \leq f(G_{i+1})$ for all i. We call such a chain an *f-central chain*. Note that all the groups G_i in an f-central chain are f-normal although of course they need not be normal in the ordinary sense.

This definition generalizes nilpotency as nilpotent groups are precisely (-1)-nil groups. We are interested in seeing to what extent the theory of nilpotent groups can be extended to f-nil groups. This of course will very much depend on the nature of the function f which must be defined on the group being considered. It is only meaningful to consider whether the specific group G which is the domain of f is f-nil, and for that reason it is usually not meaningful to consider f-nil groups as a class.

The exception to this is where f denotes a class of functions which is well-defined on every group. Examples include the power functions $(n)\colon x \mapsto x^n$ and the p-part function π_p which maps every element to its p-part where p is a prime.

We can generalize solubility in the same way. We say that f is a *solmorphism* and that G is an *f-sol group* if there is a chain of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$ with the property that $[G_i, G_i; f] \leq f(G_{i+1})$ for all i. We call such a chain an *f-chain*.

This paper will focus on nilmorphisms. Clearly every nilmorphism is a solmorphism and from f-nil follows f-sol.

We begin by looking at subgroups of f-nil groups and asking whether they are in some sense also f-nil. An obvious restriction is to subgroups which are f-normal. However even in this case the answer in general is no, and we begin with an example of this.

Example 3.1. Let $H \leq G$ and let $\{x_i\}$ be a set of coset representatives so that $\{x_iH\}$ is the set of cosets. Assume that $x_0 = 1$ so that 1 is the coset representative for the coset $H = 1H$. Then every $g \in G$ can be uniquely written in the form $g = x_ih$ for some x_i and $h \in H$. Define $f\colon G \to H$ by $f(g) = f(x_ih) = h$ so that f projects onto the H component. Then we claim that f is a nilmorphism, G is f-nil and $G \geq H \geq 1$ is an f-central series.

Proof: Note that if $g \in G$ and $h \in H$ then $f(h) = h$ and $f(gh) = f(g)h$. Clearly $[G, G; f] \leq H = f(H)$. And also $[g, h; f] = f(h)^{-1}f(g)^{-1}f(gh) = h^{-1}f(g)^{-1}f(g)h = 1$ so $[G, H; f] = 1 = f(1)$. Hence $G \geq H \geq 1$ is f-central, G is f-nil and f is a nilmorphism as claimed. □

Now consider the particular case where $G = S_3 = \langle x, y\colon x^2 = y^3 = 1,\ xy = y^2x\rangle$ and $H = \langle y\rangle$. Then $\{1, x\}$ is a set of coset representatives. Define $f\colon G \to \langle y\rangle$ via the method in Example 3.1. Then $G \geq \langle y\rangle \geq 1$ is f-central, G is f-nil and f is a nilmorphism.

Now consider $M = \langle xy\rangle$ which is a cyclic subgroup of order 2. Then $[G, M; f] = [M, M; f] = \langle y\rangle = f(M)$ so M is an f-normal subgroup of G. However clearly M is not f-nil since the only proper subgroup of M is 1 and $f(1) = 1$ does not contain $[M, M; f] = \langle y\rangle$. Hence there is no f-central series for M.

So in general the f-nil property is not inherited by subgroups. But for some functions f, it will be so inherited.

We say that a function $f\colon G \to H$ is *intersection preserving* if for subgroups $X, Y \leq G$ we have $f(X \cap Y) = f(X) \cap f(Y)$. In general we only know that $f(X \cap Y) \leq f(X) \cap f(Y)$.

Note that the function in the previous example is not intersection preserving since $f(\langle y\rangle) \cap f(\langle xy\rangle) = \langle y\rangle$ while $f(\langle y\rangle \cap \langle xy\rangle) = f(1) = 1$.

Proposition 3.2. *If $f\colon G \to H$ is intersection preserving, G is f-nil and $M \leq G$ then the restriction $f|_M\colon M \to H$ of f to M is also intersection preserving and M is $f|_M$-nil.*

Proof: Clearly $f|_M$ is intersection preserving.

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ be an f-central chain. It is enough to show that $M = M \cap G_0 \geq M \cap G_1 \geq \ldots \geq M \cap G_n = 1$ is an $f|_M$-central chain.

But $[M, M \cap G_i; f|_M] \leq [M, M; f] \leq f(M)$ and also $[M, M \cap G_i; f|_M] \leq [G, G_i; f] \leq f(G_{i+1})$. Hence $[M, M \cap G_i; f|_M] \leq f(M) \cap f(G_{i+1}) = f(M \cap G_{i+1})$ and the result follows. □

We will write f instead of $f|_M$ where this does not cause confusion, and will therefore describe this result by saying that subgroups of f-nil groups are f-nil if f is intersection preserving.

If we restrict our attention to only f-normal subgroups, then since all subgroups in a central chain are f-normal, we really only need the f-intersection property to hold for f-normal subgroups. We leave it to the reader to construct the appropriate proposition which will have essentially the identical proof.

The intersection preserving condition in these results is sufficient but not necessary. Intersection preserving is actually a very strong property for a function between finite groups. Even homomorphisms are not in general intersection preserving as can be seen by considering the quotient map $\pi\colon S_3 \to S_3/A_3$ and any two Sylow 2-subgroups of S_3. Of course if $f\colon G \to H$ is a homomorphism then all distributors are trivial so every subgroup is f-nil.

This failure of the f-nil property to be inherited by subgroups, even f-normal ones, has important consequences. It means for example that there is no f-equivalent of the descending central series. The descending central series for nilpotent groups is a central series constructed via the greedy algorithm starting at the top with $L_0 = G$ and defining $L_{i+1} = [G, L_i]$.

A similar construction via the greedy algorithm in the f-nil case would require us to define L_{i+1} to be some minimal f-cover of $[G, L_i; f]$. Unfortunately this is not uniquely determined and a choice is required to pick a minimal f-cover. In our example both $\langle y\rangle$ and $\langle xy\rangle$ are minimal f-covers of $[G, G; f]$ and therefore are possible first steps in a greedy algorithm. However only the $\langle y\rangle$ choice can be completed to give an f-central series.

If G is f-nil and $\{G_i\}$ is f-central then we know that at each step G_{i+1} is an f-cover of $[G, G_i; f]$ and therefore G_{i+1} must contain at least one minimal f-cover M_{i+1} of this group. If we replace G_{i+1} with M_{i+1} the resulting series will still be f-central. By applying this idea repeatedly (and systematically) we can obtain an f-central series where G_{i+1} is a *minimal* f-cover of $[G, G_i; f]$ at each step. In other words at least one route through the greedy algorithm will lead us to a central series. Unfortunately a wrong choice at any step may lead us to a dead end.

The greedy algorithm will stop if we arrive at an f-normal subgroup $M \le G$ where M itself is a minimal f-cover of $[G, M; f]$. We define a subgroup M of G to be *f-problematic* if M is f-normal and is a minimal f-cover of $[G, M; f]$. The subgroup $\langle xy\rangle$ in our example is f-problematic. The intersection property ensures that f-problematic subgroups cannot exist.

To see this assume f is intersection preserving and G is f-nil with f-central series $G = G_0 \ge G_1 \ge \ldots G_n = 1$. Assume $M \le G$ is f-problematic. Choose i so that $M \le G_i$ but $M \not\le G_{i+1}$. Then $[G, M; f] \le f(M) \cap f(G_{i+1}) = f(M \cap G_{i+1})$ by the intersection property. But this contradicts the f-problematic property of M since $M \cap G_{i+1}$ is a smaller cover.

If f is intersection preserving then because there are no f-problematic subgroups, the greedy algorithm will always give an f-central series no matter what choices we make at each step.

Proposition 3.3. *If G has no f-problematic subgroups then G is f-nil by the greedy algorithm, and furthermore so is every f-normal subgroup of G.*

Proof: If there are no f-problematic subgroups then the greedy algorithm will not halt until the trivial subgroup is reached. Hence it will generate an f-central series and so G is f-nil. Furthermore we can start up the greedy algorithm at any f-normal subgroup $X \le G$ thereby generating a sequence $X = X_0 \ge X_1 \ge \ldots \ge$

$X_n = 1$ with $[G, X_i; f] \leq f(X_{i+1})$. But this implies $[X, X_i; f] \leq f(X_{i+1})$ and so X is f-nil. □

4. Ascending series and the f-boundary

We next look at whether ascending f-central series exist for f-nil groups, that is whether f-central series can be constructed via the greedy algorithm working upwards from the trivial subgroup.

To facilitate the discussion in this section consider an f-central series numbered in ascending fashion with $1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$ and where therefore $[G, G_{i+1}; f] \leq f(G_i)$. If each step in such a series is maximal with this property we will say that the series is ascending. We begin by looking at the group G_1 in the first step.

Since $[G, G_1; f] = 1$ then $[g, a; f] = 1$ and hence $f(ga) = f(g)f(a)$ for all $g \in G$ and $a \in G_1$. We have met this property before. In particular the subgroup $Z^f(G)$ consists of precisely the elements a that satisfy this property. Hence $G_1 \leq Z^f(G)$, and since $Z^f(G)$ is maximal with the property that $[G, Z^f(G); f] = 1$, if our series is to be ascending we must have $G_1 = Z^f(G)$. Thus in contrast to the descending case the greedy algorithm involves no choice in the first step, which is promising.

Let us now consider a general step $G_i \leq G_{i+1}$ and what it would mean for this step to be maximal. By the f-central property the elements of G_{i+1} all have the property that $[g, a; f] \in f(G_i)$ for all $a \in G_{i+1}$ and $g \in G$. This leads us to make the following definition.

If $f\colon G \to H$ is an arbitrary function between finite groups and $M \leq G$ then the *f-boundary* $B^f(M)$ of M is the set

$$B^f(M) = \{a \in G\colon [g, a; f] \in f(M) \text{ for all } g \in G \text{ and } f(a) \in N_H(f(M))\}.$$

Note that the f-center is the f-boundary of 1. The second part of this definition is needed to ensure that $B^f(M)$ is a subgroup. In particular it allows us to show the following.

Proposition 4.1. *If M is f-normal then $M \leq B^f(M) \leq G$ and $B^f(M)$ is also f-normal.*

Proof: Suppose $a, b \in B^f(M)$. We wish to show that $ab \in B^f(M)$.

Let $g \in G$. Then the cocycle identity gives

$$[g, ab; f] = [a, b; f]^{-1}[g, a; f]^{f(b)}[ga, b; f] \tag{3}$$

and the three terms on the right are all in $f(M)$ using the definition of the f-boundary and the fact that M is f-normal.

Also $f(ab) = f(a)f(b)[a, b; f]$ where $f(a), f(b) \in N_H(f(M))$ and $[a, b; f] \in f(M) \leq N_H(f(M))$. Hence $f(ab) \in N_H(f(M))$.

We can conclude that $ab \in B^f(M)$ and hence the f-boundary is closed under product. For finite groups this is sufficient to show that $B^f(M) \leq G$.

Checking the definition and using f-normality it is straightforward to verify that $M \leq B^f(M)$. We thus have $[G, B^f(M); f] \leq f(M) \leq f(B^f(M))$ and so $B^f(M)$ is also f-normal. □

We can apply this result repeatedly to construct an f-central series by setting $G_0 = 1$ and defining $G_{i+1} = B^f(G_i)$. We call this the *ascending boundary series*, and if it ascends all the way to G then G will be f-nil.

What is less clear however is whether, when G is f-nil, the ascending boundary series must always ascend to the top. To understand the issue consider an f-central series $G = G_n \geq G_{n-1} \geq \ldots \geq G_1 \geq G_0 = 1$ for the f-nil group G. Let us look at what this tells us about boundaries. We have $[G, G_{i+1}; f] \leq G_i$ which suggests that we might have $G_{i+1} \leq B^f(G_i)$. If this were true it would force the ascending boundary series to reach the top. However to prove this we would also need $f(G_{i+1}) \leq N_H(f(G_i))$, and it is not at all clear why this should be true.

We do not however know of a counterexample, so the question remains open.

Nilpotent groups are (-1)-nil groups; and (-1)-normal subgroups are simply normal. However in general f-normal and normal are not the same. The awkwardness with the ascending boundary series arises because of this clash between the two types of normality. The f-normal property is the one linked most closely to the function f, but ordinary normality remains vitally important for considerations relating to subgroups and quotients. Hence we cannot simply replace every mention of normality in the theory of nilpotency with f-normality and expect everything to work.

The ascending central series for nilpotent groups for example is usually defined in terms of quotients and thus makes heavy use of this coincidence between (-1)-normality and normality. Since we cannot expect in general to be able to take the quotient of an f-normal subgroup we cannot use this approach in the general case.

The question of how f-nil groups behave under quotient is however worthy of study. In particular we would like to answer such basic questions as whether the quotient of an f-nil group must in some sense be f-nil.

5. Quotients

If $f\colon G \to H$ and $N \trianglelefteq G$ then in the case that G is f-nil we would like to know what we can conclude about G/N. To consider this question we first need to obtain from f a function defined on G/N. Clearly $f(xN) = f(x)$ need not be a well defined function on the quotient group, however we may hope to make it well defined by combining it with a projection map $\pi\colon H \to H/K$ for some suitable $K \trianglelefteq H$.

In order that πf gives a well defined function on G/N we need $\pi f(g) = \pi f(gn)$ for all $g \in G$ and $n \in N$. It follows that we must have $f(g)^{-1} f(gn) = f^g(n) \in K$, and so $f^G(N) \leq K$. As it is not necessarily the case that $f^G(N) \trianglelefteq H$ we will also need to close it under conjugation in H. Hence we choose $K = (f^G(N))^H$, the subgroup generated by all conjugates of $f^G(N)$ in H. If π denotes the quotient by this subgroup, then $\overline{f}\colon gN \mapsto \pi f(g)$ gives a well defined function from G/N to H/K.

Proposition 5.1. *With the notation above, if G is f-nil then G/N is $\overline{f}$-nil.*

Proof: Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ be an f-central chain. We claim that $G/N = G_0N/N \geq G_1N/N \geq \ldots \geq G_nN/N$ is an $\overline{f}$-central chain.

It is enough to show that $[G/N, G_iN/N; \overline{f}] \leq \overline{f}(G_{i+1}N/N)$. But if $g \in G$ and $x \in G_i$ then

$$\begin{aligned}[gN, xN; \overline{f}] &= \left(\overline{f}(xN)\right)^{-1}\left(\overline{f}(gN)\right)^{-1}\overline{f}(gN.xN)\\ &= \pi\left(f(x)^{-1}f(g)^{-1}f(gx)\right)\\ &= \pi[g, x; f] \in \pi\left(f(G_{i+1})\right)\end{aligned}$$

and $\pi(f(G_{i+1})) = \langle \pi f(x)\colon x \in G_{i+1}\rangle = \langle \overline{f}(xN)\colon x \in G_{i+1}\rangle = \overline{f}(G_{i+1}N/N)$. □

Note that if $f(G) \lneqq H$ then first restricting f to map into $f(G)$ may strengthen the result by allowing a smaller normal subgroup K. It is perhaps rather surprising that the f-nil property is inherited by quotients when it is not inherited by subgroups.

A limited converse is also of interest.

Proposition 5.2. *Let $f\colon G \to H$ be arbitrary and let $N \trianglelefteq G$ with $f(N) \trianglelefteq f(G)$. Assume $f(G) = H$ by restricting the range if necessary. Let $K = \left(f^G(N)\right)^H \trianglelefteq H$ and define $\overline{f}\colon G/N \to H/K$ by composition of f with the quotient into H/K as described above.*

If G/N is $\overline{f}$-nil and N is f-central then G is f-nil.

Proof: Since N is f-central then $[G, N; f] = 1$ and $f^G(N) = f(N)$. Also $f(N) \trianglelefteq f(G) = H$ by assumption. Hence $K = f(N)$.

Let $G/N = G_0/N \geq G_1/N \geq \ldots \geq G_n/N = N/N$ be an $\overline{f}$-central chain for G/N. We claim that $G = G_0 \geq G_1 \geq \ldots \geq G_n = N \geq 1$ is an f-central chain and therefore G is f-nil.

We already noted that $[G, N; f] = 1$ so it is enough to show that $[G, G_i; f] \leq f(G_{i+1})$. Let $g \in G$ and $x \in G_i$. Then $[gN, xN; \overline{f}] \in \overline{f}(G_{i+1})$ and hence $[g, x; f] \in f(G_{i+1})K$. But $K = f(N) \leq f(G_{i+1})$ so $[g, x; f] \in f(G_{i+1})$. It follows that $[G, G_i; f] \leq f(G_{i+1})$ as claimed. □

This generalizes the result that a group is nilpotent if and only if $G/Z(G)$ is nilpotent. However our generalization is much weaker. Whereas every nilpotent group admits a decomposition into a center and a nilpotent quotient, f-nil groups cannot always be decomposed in this way.

One problem is that the f-center $Z^f(G)$ need not be normal in G so we cannot always take a quotient. And another problem is that even when $Z^f(G)$ is normal, unless we also have $f(Z^f(G)) \trianglelefteq f(G)$ the kernel K will be too big to allow us to make inferences about f from the properties of $\overline{f}$. The two assumptions in the proposition, that $N \trianglelefteq G$ and that $f(N) \trianglelefteq f(G)$ are required to deal with these issues.

6. Nilmorphisms as a set of functions

As well as studying f-nil groups as a generalization of nilpotency these definitions also open up the study of nilmorphisms as a generalization of homomorphism. The most obvious question in this area is whether the composition of nilmorphisms must be a nilmorphism and consequently whether the autonilmorphisms of G form a group.

Let us look at the first of these questions, namely whether the composition of nilmorphisms is a nilmorphism. Consider a nilmorphism $f\colon G \to H$. Let $\iota\colon K \to G$ be the injection homomorphism for some subgroup $K \leq G$. Then ι is a nilmorphism since homomorphisms are nilmorphisms. In this situation the composition ιf will be a nilmorphism if and only if K is f-nil as a subgroup of G.

So if the composition of nilmorphisms was always a nilmorphism we would be able to prove that subgroups of f-nil groups were always f-nil; something which we know to be false because we have constructed a counterexample. We conclude that the composition of two nilmorphisms need not be a nilmorphism.

The fact that nilmorphisms are not closed under composition suggests that we might try extending the definition by closing the collection of nilmorphisms under composition.

Definition 6.1. A *subnilmorphism*, also alled a composite nilmorphism, is a function which can be written as a composition of nilmorphisms.

The collection of finite groups and composite nilmorphisms forms a category. However we pay a price for this in terms of computability. Determining whether a particular function is a subnilmorphism requires us to consider chains of functions between finite groups whose composition gives us that function. Since this involves arbitrary embeddings there are an infinite number of such chains. We therefore cannot be assured of a finite halting algorithm for determining whether a particular map between groups is a subnilmorphism.

If however we restrict our attention only to endomorphisms of G then things are a little bit nicer.

Definition 6.2. An *autosubnilmorphism* is a composition of invertible nilmorphisms from G to G.

Autosubnilmorphisms are composite nilmorphisms, but the converse may not be true. As autosubnilmorphisms of a finite group G are constructed within the finite set of invertible functions from G to G there are no issues of computability in this definition. The invertible autosubnilmorphisms from G to G form a group which we call the *autosubnilmorphism group*. It contains the automorphism group of G and is contained within the group of invertible functions from G to G.

7. Conclusion

The generalization of nilpotency to f-nil groups defined in this paper seems like a fruitful and potentially useful one. While this paper has concentrated on the basic questions which might be described as establishing a general theory of nilmorphisms and f-nil groups, applications are most likely to arise with respect to a specific function f, or a specific class of functions chosen to suit a particular group or context. Nilpotency is the specific case where $f = (-1)$. It is reasonable to hope that other choices of f might also prove useful.

References

[1] Dembowski P., *Finite Geometries*, Springer, Berlin, 1968.
[2] Hawthorn I., Guo Y., *Arbitrary functions in group theory*, New Zealand J. Math. **45** (2015), 1–9.
[3] Pott A., *Finite Geometry and Character Theory*, Springer, Berlin, 1995.

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