Some versions of second countability of metric spaces in ZF and their role to compactness

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Abstract. In the realm of metric spaces we show in ZF that:

(i) A metric space is compact if and only if it is countably compact and for every $\varepsilon > 0$, every cover by open balls of radius ε has a countable subcover.

(ii) Every second countable metric space has a countable base consisting of open balls if and only if the axiom of countable choice restricted to subsets of $\mathbb{R}$ holds true.

(iii) A countably compact metric space is separable if and only if it is second countable.

Keywords: axiom of choice; compact space; countably compact space; totally bounded space; Lindelöf space; separable space, second countable metric space

Classification: 54E35, 54E45

1. Notation and terminology

Let $\mathbf{X} = (X, d)$ be a metric space and $\mathcal{U}$ be an open cover of $\mathbf{X}$. We say that $\mathcal{U}$ has a *Lebesgue number* $\delta > 0$ if for every $A \subseteq X$ with diameter $\delta(A) < \delta$ there exists $U \in \mathcal{U}$ with $A \subseteq U$.

Given $\varepsilon > 0$, a subset D of X is called ε-*dense* if for every $x \in X$, $B(x, \varepsilon) \cap D \neq \emptyset$. A finite ε-dense set D of $\mathbf{X}$ is called ε-*net*.

The space $\mathbf{X}$ is said to be *almost separable*, AS for abbreviation if for every $\varepsilon > 0$ there is a countable ε-subset D of X.

The space $\mathbf{X}$ is *quasi separable*, QS for abbreviation (or ω-QS for ω-*quasi separable*) if $\mathbf{X}$ has a dense subset which is expressible as a countable union of finite (or at most countable, respectively) sets.

The space $\mathbf{X}$ is *quasi second countable*, QSC for abbreviation (or ω-QSC for ω-*quasi second countable*) if $\mathbf{X}$ has a base $\mathcal{B}$ which can be written as a countable union of finite (or countably infinite, respectively) sets.

The space $\mathbf{X}$ is called *almost second countable*, ASC for abbreviation if for every $n \in \mathbb{N}$, there is a countable family $\mathcal{B}_n$ of open balls of $\mathbf{X}$ such that for every open set O in $\mathbf{X}$, for every $x \in O$ with $d(x, O^c) \geq 1/n$ there is a $B \in \mathcal{B}_n$ with $x \in B \subseteq O$. The family $\mathcal{B}_n$ is called $1/n$-*almost base* of $\mathbf{X}$.

DOI 10.14712/1213-7243.2015.229

The space $\mathbf{X}$ is said to be *strongly almost second countable*, SASC for abbreviation if and only if it admits a sequence $(\mathcal{B}_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is a $1/n$-almost base for $\mathbf{X}$. Clearly, $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a base for $\mathbf{X}$.

The space $\mathbf{X}$ is called *almost Lindelöf*, AL for abbreviation if for every $\varepsilon > 0$, every open cover of $\mathbf{X}$ consisting of open balls of radius ε has a countable subcover.

The space $\mathbf{X}$ is said to be *preLindelöf* if and only if for every $\varepsilon > 0$, $\mathbf{X}$ can be covered by countably many open discs of radius ε.

We call attention to the fact that for a metric space $\mathbf{X}$ to be preLindelöf, it is enough that for every given $\varepsilon > 0$, the cover of all open balls of $\mathbf{X}$ of radius ε has a countable subcover. On the other hand, in order for $\mathbf{X}$ to be almost Lindelöf we require that for every $\varepsilon > 0$, every open cover of $\mathbf{X}$ consisting of open balls of radius ε has a countable subcover. Likewise, in order for $\mathbf{X}$ to be almost second countable it is enough that for every $n \in \mathbb{N}$, $\mathbf{X}$ has a $1/n$-almost base whereas, for $\mathbf{X}$ to be strongly almost second countable, we require, in addition, that the family $\mathcal{A}_n = \{\mathcal{B}$ is a $1/n$-almost base of $\mathbf{X}\}$, $n \in \mathbb{N}$ has a choice set.

The rest of the topological notions used in this paper are standard and can be found in any textbook of general topology such as [13].

An infinite set X is said to be:

- *Dedekind-infinite*, denoted by $\mathrm{DI}(X)$, if X contains a countably infinite subset. Otherwise is said to be *Dedekind-finite*.
- *Weakly Dedekind-infinite*, denoted by $\mathrm{WDI}(X)$, if $\mathcal{P}(X)$ contains a countably infinite set. Otherwise is said to be *weakly Dedekind-finite*.

By universal quantifying over X, $\mathrm{DI}(X)$ gives rise to the choice principle IDI for all X (X infinite $\to \mathrm{DI}(X)$) that is, "every infinite set is Dedekind-infinite". Similarly one defines IWDI.

CAC will denote the countable axiom of choice, $\mathrm{CAC_{fin}}$ (or $\mathrm{CAC}(\mathbb{R})$, $\mathrm{CAC}_\omega(\mathbb{R})$) will stand for the restriction of CAC to finite sets (or subsets of $\mathbb{R}$, countable subsets of $\mathbb{R}$, respectively). Finally, CMC_ω will stand for the proposition: Every countable family $\mathcal{A} = \{A_i : i \in \omega\}$ of countable sets has a multiple choice set, i.e., a family $\mathcal{B} = \{B_i : i \in \omega\}$ of nonempty finite sets such that for all $i \in \omega$, $B_i \subseteq A_i$.

2. Introduction and some preliminary and known results

Ordinarily topology is dealt with in the setting of ZFC, i.e., Zermelo-Fraenkel set theory including AC, the axiom of choice. Although AC is neither evidently true nor evidently false, this adherence to AC seems to be based on a general belief that adoption of AC enables topologists to prove more and better theorems. Aside from the trivial observation that no theorem T in ZFC is lost in ZF (Zermelo-Fraenkel set theory without AC), — it simply turns into the implication AC $\to$ T which often enough can be

even improved to an equivalence WC $\leftrightarrow$ T for a suitable weak form WC of AC.
Horst Herrlich[1]

In this paper, the intended context for reasoning and statements of theorems will be ZF unless otherwise noted. In order to stress that a result is proved in ZF (or ZF+CAC) we shall write in the beginning of the statements of the theorems and propositions (ZF) (or (ZF + CAC), respectively).

There are several forms of compactness of metric spaces which are equivalent to Heine-Borel compactness (open covers have finite subcovers) in ZFC but not in ZF. The following theorem from [11] gives some additional properties of metric spaces under which certain weak forms of compactness of metric spaces are equivalent to Heine-Borel compactness.

Theorem 1 (ZF [11]). (i) *Every separable, complete and totally bounded metric space is compact.*
(ii) *Every quasi separable, limit point compact metric space is compact.*
(iii) *Every ω-quasi separable, countably compact metric space is compact.*

Each of the following is a potential property of metric spaces under which a countably compact metric space is compact: AL, 2C (abbreviates second countability), ASC, SASC, QSC, ω-QSC, **S**, QS and ω-QS. It is easy to see that in ZF + CAC each of the latter properties is equivalent to separability. The forthcoming Table 1 shows that they are all distinct in ZF.

Since the axioms of countability play a prominent role in the theory of metric spaces we believe that studying their set theoretic strength as well as the interrelations between them in the absence of AC is important so that we may conceive better our limitations without AC within this part of topology. As an illuminating example we note that in contrast to second countable metric spaces a quasi second countable (or quasi separable) metric space need not have size less than or equal to $|\mathbb{R}|$. Indeed, if $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is a family of finite nonempty sets without a choice set then $X = \bigcup \mathcal{A}$ endowed with the discrete metric d is an example of a quasi second countable (or quasi separable) metric space which fails to have size less than or equal to $|\mathbb{R}|$ or greater than or equal to $|\mathbb{R}|$ as in the opposite case $\{\min(A_n) : n \in \mathbb{N}\}$ would be a choice set for $\mathcal{A}$. Hence, the statement: "Every quasi second countable (or quasi separable) metric space has size less than or equal to $|\mathbb{R}|$" implies $\mathrm{CAC}_{\mathrm{fin}}$ a statement which is not provable in ZF.

The following ZF implications

$$\mathrm{S} \to 2\mathrm{C}, \quad \mathrm{S} \to \mathrm{QS}, \quad \mathrm{S} \to \omega\text{-QS}, \quad 2\mathrm{C} \to \mathrm{QSC}, \quad 2\mathrm{C} \to \omega\text{-QSC},$$
$$\mathrm{QS} \to \mathrm{QSC}, \quad \omega\text{-QS} \to \omega\text{-QSC}, \quad \mathrm{SASC} \to \mathrm{ASC}$$

are straightforward and are left as a warm up exercise for the reader.

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[1] See [5].

Remark 1. We would like to remark here that if for some property $p \in \{\mathrm{AL}, 2\mathrm{C}, \mathrm{ASC}, \mathrm{SASC}, \mathrm{QSC}, \omega\text{-QSC}, \mathrm{S}, \mathrm{QS}, \omega\text{-QS}\}$,

$$\begin{array}{rll} & \mathrm{QS} \nrightarrow p & (\text{or } \omega\text{-QS} \nrightarrow p, \quad \mathrm{SASC} \nrightarrow p) \\ \text{then} & \mathrm{QSC} \nrightarrow p & (\text{or } \omega-\mathrm{QSC} \nrightarrow p, \quad \mathrm{ASC} \nrightarrow p, \text{ respectively}). \end{array}$$

and,

$$\begin{array}{rll} \text{if} & p \nrightarrow \mathrm{QSC} & (\text{or } p \nrightarrow \omega\text{-QSC}, \quad p \nrightarrow \mathrm{ASC}) \\ \text{then} & p \nrightarrow \mathrm{QS} & (\text{or } p \nrightarrow \omega\text{-QS}, \quad p \nrightarrow \mathrm{SASC}). \end{array}$$

We recall that in the basic Cohen model $\mathcal{M}1$ in [8] the set A of all added Cohen reals is a dense Dedekind finite subset of $\mathbb{R}$. We observe that:

1) Since $\mathbb{R}$ is second countable, the restriction of any countable base of $\mathbb{R}$ to A is a countable base for the subspace $\mathbf{A}$ of $\mathbb{R}$. Hence, in $\mathcal{M}1$, $\mathbf{A}$ has all the properties listed in $B = \{2\mathrm{C}, \mathrm{QSC}, \omega\text{-QSC}\}$. Since A is Dedekind finite, it follows that A is not quasi separable, ω-quasi separable, almost second countable and strongly almost second countable. (If $\mathcal{B}$ is a countable $1/n$-almost base for $\mathbf{A}$ for some $n \in \mathbb{N}$, then since A is dense in $\mathbb{R}$, every ball in $\mathcal{B}$ has a unique center. Hence, A has a countably infinite subset contradicting the fact that it is Dedekind finite). Furthermore, the open cover $\mathcal{U} = \{(a - 1/2, a + 1/2) \cap A \colon a \in A\}$ of $\mathbf{A}$ has no countable subcover. Indeed, if $\mathcal{V} = \{V_n \colon n \in \mathbb{N}\}$ is a countable subcover of $\mathcal{U}$ then $\{a_n \colon n \in \mathbb{N}\}$ where for every $n \in \mathbb{N}$, a_n is the center of the ball V_n (each V_n has a unique center) is a countably infinite subset of A contradicting the fact that A is Dedekind finite. Hence, $\mathbf{A}$ is not almost Lindelöf. Thus, all non implications $p \nrightarrow q$, $p \in B$, $q \in \{\mathrm{AL}, \mathrm{ASC}, \mathrm{SASC}, \mathrm{QS}, \omega\text{-QS}\}$ are consistent with ZF.

2) In $\mathcal{M}1$, the subspace $\mathbf{X}$, $X = A \cup \mathbb{Q}$ of $\mathbb{R}$ is separable hence, quasi separable and ω-quasi separable. In addition, $\mathbf{X}$ is strongly almost second countable (for every $n \in \mathbb{N}$, $\mathcal{B}_n = \{B(q, 1/m) \colon m > n,\ q \in \mathbb{Q}\}$ is easily seen to be a $1/n$-almost base for $\mathbf{X}$). However, $\mathbf{X}$ is not almost Lindelöf as the open cover $\mathcal{U} = \{(a - 1/2, a + 1/2) \colon a \in A\}$ of $\mathbf{X}$ has no countable subcover. Hence, the non implications $\mathrm{QS} \nrightarrow \mathrm{AL}$, $\omega\text{-QS} \nrightarrow \mathrm{AL}$, $\mathrm{ASC} \nrightarrow \mathrm{AL}$ and $\mathrm{SASC} \nrightarrow \mathrm{AL}$ are consistent with ZF.

3) Subspaces of separable (or quasi separable, ω-quasi separable) metric spaces need not be separable (or quasi separable, ω-quasi separable, respectively). (In the model $\mathcal{M}1$, $\mathbb{R}$ is separable hence, quasi separable and ω-quasi separable also but its subspace $\mathbf{A}$ has none of the latter properties.) The interested reader can easily verify that "every subspace of a separable metric space is separable" is equivalent to $\mathrm{CAC}(\mathbb{R})$, "every subspace of an ω-quasi separable metric space is ω-quasi separable" implies $\mathrm{IDI}(\mathbb{R})$ and, "every subspace of a quasi separable metric space is quasi separable" implies $\mathrm{CAC}(\mathbb{R})$.

Since $\mathrm{S} \rightarrow 2\mathrm{C}$ and $2\mathrm{C} \rightarrow \mathrm{S}$ if and only if $\mathrm{CAC}(\mathbb{R})$, see e.g. [4], we shall be concentrated mainly on the following seven properties: AL, ASC, SASC, QSC,

ω-QSC, QS and ω-QS. The following Table 1 records the ZF implications\non implications between these properties obtained in this project.

	AL	ASC	SASC	QSC	ω-QSC	QS	ω-QS
AL	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
ASC	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
SASC	$\not\rightarrow$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
QSC	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
ω-QSC	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
QS	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$
ω-QS	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\rightarrow$

Table 1.

The non implications in Table 1 are obtained by using the forthcoming Proposition 6, Theorems 9, 10 and 11 respectively and Remark 1.

Theorem 2 ([6]). *In* ZF *the following conditions are equivalent:*

(i) $\mathbb{N}$ *is a Lindelöf space;*
(ii) *every second countable metric space is Lindelöf;*
(iii) CAC*(*$\mathbb{R}$*).*

Clearly in ZF, a second countable (or Lindelöf) metric space is countably compact if and only if it is compact. In view of Theorem 2, and Example 1.6 from [1] (a ZF example of Lindelöf metric space which fails to be second countable) it follows that the properties being second countable and Lindelöf are independent of each other in ZF. In view of this and Theorem 1 it is plausible to ask:

Question 1. Are there any other properties p of metric spaces weaker than Lindelöfness (or second countability) such that every countably compact metric space having the property p is compact?

Regarding Question 1, one may think that in view of part (b) of the following theorem separability might be an answer.

Theorem 3. *Let* $\mathbf{X} = (X, d)$ *be a metric space. Then:*

(a) *(See* [9], ZF + CAC*.) The following are equivalent:*
  (i) *The space* $\mathbf{X}$ *is compact.*
  (ii) *The space* $\mathbf{X}$ *is sequentially compact.*
  (iii) *The space* $\mathbf{X}$ *is countably compact.*
(b) *(See* [9], ZF*.) If* $\mathbf{X}$ *is separable then* (i)–(iii) *are equivalent.*
(c) *(See* [13], ZF*.) The space* $\mathbf{X}$ *is compact if and only it is totally bounded and Lebesgue.*
(d) *(See* [10], ZF*.) If* $\mathbf{X}$ *is countably compact then it is Lebesgue.*
(e) *(See* [13], ZF*.) The space* $\mathbf{X}$ *is sequentially compact if and only it is complete and every sequence in* $\mathbf{X}$ *admits a Cauchy subsequence (*$\mathbf{X}$ *is sequentially bounded).*

However, separability of metric spaces is stronger than second countability in ZF. (In the model $\mathcal{M}1$ the subspace $\mathbf{A}$ of $\mathbb{R}$ is second countable but not separable.)

Theorem 4. *Let $\mathbf{X} = (X, d)$ be a metric space. The following are equivalent.*

(i) *The space $\mathbf{X}$ is totally bounded.*

(ii) *For every $n \in \mathbb{N}$, there exists a finite family of open balls $\mathcal{B}_n$ such that for every $x \in X$ there is a $B_x \in \mathcal{B}_n$ with $x \in B_x \subseteq B(x, 1/n)$.*

(iii) *For every $n \in \mathbb{N}$, there exists a finite family of open sets $\mathcal{B}_n$ such that for every $x \in X$ there is a $B_x \in \mathcal{B}_n$ with $x \in B_x \subseteq B(x, 1/n)$.*

Proof: (i) $\rightarrow$ (ii) Fix $k \in \mathbb{N}$ and let, by our hypothesis, $S = \{x_i : i \leq m\}$ be a $1/3k$-net of $\mathbf{X}$. We claim that $\mathcal{B}_k = \{B(x_i, 1/3k) : i \leq m\}$ satisfies: For every $x \in X$ there is a $B_x \in \mathcal{B}_k$ with $x \in B_x \subseteq B(x, 1/k)$. Fix $x \in X$. Since $d(x, S) < 1/3k$ there exists $i \leq m$ such that $d(x, x_i) < 1/3k$. We claim that $x \in B(x_i, 1/3k) \subseteq B(x, 1/k)$. Indeed, if $y \in B(x_i, 1/3k)$ then $d(x, y) \leq d(x, x_i) + d(x_i, y) < 1/3k + 1/3k < 1/k$. Hence, $x \in B(x_i, 1/3k) \subseteq B(x, 1/k)$ as claimed.

(ii) $\rightarrow$ (iii) This is straightforward.

(iii) $\rightarrow$ (i) Fix $n \in \mathbb{N}$. We show that $\mathbf{X}$ has $1/n$-nets. By our hypothesis, there exists a finite family of open sets $\mathcal{B}_n$ of $\mathbf{X}$ such that for every $x \in X$ there is a $B_x \in \mathcal{B}_n$ with $x \in B_x \subseteq B(x, 1/n)$. For every $B \in \mathcal{B}_n$ fix $x_B \in X$ such that $B \subseteq B(x_B, 1/n)$. Clearly, $\{x_B : B \in \mathcal{B}_n\}$ is a $1/n$-net of $\mathbf{X}$. Hence $\mathbf{X}$ is totally bounded as required. □

Theorem 5. *Every second countable metric space has a countable base consisting of open balls if and only if* CAC($\mathbb{R}$).

Proof: ($\leftarrow$) This is straightforward, in view of the fact that second countable metric spaces have size less than or equal to $|\mathbb{R}|$ (if $\mathcal{B}$ is a countable base for the metric space $\mathbf{X} = (X, d)$ then the mapping $x \rightarrow \{B \in \mathcal{B} : x \in B\}$ from X to $\mathcal{P}(\mathcal{B})$ is one-to-one).

($\rightarrow$) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of nonempty dense subsets of $\mathbb{R}$. Since for every $x, y \in \mathbb{R}$, $x < y$ a homeomorphism $f : \mathbb{R} \rightarrow (x, y)$ can be defined in ZF, we may assume that for all $n \in \mathbb{N}$, $A_n \subseteq (1/(n+1), 1/n)$ is dense in the subspace $(1/(n+1), 1/n)$ of $\mathbb{R}$. Let $X = \bigcup\{A_n : n \in \mathbb{N}\}$ and $d : X \times X \rightarrow \mathbb{R}$ be the restriction of the usual metric of $\mathbb{R}$ to X. Since $\mathbb{R}$ is second countable, it follows that $\mathbf{X}$ is also second countable. Fix, by our hypothesis, a countable base $\mathcal{B}$ of $\mathbf{X}$ consisting of open balls. Since for every $n \in \mathbb{N}$, A_n is open, we may assume that each member B of $\mathcal{B}$ is included in some A_n. Since each A_n is dense in $(1/(n+1), 1/n)$ it follows easily that each member B of $\mathcal{B}$ has a unique center d_B. On the basis of $D = \{d_B : B \in \mathcal{B}\}$ we can readily define a choice set for $\mathcal{A}$. □

3. Countable compactness, almost Lindelöfness and second countability like properties of metric spaces

In our first result in this section we observe, via construction of ZF examples of metric spaces, that the following ZF implications:

- ∘ second countable → quasi second countable → ω-quasi second countable;
- ∘ ω-quasi separable → ω-quasi second countable, strongly almost second countable → ω-quasi second countable; and
- ∘ Lindelöf → almost Lindelöf

are not reversible in ZF. In addition, the properties ω-quasi separable and second countable are independent of each other in ZF.

Proposition 6. (i) *"Every quasi separable (or ω-quasi separable) metric space is second countable" implies* CAC_{fin} *(or* CUT, *respectively).*

(ii) *"Every second countable metric space is ω-quasi separable" implies* $\mathrm{IDI}(\mathbb{R})$*. In particular, "every subspace of $\mathbb{R}$ is ω-quasi separable" implies* $\mathrm{IDI}(\mathbb{R})$*.*

(iii) *"Every quasi second countable (or ω-quasi second countable) metric space is second countable" implies* $\mathrm{CAC}_{\mathrm{fin}}$ *(or* CUT, *respectively).*

(iv) *"Every ω-quasi separable metric space is quasi second countable" implies* CMC_{ω}.

(v) *"Every almost Lindelöf metric space is Lindelöf" implies* $\mathrm{CAC}(\mathbb{R})$.

(vi) *"Every ω-quasi second countable metric space is ω-quasi separable" implies* $\mathrm{IDI}(\mathbb{R})$.

(vii) *"Every ω-quasi separable metric space is almost second countable" implies* CUT.

(viii) *Each of the statements "Every quasi separable metric space is ω-quasi second countable" and "every quasi separable metric space is ω-quasi separable" implies* $\mathrm{CAC}_{\mathrm{fin}}$.

(ix) *"Every almost Lindelöf metric space is ω-quasi separable" implies* $\mathrm{CAC}_{\mathrm{fin}}$.

(x) *"Every strongly almost second countable metric space is quasi second countable" implies* CMC_{ω} *and* $\mathrm{CAC}_{\omega}(\mathbb{R})$.

(xi) *"Every strongly almost second countable metric space is ω-quasi second countable" implies* $\mathrm{CAC}_{\mathrm{fin}}$.

Proof: (i) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of finite (or countable) sets. Put $X = \bigcup \mathcal{A}$ and let d be the discrete metric on X. Clearly, $\mathbf{X}$ is quasi separable (or ω-quasi separable). By our hypothesis, $\mathbf{X}$ has a countable base $\mathcal{B}$. Since, $\{\{x\} : x \in X\} \subseteq \mathcal{B}$ it follows that X is countable.

(ii) Assume the contrary and fix a Dedekind finite subset X of $\mathbb{R}$. Clearly, the subspace $\mathbf{X}$ of $\mathbb{R}$ is second countable but not ω-quasi separable, contradicting our hypothesis.

(iii) This can be proved as in (i).

(iv) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of countable sets and let X and d be as in the proof of part (i). Clearly, $\mathbf{X}$ is ω-quasi separable ($X = \bigcup\{A_n : n \in \mathbb{N}\}$

is dense in $\mathbf{X}$). By our hypothesis $\mathbf{X}$ has a base $\mathcal{Q}$ which can be written as $\bigcup\{Q_n : n \in \mathbb{N}\}$ where for every $n \in \mathbb{N}$, Q_n is finite. Since $\mathcal{B} = \{\{x\} : x \in X\} \subseteq \mathcal{Q}$ it follows that $\mathcal{B}$, and consequently X, can be expressed as a countable union of finite sets. Using this information we can easily define a multiple choice set for $\mathcal{A}$.

(v) This follows from Theorem 2 ($\mathbb{N}$ is almost Lindelöf).

(vi) This can be proved as in (ii).

(vii) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of countable sets. Let $X = \bigcup \mathcal{A}$ and d be the discrete metric on X. Since $\mathbf{X}$ is ω-quasi separable, it follows by our hypothesis that $\mathbf{X}$ has a countable $1/2$-almost base $\mathcal{C}$. Since, for every $x \in X$, $d(x, \{x\}^c) = 1$, we see that there exists an open ball $C \in \mathcal{C}$ with $x \in C \subseteq \{x\}$. Hence, $C = \{x\}$ and $\mathcal{B} = \{\{x\} : x \in X\} \subseteq \mathcal{C}$. Thus, $\mathcal{B}$ and consequently X is countable.

(viii) Fix a family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of disjoint nonempty finite sets. Let X and d be as in the proof of (i). Fix, by our hypothesis, a base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ for $\mathbf{X}$ such that for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is countably infinite. Since every element of X is isolated it follows that $\{\{x\} : x \in X\} \subseteq \mathcal{B}$. Thus X is Dedekind infinite and $\mathrm{CAC_{fin}}$ holds true as required.

The second assertion can be proved similarly.

(ix) Assume the contrary and fix $\mathcal{A}$ as in (viii). Put $X = \bigcup \mathcal{A} \cup \{\infty\}$ for some element $\infty \notin \bigcup \mathcal{A}$ and let $d\colon X \times X \to \mathbb{R}$ be the metric given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \dfrac{1}{n} & \text{if } x \in A_n,\ y \in A_v \text{ and } n \leq v, \\ \dfrac{1}{n} & \text{if } x \in A_n \text{ and } y = \infty. \end{cases} \tag{1}$$

Since $\mathbf{X}$ is compact, it follows that it is almost Lindelöf as well. Fix, by our hypothesis, a dense set $D = \bigcup\{D_n : n \in \mathbb{N}\}$ of $\mathbf{X}$ such that for every $n \in \mathbb{N}$, D_n is countably infinite. It follows that $Y = \bigcup \mathcal{A}$ is Dedekind infinite. Hence $\mathcal{A}$ has a partial choice set as required.

(x) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of countable sets and let $X = \bigcup \mathcal{A}$. Let $d\colon X \times X \to \mathbb{R}$ be the metric given by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \dfrac{1}{n} & \text{if } x \in A_n,\ y \in A_m \text{ and } n \leq m. \end{cases} \tag{2}$$

We claim that for every $n \in \mathbb{N}$, $\mathcal{B}_n = \{\{x\} : x \in \bigcup\{A_i : i \leq n\}\} \cup \{\bigcup\{A_i : i > n\}\}$ is a $1/n$-almost base for $\mathbf{X}$. To see this, fix an open set O of $\mathbf{X}$ and let $x \in O$ be such that $d(x, O^c) \geq 1/n$. We consider the following two cases:

(1) $x \in A_k$ for some $k \leq n$. Clearly, $\{x\} \subseteq B(x, 1/n) \subseteq B(x, 1/k) = \{x\} \in \mathcal{B}_n$. Hence, $x \in B(x, 1/n) \subseteq O$.

(2) $x \in A_k$ for some $k > n$. Since, $B(x, 1/n) = \bigcup\{A_i : i > n\} \in \mathcal{B}$ it follows that $x \in B(x, 1/n) \subseteq O$.

From cases (1) and (2) it follows that for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is a $1/n$-almost base for $\mathbf{X}$. Hence, $\mathbf{X}$ is strongly almost second countable. Therefore, by our hypothesis, there exists a base $\mathcal{B}$ for $\mathbf{X}$ which is expressible as a countable union of finite sets. Since d produces the discrete topology on X, it follows that $\{\{x\} : x \in X\} \subseteq \mathcal{B}$. On the basis of $\mathcal{B}$ we can define a multiple choice set for $\mathcal{A}$.

(xi) Fix a disjoint family $\mathcal{A} = \{A_n \colon n \in \mathbb{N}\}$ of finite nonempty sets. We show that $\mathcal{A}$ has a partial choice set. To this end, it suffices to show that $X = \bigcup \mathcal{A}$ is Dedekind infinite. Let $d \colon X \times X \to \mathbb{R}$ be the metric given by (2). By the proof of (x), $\mathbf{X}$ is strongly almost second countable. Hence, by our hypothesis, $\mathbf{X}$ has a base $\mathcal{B}$ which can be written as $\bigcup\{\mathcal{B}_n \colon n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is countably infinite. Since $\{\{x\} \colon x \in X\} \subseteq \mathcal{B}$ it follows $\mathbf{X}$ is Dedekind infinite as required. □

By Proposition 6, ω-quasi separability is strictly weaker than second countability. Hence, by Theorem 1 ω-quasi separability is an answer to Question 1. Next, in view of Proposition 6, we observe that the properties ω-quasi second countable and almost Lindelöf are also answers to Question 1.

Proposition 7. (i) *Let $\mathbf{X} = (X, d)$ be an ω-quasi second countable metric space. Then $\mathbf{X}$ is compact if and only if it is countably compact.*

(ii) *Let $\mathbf{X} = (X, d)$ be an almost Lindelöf metric space. Then $\mathbf{X}$ is compact if and only if it is countably compact.*

(iii) *Let $\mathbf{X} = (X, d)$ be an almost second countable metric space. Then $\mathbf{X}$ is compact if and only if it is countably compact.*

Proof: (i) $(\rightarrow)$ This is straightforward.

$(\leftarrow)$ Fix a countably compact metric space $\mathbf{X}$ and let $\mathcal{B} = \bigcup\{\mathcal{B}_n \colon n \in \mathbb{N}\}$ be a base for $\mathbf{X}$ such that for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is countable. We show that every cover $\mathcal{U} \subseteq \mathcal{B}$ of $\mathbf{X}$ has a finite subcover. For every $n \in \mathbb{N}$, let $O_n = \bigcup \mathcal{U} \cap \mathcal{B}_n$. Clearly, $\mathcal{W} = \{O_n \colon n \in \mathbb{N}\}$ is a countable open cover of $\mathbf{X}$. By our hypothesis $\mathcal{W}$ has a finite subcover $\mathcal{V} = \{O_{n_i} \colon i \leq k\}$. Since, $\mathcal{Q} = \bigcup\{\mathcal{U} \cap \mathcal{B}_{n_i} \colon i \leq k\}$ is clearly a countable subcover of $\mathcal{U}$ it follows by the countable compactness of $\mathbf{X}$ that $\mathcal{Q}$ has a finite subcover $\mathcal{G}$. Clearly, $\mathcal{G}$ is a subcover of $\mathcal{U}$ and $\mathbf{X}$ is compact as required.

(ii) $(\rightarrow)$ This is obvious.

$(\leftarrow)$ Fix a countably compact and almost Lindelöf metric space $\mathbf{X}$. By Theorem 3 (c) and (d) it is enough to show that $\mathbf{X}$ is totally bounded. Fix $\varepsilon > 0$ and let $\mathcal{U} = \{B(x, \varepsilon) \colon x \in X\}$. Since $\mathbf{X}$ is almost Lindelöf, $\mathcal{U}$ has a countable subcover $\mathcal{V}$. Since $\mathbf{X}$ is countably compact $\mathcal{V}$ has a finite subcover, meaning that $\mathbf{X}$ is totally bounded as required.

(iii) Fix $\mathbf{X} = (X, d)$ an almost second countable and countably compact metric space. In order to show that $\mathbf{X}$ is compact it suffices in view of Theorem 3 to prove that $\mathbf{X}$ is totally bounded. Fix $n \in \mathbb{N}$ and let $\mathcal{B}_n$ be a countable $1/n$-base of $\mathbf{X}$. Clearly, for every $x \in X$, there exist $B_x \in \mathcal{B}_n$ such that $x \in B_x \subseteq B(x, 1/n)$. Since $\mathbf{X}$ is countably compact it follows that finitely many members of $\mathcal{B}_n$ cover X.

Hence, there exist $x_i \in X$, $i \leq k$ such that $\bigcup\{B(x_i, 1/n) \colon i \leq k\} = X$. Thus, $\mathbf{X}$ is totally bounded as required. $\square$

Even though second countability is strictly weaker than separability in ZF, we show next that in the class of countably compact metric spaces S $\leftrightarrow$ 2C.

Theorem 8. *Let $\mathbf{X} = (X, d)$ be a countably compact metric space. Then $\mathbf{X}$ is separable if and only if it is second countable.*

Proof: ($\rightarrow$) This is straightforward.

($\leftarrow$) Let $\mathcal{B} = \{B_n \colon n \in \mathbb{N}\}$ be a base for $\mathbf{X}$. Without loss of generality we may assume that the members of $\mathcal{B}$ are bounded. For every $n \in \mathbb{N}$, via a straightforward induction, we construct a strictly descending chain $(B_{n_k})_{k\in\mathbb{N}} \subseteq \mathcal{B}$ such that for every $k \in \mathbb{N}$,

$$\overline{B_{n_{k+1}}} \subset B_{n_k} \subseteq B_n \qquad \text{and} \qquad \delta(B_{n_{k+1}}) < \frac{1}{2}\,\delta(B_{n_k}).$$

It follows by the countable compactness of $\mathbf{X}$ that $\emptyset \neq \bigcap\{\overline{B_{n_k}} \colon k \in \mathbb{N}\} \subseteq B_n$. Since $\delta(\bigcap\{\overline{B_{n_k}} \colon k \in \mathbb{N}\}) = 0$, it follows that $\bigcap\{\overline{B_{n_k}} \colon k \in \mathbb{N}\}$ is a singleton of X, say $\{d_n\}$. Then $D = \{d_n \colon n \in \mathbb{N}\}$ is a countable dense subset of $\mathbf{X}$ and $\mathbf{X}$ is separable as required. $\square$

Remark 2. We would like to remark here that if $\mathcal{A} = \{A_n \colon n \in \mathbb{N}\}$ is a disjoint family of finite nonempty sets without a choice set, $X = \bigcup\mathcal{A} \cup \{\infty\}$, $\infty \notin \bigcup\mathcal{A}$, and $d \colon X \times X \rightarrow \mathbb{R}$ is the metric given by (1) then $\mathbf{X}$ is compact, quasi second countable and quasi separable but not separable. Thus, in the class of countably compact metric spaces quasi second countability and quasi separability are strictly weaker than second countability.

If $\mathbf{X} = (X, d)$ is a countably compact and ω-quasi separable metric space then by Theorem 1 $\mathbf{X}$ is compact, hence almost Lindelöf as well. So, in the class of countably compact metric spaces ω-quasi separability implies almost Lindelöfness and one may ask if the latter implication is reversible. We show next that the answer is no.

Theorem 9. (i) *The negation of the statement: "Every countably compact and almost Lindelöf metric space is quasi separable (or ω-quasi separable)" is consistent with* ZF. *In particular, in the class of countably compact metric spaces almost Lindelöfness is strictly weaker than ω-quasi separability.*

(ii) *"Every quasi separable (or ω-quasi separable) metric space is almost Lindelöf" implies* IDI($\mathbb{R}$).

Proof: (i) We recall the concentric circles permutation model $\mathcal{N}$ given in [12]. The set of atoms A is expressed as $\bigcup\{A_n \colon n \in \mathbb{N}\}$, where for all $n \in \mathbb{N}$,

$$A_n = \Big\{a_{nx} \colon x \in S\Big(0, \frac{1}{2}\Big)\Big\}$$

and $S(0, 1/n)$ is the circle of the Euclidean plane $(\mathbb{R}^2, \varrho)$ of radius $1/n$ centered at 0. The group of permutations $\mathcal{G}$ is the group of all permutations on A which rotate the A_n's by an angle $\theta_n \in \mathbb{R}$ and supports are finite. In [12] it has been shown that the family $\{A_n : n \in \mathbb{N}\}$ does not have a multiple choice set. It is not hard to see that $\{A_n : n \in \mathbb{N}\}$ has also no Kinna-Wagner choice set, i.e., a family $\{B_n : n \in \omega\}$ of nonempty sets such that for all $n \in \omega$, $B_n \subseteq A_n$ and $B_n \neq A_n$.

Let $X = A \cup \{\infty\}$, $\infty \notin A$. Clearly, the function $d\colon X \times X \to \mathbb{R}$ given by the rule: $d(a_{nx}, \infty) = 1/n$ and,

$$d(a_{nx}, a_{my}) = \begin{cases} \varrho(x, y) & \text{if } n = m, \\ \dfrac{1}{n} & \text{if } n < m, \end{cases}$$

is a metric on X such that $\mathbf{X}$ is compact. Hence, $\mathbf{X}$ is almost Lindelöf. However, $\mathbf{X}$ is neither quasi separable nor ω-quasi separable as otherwise $\{A_n : n \in \mathbb{N}\}$ would have a multiple choice set and a Kinna-Wagner choice set respectively.

The model $\mathcal{N}$ is a ZF^0 (= ZF minus the axiom of regularity) model but the ZF version $\mathcal{M}$ of $\mathcal{N}$ has been constructed in [14].

(ii) Assume the contrary and fix A an infinite Dedekind finite subset of $\mathbb{R}$. N. Brunner has shown in [2] that if there exists a Dedekind finite subset of $\mathbb{R}$ then there exists a dense one also. So we may assume that A is dense in $\mathbb{R}$. Clearly the subspace $\mathbf{X}$, $X = A \cup \mathbb{Q}$ of $\mathbb{R}$ is separable hence, quasi separable and ω-quasi separable. However, $\mathbf{X}$ is not almost Lindelöf as the open cover $\mathcal{U} = \{(a - 1/2, a + 1/2) : a \in A\}$ of $\mathbf{X}$ has no countable subcover. This contradicts our hypothesis. □

Remark 3. From Theorem 9 it follows that the properties of being almost Lindelöf and quasi separability (or ω-quasi separability) are independent of each other in ZF. However in ZF + CAC, almost Lindelöf = Lindelöf = separable = quasi separable = ω-quasi separable.

4. More on second countability like properties of metric spaces

In this section we are going to eliminate some more question marks from Table 1 by constructing more ZF examples of metric spaces having a certain property p but not the property q from Table 1.

Theorem 10. (i) *"Every almost Lindelöf (or strongly almost second countable) metric space is second countable" implies* CUT *and, "every almost Lindelöf metric space (or ω-quasi separable, respectively) is quasi second countable" implies* CMC_ω.

(ii) *"Every almost Lindelöf (or strongly almost second countable) metric space is separable" implies* CUT.

(iii) *"Every almost Lindelöf (or ω-quasi separable) metric space is almost second countable" implies* CUT.

(iv) *"Every quasi separable metric space is almost second countable" implies* $\mathrm{CAC}_{\mathrm{fin}}$.

Proof: (i) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of countable sets. Let $X = \bigcup\{A_n : n \in \mathbb{N}\}$ and $d: X \times X \to \mathbb{R}$ be the metric given by (2). We claim that $\mathbf{X}$ is:

(a) Almost Lindelöf. Fix $\varepsilon > 0$ and let $\mathcal{U}$ be an open cover of $\mathbf{X}$ consisting of open balls of radius ε. We show that $\mathcal{U}$ has a countable subcover. Let

$$K = \{x \in X : x \text{ is a center of some } U \in \mathcal{U}\}.$$

If $K \subseteq \bigcup\{A_i : i \leq m\}$ for some $m \in \mathbb{N}$ then $\mathcal{U}$ is countable and there is nothing to show. So assume that $K \cap A_m \neq \emptyset$ for infinitely many $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ satisfy $1/m < \varepsilon$, $K \cap A_m \neq \emptyset$ and fix $x \in K \cap A_m$. Clearly, $\bigcup\{A_n : n > m\} \cup \{x\} = B(x, 1/m) \subseteq B(x, \varepsilon)$. Since, $C = \bigcup\{A_n : n \leq m\}$ is countable, it follows that we need countably many more members of $\mathcal{U}$ to cover C. Hence, $\mathbf{X}$ is almost Lindelöf as claimed.

(b) Strongly almost second countable. This has been established in the proof of part (x) of Proposition 6.

Fix, by our hypothesis, a countable base $\mathcal{B}$ for $\mathbf{X}$. Since d produces the discrete topology on X it follows that $\{\{x\} : x \in X\} \subseteq \mathcal{B}$. Hence, X is a countable set and CUT holds true as required.

For the second assertion we note that $\mathbf{X}$ in addition to being almost Lindelöf, is ω-quasi separable as well. Fix, by our hypothesis, a countable base $\mathcal{B}$ for $\mathbf{X}$ such that $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite set. Since d produces the discrete topology on $\mathbf{X}$, it follows that $\{\{x\} : x \in X\} \subseteq \mathcal{B}$. It is easy to see that $\mathcal{A}$ has a multiple choice set. In fact, something stronger holds true here. Namely, $\bigcup \mathcal{A}$ can be expressed as a countable union of finite sets.

(ii) This in view of the proof of (i) is straightforward.

(iii) Fix $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ a disjoint family of countable sets. For every $n \in \mathbb{N}$, let $X_n = A_n \cup \{n\}$. Without loss of generality we may assume that $n \notin A_n$. Let $d_n : X_n \times X_n \to \mathbb{R}$ be the metric given by

$$d_n(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \dfrac{1}{n} & \text{if } x, y \in A_n, \\ 1 & \text{if } x \in A_n \text{ and } y = n. \end{cases}$$

Put $X = \prod_{n \in \mathbb{N}} X_n$ and let $d: X \times X \to \mathbb{R}$ be the sup metric, i.e.,

$$d(x, y) = \sup\{d_n(x(n), y(n)) : n \in \mathbb{N}\}.$$

We claim that

(a) $\mathbf{X}$ is almost Lindelöf. To see this, fix $n \in \mathbb{N}$. Clearly, for every $x, y \in X$ with $x(i) = y(i)$ for all $i \leq n$, $B(x, 1/n) = B(y, 1/n)$. Since $Y_n = \prod_{i \leq n} X_i$ is countable, it follows that there are at most countably many open discs of radius $1/n$ and consequently $\mathbf{X}$ is almost Lindelöf as claimed.

(b) $\mathbf{X}$ is ω-quasi separable. Clearly, for every $n \in \mathbb{N}$

$$D_n = \prod_{i \leq n} X_i \times \prod_{i>n} \{i\}$$

is a countable set. To complete the claim it suffices to show that $D = \bigcup\{D_n : n \in \mathbb{N}\}$ is dense in $\mathbf{X}$. To see this fix $x \in X \backslash D$ and consider the open ball $B(x, 1/n)$. Let $z \in X$ be the element given by:

$$z \mid n+1 = x \mid n+1 \quad \text{ and } \quad z(i) = i \qquad \text{for all } i > n.$$

Clearly, $z \in D \cap B(x, 1/n)$ meaning that D is dense in $\mathbf{X}$.

Let, by our hypothesis, $\mathcal{B}$ be a countable 1-almost base for $\mathbf{X}$. Without loss of generality we may assume that for every $B \in \mathcal{B}$, $B = B(x, 1/m)$ for some $x \in X$ and $m \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$O_n = \prod_{i \leq n} A_i \times \prod_{i>n} X_i.$$

Clearly, for every $x \in O_n$ is $d(x, O_n^c) = 1$. Hence, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O_n$. Let B_x be the first such ball of $\mathcal{B}$. We claim that $B_x = B(x, 1/m)$ for some $m \geq n$. Clearly, $B_x = B(y, 1/m)$ for some $m \in \mathbb{N}$ and $y \in X$. If $m < n$ then the element $z \in X$ given by: $z(m+1) = m+1$ and for all $i \in \mathbb{N}$, $i \neq m+1$, $z(i) = y(i)$ is in $B_x \backslash O_n$ contradicting the fact that $B_x \subseteq O_n$. Thus, $m \geq n$. Since $x \in B(y, 1/m)$ it follows that $d(x, y) = \sup\{d_n(x(n), y(n)) \colon n \in \mathbb{N}\} < 1/m$. Therefore, for all $n \leq m$, $d_n(x(n), y(n)) < 1/m$. If for some $n \leq m$, $x(n) \neq y(n)$ then $1/n = d_n(x(n),\, y(n)) < 1/m \leq 1/n$ which is a contradiction. Thus, for all $n \leq m$, $x(n) \neq y(n)$ and consequently $B_x = B(y, 1/m) = B(x, 1/m)$ as claimed.

For every $n \in \mathbb{N}$, let $W_n = \prod_{i \leq n} A_i \times \prod_{i>n}\{i\}$. Clearly, for every $n \in \mathbb{N}$ and $x, y \in W_n$ with $x \neq y$ there exists $i \leq n$ such that $x(i) \neq y(i)$. Hence, $B_x \neq B_y$ and consequently for every $n \in \mathbb{N}$, the function $f_n \colon W_n \to \mathcal{B}$, $f_n(x) = B_x$ is one-to-one. Thus $W = \bigcup\{W_n \colon n \in \mathbb{N}\}$ (or $H = \bigcup\big\{\prod_{i \leq n} A_i \colon n \in \mathbb{N}\big\}$) is a countable set. On the basis of H one can readily define a one-to-one function $f \colon \bigcup \mathcal{A} \to H$ meaning that $\bigcup \mathcal{A}$ is countable as required.

(iv) This, in view of the proof of (iii), is straightforward. So, we leave the proof as an easy exercise for the reader. □

Theorem 11. (i) *The negation of the statement: "Every almost second countable metric space is strongly almost second countable" is consistent with* ZF.

(ii) *The negation of the statement: "Every almost second countable metric space is quasi second countable (or ω-quasi second countable)" is consistent with* ZF.

Proof: We recall that the Good/Tree/Watson Model II $\mathcal{N}$ in [8] is a permutation model constructed as follows: The set of atoms A is $\bigcup\{A_n \colon n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$, $A_n = \{a_{n,x} \colon x \in \mathbb{R}\}$. The group of permutations $\mathcal{G}$ is the group of all permutations π on A such that for every $n \in \mathbb{N}$, π is the identity of A_n, or there exists $r \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, $\pi(a_{n,x}) = a_{n,x+r}$, i.e., π is a translation on A_n. Supports are finite. The ZF version of the model $\mathcal{N}$ has been constructed in [3] and the ZF^0 was used in [7]. The fact that it is easier in general to work with permutation models explains our preference for the permutation version of the model. For every $n \in \mathbb{N}$, let $d_n\colon A_n \times A_n \to \mathbb{R}$ be the metric given by:

$$d_n(a_{n,x}, a_{n,y}) = |x - y|.$$

Clearly, the set $D = \{d_n\colon n \in \mathbb{N}\}$ has empty support. Hence, $D \in \mathcal{N}$. Let d be the metric on A which is defined by the rule:

$$d(a_{n,x}, a_{m,y}) = \begin{cases} \min\left\{\dfrac{1}{n}, d_n(a_{n,x}, a_{n,y})\right\} & \text{if } n = m, \\ \dfrac{1}{n} & \text{if } n \leq m. \end{cases}$$

We claim that $\mathbf{A} = (A, d)$ is almost second countable. To see this, fix $k \in \mathbb{N}$. Clearly, for every $i = 1, 2, \ldots, k$, $\mathcal{B}_i = \{B(a_{i,r}, 1/2k)\colon r \in \mathbb{Q}\}$ is a countable $1/k$-almost base for the (open) subspace $\mathbf{A}_i$ of $\mathbf{A}$. Indeed, if O is an open subset of $\mathbf{A}_i$ and $a_{i,x} \in O$ is such that $B(a_{i,x}, 1/k) \subseteq O$ then for every $q \in \mathbb{Q}$ with $|q - x| < 1/2k$ we have $B(a_{i,q}, 1/2k) \subseteq B(a_{i,x}, 1/k)$. (If $a_{i,p} \in B(a_{i,q}, 1/2k)$ then $d(a_{i,p}, a_{i,x}) \leq d(a_{i,p}, a_{i,q}) + d(a_{i,q}, a_{i,x}) < 1/2k + 1/2k = 1/k$.) Hence, $B(a_{i,q}, 1/2k) \subseteq O$ as required. We claim that

$$\mathcal{B} = \bigcup\{\mathcal{B}_i\colon i \leq k\} \cup \left\{\bigcup\{A_i\colon i > k\}\right\}$$

is a countable $1/k$-almost base for $\mathbf{A}$. The fact that $\mathcal{B}$ is countable is obvious. To see that $\mathcal{B}$ is a $1/k$-almost base for $\mathbf{A}$, we fix an open set O in $\mathbf{A}$ and $a_{i,x} \in O$ such that $B(a_{i,x}, 1/k) \subseteq O$. We consider the following two cases:

(1) $i \leq k$. Since $B(a_{i,x}, 1/k) \subseteq O \cap A_i$, it follows that there exists $B \in \mathcal{B}_i \subseteq \mathcal{B}$ with $a_{i,x} \in B \subseteq B(a_{i,x}, 1/k) \subseteq O \cap A_i \subseteq O$.

(2) $i > k$. In this case we have $B(a_{i,x}, 1/k) = \bigcup\{A_i\colon i > k\} \in \mathcal{B}$. Hence, there exists $B \in \mathcal{B}$ with $a_{i,x} \in B \subseteq B(a_{i,x}, 1/k) \subseteq O$.

From (1) and (2) it follows that $\mathcal{B}$ is a $1/k$-almost base for $\mathbf{A}$ as claimed.

Since any finite subset E of A meeting non-trivially each A_i, $i = 1, 2, \ldots, k$ is a support of $\mathcal{B}$ it follows that $\mathcal{B} \in \mathcal{N}$. Hence, $\mathbf{A}$ is almost second countable as required.

(i) We claim that $\mathbf{A}$ is not strongly almost second countable. To see this, we assume the contrary and let $\mathcal{B} = \bigcup\{\mathcal{B}_n\colon n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is a countable $1/n$-almost base for $\mathbf{A}$. Clearly, for every $n \in \mathbb{N}$, A_n is an open subset such that for every $x \in \mathbb{R}$ there exists an open ball $B \in \mathcal{B}_{n+1}$ with $a_{n,x} \in B \subseteq A_n$. Let

$$C_n = \{a_{n,x} \in A_n\colon\ a_{n,x} \text{ is a center of an open ball } B \in \mathcal{B}_{n+1} \text{ with } B \subseteq A_n\}.$$

Since the members of every $\mathcal{B}_n$ may be viewed as open finite intervals of $\mathbb{R}$, they have unique centers. Hence, $C = \{C_n : n \in \mathbb{N}\}$ is a family of countable sets in $\mathcal{N}$ such that for every $n \in \mathbb{N}$, $C_n \subseteq A_n$. Let E be a support for C and fix $n \in \mathbb{N}$ such that $E \cap A_n = \emptyset$. Fix $a_{n,y} \in A_n \backslash C_n$, $a_{n,z} \in C_n$ and let $r = y - z$. Let π be the permutation of A which is the identity on each A_i, $i \neq n$ and for every $a_{n,x} \in A_n$, $\pi(a_{n,x}) = a_{n,x+r}$. We have $C = \pi(C) = \{\pi(C_i) : i \in \mathbb{N}\}$. Hence, $C_n = \pi(C_n)$. Since $\pi(a_{n,z}) = a_{n,z+y-z} = a_{n,y} \in \pi(C_n)$ it follows that $a_{n,y} \in C_n$ which contradicts the fact that $a_{n,y} \in A_n \backslash C_n$. Hence, $\mathbf{A}$ is not strongly almost second countable as claimed.

(ii) Assume, aiming for a contradiction, that $\mathbf{A}$ is quasi second countable (or ω-quasi second countable) and fix a base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ for $\mathbf{A}$ such that for every $n \in \mathbb{N}$, $\mathcal{B}_n$ is finite (or countably infinite, respectively). Without loss of generality we may assume that for every $B \in \mathcal{B}$, there exists $n \in \mathbb{N}$ with $B \subseteq A_n$ and $B \neq A_n$. Let E be a support for $\mathcal{B}$. For every $n \in \mathbb{N}$, let

$$k_n = \min\{m \in \mathbb{N} : B \subseteq A_n \text{ for some } B \in \mathcal{B}_m\}$$

and put $W_n = \{B \in \mathcal{B}_{k_n} : B \subseteq A_n\}$. Clearly, E is a support for each W_n, $n \in \mathbb{N}$. Since for every $n \in \mathbb{N}$, $f_n : \mathbb{R} \to A_n$, $f_n(x) = a_{n,x}$ is a homeomorphism, it follows that $\mathbf{A}_n$ is connected and consequently each $B \in W_n$ has countably many components, i.e., maximal connected subsets. Since the components of open subsets of $\mathbb{R}$ are countably many open intervals, it follows that for every $n \in \mathbb{N}$, the set C_n of all endpoints of the components of members of W_n is countable. Indeed, for every $n \in \mathbb{N}$, $|A_n| = |\mathbb{R}|$ and $\mathbb{R}$ is well orderable in every permutation model. So A_n is well orderable and the union of countably many countable sets in A_n is countable. Clearly, E is a support of $C = \{C_n : n \in \mathbb{N}\}$. So, $C \in \mathcal{N}$. Working as in part (i) we can show that $C \notin \mathcal{N}$. This leads to a contradiction and finishes the proof of (ii). $\square$

Acknowledgment. Many thanks to the referee for several astute comments about our paper.

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(*Received* September 15, 2016, *revised* October 25, 2017)