On a class of abstract degenerate fractional differential equations of parabolic type

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Abstract. In this paper, we investigate a class of abstract degenerate fractional differential equations with Caputo derivatives. We consider subordinated fractional resolvent families generated by multivalued linear operators, which do have removable singularities at the origin. Semi-linear degenerate fractional Cauchy problems are also considered in this context.

Keywords: abstract degenerate fractional differential equations; infinitely differentiable fractional resolvent families; multivalued linear operators; semi-linear degenerate fractional Cauchy problems; Caputo fractional derivatives

Classification: 47D03, 47D06, 47D62, 47D99, 47G20

1. Introduction and preliminaries

In [12, Chapter III], A. Favini and A. Yagi have considered a class of infinitely differentiable semigroups generated by the multivalued linear operators satisfying the following condition:

(P) There exist finite constants $c,\ M>0$ and $\beta\in(0,1]$ such that

$$\Psi:=\{\lambda\in\mathbb{C}\colon \Re\lambda\geq -c(|\Im\lambda|+1)\}\subseteq\rho(\mathcal{A})$$

and

$$\|R(\lambda:\mathcal{A})\|\leq M(1+|\lambda|)^{-\beta},\qquad \lambda\in\Psi.$$

In this paper, we consider fractional resolvent families subordinated to these semigroups and apply our results in the analysis of existence and uniqueness of solutions for a class of abstract degenerate (semilinear) fractional differential equations with Caputo derivatives. As mentioned in the abstract, fractional resolvent families under our consideration have removable singularities at the origin.

Following the methods proposed in the doctoral dissertation of E. Bazhlekova, see [2] and the monograph [12], we extend the results of R.-N. Wang, D.-H. Chen and T.-J. Xiao, see [38], to abstract degenerate fractional differential equations.

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Even in single-valued case, we establish proper extensions of [38, Theorem 3.3 (ii), Theorem 3.4 (iii), Theorem 4.1]; we can freely say that this is the major novelty of our work. Besides that, we present a great number of new analytical properties of subordinated fractional resolvent families and transfer the results of F. Periago in [29] to abstract degenerate semilinear (fractional) differential equations.

We use the standard terminology throughout the paper. Unless specified otherwise, we assume henceforth that $(E, \|\cdot\|)$ is a complex Banach space. In the case that X is also a complex Banach space, then we denote by $L(E, X)$ the space consisting of all continuous linear mappings from E into X; $L(E) \equiv L(E, E)$. If A is a closed linear operator acting on E, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph.

Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} \colon s \leq l\}$. Define $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} \colon |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s) g(s)\, \mathrm{d}s$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$, $\zeta > 0$, $t > 0$, and $g_0(t) :=$ the Dirac δ-distribution.

Let $0 < \tau \leq \infty$, let $m \in \mathbb{N}$, and let $I = (0, \tau)$. Then the Sobolev space $W^{m,1}(I : E)$ can be introduced in the following way (see e.g. [2, page 7]):

$$W^{m,1}(I : E) := \Biggl\{ f \colon\ \exists \varphi \in L^1(I : E)\ \exists c_k \in \mathbb{C},\ 0 \leq k \leq m-1,$$
$$f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + (g_m * \varphi)(t) \ \text{ for a.e. } \ t \in (0, \tau) \Biggr\}.$$

Then $\varphi(t) = f^{(m)}(t)$ in distributional sense, and $c_k = f^{(k)}(0)$, $0 \leq k \leq m-1$.

We refer the reader to [7], [12], [26] and [36] for the basic source of information on abstract degenerate differential equations with integer order derivatives.

Fairly complete information about fractional calculus and fractional differential equations can be obtained by consulting [2], [8], [16]–[17] and [31]–[34]. In this paper, we will use the following notion of Caputo fractional derivatives of order $\gamma \in (0,1)$. The Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those functions $u \colon [0, T] \to E$ for which $u_{|(0,T]}(\cdot) \in C((0, T] : E)$, $u(\cdot) - u(0) \in L^1((0, T) : E)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0, T) : E)$, by

$$\mathbf{D}_t^\gamma u(t) = \frac{\mathrm{d}}{\mathrm{d}t} [g_{1-\gamma} * (u(\cdot) - u(0))](t), \qquad t \in (0, T].$$

The Wright function $\Phi_\gamma(z)$ is defined by

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(1 - \gamma - n\gamma)}, \qquad z \in \mathbb{C},\ \gamma \in (0,1).$$

Let us recall that $\Phi_\gamma(t) \geq 0$, $t \geq 0$, and that the following identity holds:

(a1) $\int_0^\infty t^r \Phi_\gamma(t)\,\mathrm{d}t = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$.

The asymptotic expansion of the Wright function $\Phi_\gamma(z)$, as $|z| \to \infty$ in the sector $|\arg(z)| \leq \min((1-\gamma)3\pi/2, \pi) - \epsilon$, is given by

$$\Phi_\gamma(z) = Y^{\gamma-1/2}\mathrm{e}^{-Y}\left(\sum_{m=0}^{M-1} A_m Y^{-M} + O\big(|Y|^{-M}\big)\right), \tag{1}$$

where $Y = (1-\gamma)(\gamma^\gamma z)^{1/(1-\gamma)}$, $M \in \mathbb{N}$ and A_m are certain real constants.

We refer the reader to [1], [39, Chapter 1], [17, Section 1.2] and [22] for further information concerning the vector-valued Laplace transform. In this paper, we are working in the setting of complex Banach spaces and we are following the usually considered approach from [1].

2. Multivalued linear operators

A multivalued map (multimap) $\mathcal{A}\colon E \to P(E)$ is said to be a multivalued linear operator (MLO in E, or simply, MLO) if and only if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of E;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

An almost immediate consequence of definition is that for every $x, y \in D(\mathcal{A})$ and for every $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x \colon x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) \colon 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted by $N(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) \colon y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. It is worth noting that $\mathcal{A}x = \mathcal{A}y$ for some two elements x and $y \in D(\mathcal{A})$, if and only if $\mathcal{A}x \cap \mathcal{A}y \neq \emptyset$; moreover, if $\mathcal{A}$ is injective, then the equality $\mathcal{A}x = \mathcal{A}y$ holds if and only if $x = y$ (for more details about multivalued linear operators, we refer the reader to the monographs [6] and [12]).

If $\mathcal{A}, \mathcal{B}$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It is clear that $\mathcal{A}+\mathcal{B}$ is an MLO. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) \colon D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A}$ is an MLO in E and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. We write $\mathcal{A} \subseteq \mathcal{B}$ if and only if $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

The scalar multiplication of an MLO $\mathcal{A}$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A}$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z, \omega \in \mathbb{C}$.

Assume now that a linear single-valued operator $S\colon D(S) \subseteq E \to E$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A}$ is an MLO in E. Then S is called a section of $\mathcal{A}$; if this is the case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

We say that an MLO operator $\mathcal{A}$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in E such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. If the MLO $\mathcal{A}$ is not closed, then we can simply prove that its closure $\mathcal{A}$, defined as usual, is a closed MLO in E.

Denote by Ω a locally compact and separable metric space and by μ we denote a locally finite Borel measure defined on Ω. Then we have the following ([20]):

Lemma 2.1. *Suppose that $\mathcal{A}$ is a closed MLO in E. Let $f\colon \Omega \to E$ and $g\colon \Omega \to E$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\,\mathrm{d}\mu \in D(\mathcal{A})$ and $\int_\Omega g\,\mathrm{d}\mu \in \mathcal{A}\int_\Omega f\,\mathrm{d}\mu$.*

In the remaining part of this section, we will consider the resolvent sets of MLOs. Our standing assumptions will be that $\mathcal{A}$ is an MLO in E. Then the resolvent set of $\mathcal{A}$, $\rho(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(\lambda - \mathcal{A}) = E$;
(ii) $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ is a single-valued bounded operator on E.

It is well known that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$. The operator $\lambda \mapsto R(\lambda : \mathcal{A})$ is called the resolvent of $\mathcal{A}$, $\lambda \in \rho(\mathcal{A})$. If $\rho(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is closed and for every $\lambda \in \rho(\mathcal{A})$ we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1})$.

We need the following useful lemma (see [12]).

Lemma 2.2. *We have*

$$(\lambda - \mathcal{A})^{-1}\mathcal{A} \subseteq \lambda(\lambda - \mathcal{A})^{-1} - I \subseteq \mathcal{A}(\lambda - \mathcal{A})^{-1}, \qquad \lambda \in \rho(\mathcal{A}).$$

The operator $(\lambda-\mathcal{A})^{-1}\mathcal{A}$ is single-valued on $D(\mathcal{A})$ and $(\lambda-\mathcal{A})^{-1}\mathcal{A}x = (\lambda-\mathcal{A})^{-1}y$, whenever $y \in \mathcal{A}x$ and $\lambda \in \rho(\mathcal{A})$.

2.1 Fractional powers. In this subsection, we assume that $(-\infty, 0] \subseteq \rho(\mathcal{A})$ as well as that there exist finite numbers $M \geq 1$ and $\beta \in (0, 1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leq M(1 + |\lambda|)^{-\beta}, \qquad \lambda \leq 0.$$

Then the resolvent set of $\mathcal{A}$ contains an open region Ω of complex plane around the nonpositive half-line $(-\infty, 0]$, and we are in position to define the fractional power

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \lambda^{-\theta}(\lambda - \mathcal{A})^{-1}\,\mathrm{d}\lambda \in L(E)$$

for $\theta > 1 - \beta$, where Γ is an appropriately chosen contour belonging to Ω (cf. [12, page 25] for more details). Set $\mathcal{A}^\theta := (\mathcal{A}^{-\theta})^{-1}$, $\theta > 1 - \beta$. Then the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1+\theta_2)}$ and $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1+\theta_2}$ hold for $\theta_1, \theta_2 > 1 - \beta$ (it is worth noting here that the fractional power $\mathcal{A}^\theta$ need not be injective and that the meaning of $\mathcal{A}^\theta$ is understood in the MLO sense for $\theta > 1 - \beta$).

We endow the vector space $D(\mathcal{A})$ with the norm

$$\|\cdot\|_{[D(\mathcal{A})]} := \inf_{y \in \mathcal{A}\cdot} \|y\|.$$

Then $(D(\mathcal{A}), \|\cdot\|_{[D(\mathcal{A})]})$ is a Banach space and, since $0 \in \rho(\mathcal{A})$, the norm $\|\cdot\|_{[D(\mathcal{A})]}$ is equivalent with the following one $\|\cdot\| + \|\cdot\|_{[D(\mathcal{A})]}$ (cf. the proof of [12, Proposition 1.1]). Since $0 \in \rho(\mathcal{A}^\theta)$, $(D(\mathcal{A}^\theta), \|\cdot\|_{[D(\mathcal{A}^\theta)]})$ is likewise a Banach space and we have the equivalence of norms $\|\cdot\|_{[D(\mathcal{A}^\theta)]}$ and $\|\cdot\| + \|\cdot\|_{[D(\mathcal{A}^\theta)]}$ for $\theta > 1 - \beta$. In our further work, we will use the fact that (see e.g. [10, (3.3)])

$$\mathcal{A}^{-\theta} x = \frac{\sin(\theta\pi)}{\pi} \int_0^\infty s^{-\theta}(s + \mathcal{A})^{-1} x \, \mathrm{d}s, \qquad 1 > \theta > 1 - \beta,\ x \in E. \tag{2}$$

For any $\theta \in (0,1)$, the vector space

$$E_{\mathcal{A}}^\theta := \Big\{ x \in E\colon \sup_{\xi > 0} \xi^\theta \|\xi(\xi + \mathcal{A})^{-1} x - x\| < \infty \Big\}$$

becomes one of Banach's when endowed with the norm

$$\|\cdot\| E_{\mathcal{A}}^\theta := \|\cdot\| + \sup_{\xi > 0} \xi^\theta \|\xi(\xi + \mathcal{A})^{-1} \cdot - \cdot \|.$$

It is clear that $E_{\mathcal{A}}^\theta$ is continuously embedded in E. For more details about fractional powers of multivalued linear operators, the interpolation spaces and their mutual relations, we refer the reader to [10] (see, especially, the equations [10, (1.2)–(1.5)]), [12, Section 1.4], [25] and [27].

3. Subordinated fractional resolvent families with removable singularities at the origin

In this section we investigate the subordinated fractional resolvent families with removable singularities at zero. Unless stated otherwise, we assume that $c, M > 0$, $\beta \in (0,1]$, $\mathcal{A}$ is an MLO and the condition (P) holds.

Let the contour $\Gamma := \{\lambda = -c(|\eta| + 1) + \mathrm{i}\eta\colon \eta \in \mathbb{R}\}$ be oriented so that $\Im\lambda$ increases along Γ. Set $T(0) := I$ and

$$T(t)x := \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{t\lambda}(\lambda - \mathcal{A})^{-1} x \, \mathrm{d}\lambda, \qquad t > 0,\ x \in E.$$

Then $(T(t))_{t \geq 0} \subseteq L(E)$ is a semigroup on E, and we have the following estimate

$$\|T(t)\| = O(t^{\beta - 1}), \quad t > 0; \tag{3}$$

furthermore for every $\theta \in (0,1)$

$$\|T(t)\|_{L(E, E_{\mathcal{A}}^\theta)} = O(t^{\beta - \theta - 1}), \qquad t > 0. \tag{4}$$

Concerning the strong continuity of $(T(t))_{t\geq 0}$ at zero, it is necessary to remind ourselves of the fact that ([12]):

(C) $T(t)x \to x$, $t \to 0_+$ for any $x \in E$ belonging to the space $D((-\mathcal{A})^\theta)$ with $\theta > 1-\beta$ ($x \in E^\theta_{\mathcal{A}}$ with $1 > \theta > 1-\beta$).

By [10, Lemma 3.9], we have that

$$R(\lambda : \mathcal{A})x = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x \, \mathrm{d}t, \qquad \Re\lambda > 0,\ x \in E. \tag{5}$$

From now on, we assume that $0 < \gamma < 1$. Set for every $\nu > -\beta$

$$T_{\gamma,\nu}(t)x := t^{-\gamma} \int_0^\infty s^\nu \Phi_\gamma(st^{-\gamma}) T(s)x \, \mathrm{d}s, \quad t > 0,\ x \in E \ \text{ and } \ T_{\gamma,0}(0) := I. \tag{6}$$

Since

$$T_{\gamma,\nu}(t)x = t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T(st^\gamma)x \, \mathrm{d}s, \qquad t > 0,\ x \in E, \tag{7}$$

the estimates (3)–(4) combined with (a1) imply that the integral which defines the operator $T_{\gamma,\nu}(t)$ is absolutely convergent as well as that

$$\|T_{\gamma,\nu}(t)\| = O(t^{\gamma(\nu+\beta-1)}), \quad t > 0,$$

and that for every $\theta \in (0,1)$ and $\nu > \theta - \beta$

$$\|T_{\gamma,\nu}(t)\|_{L(E,E^\theta_{\mathcal{A}})} = O(t^{\gamma(\nu+\beta-\theta-1)}), \qquad t > 0. \tag{8}$$

Further on, (7) taken together with (a1) implies that for every $\nu > -\beta$

$$\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^\infty s^\nu \Phi_\gamma(s)[T(st^\gamma)x - x] \, \mathrm{d}s, \quad t > 0,\ x \in E. \tag{9}$$

Using the dominated convergence theorem, (a1), (3), (9) and (C), we can deduce the following:

(b1) $\frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x \to \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, $t \to 0_+$ provided that $\theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$, or that $1 > \theta > 1-\beta$ and $x \in E^\theta_{\mathcal{A}}$, $\nu > -\beta$.

By the proof of [2, Theorem 3.1] and an elementary argumentation involving (5), we get that:

(b2) $\int_0^\infty \mathrm{e}^{-\lambda t} T_{\gamma,0}(t)x \, \mathrm{d}t = \lambda^{\gamma-1} \int_0^\infty \mathrm{e}^{-\lambda^\gamma t} T(t)x \, \mathrm{d}t = \lambda^{\gamma-1}(\lambda^\gamma - \mathcal{A})^{-1}x$, $\Re\lambda > 0$, $x \in E$.

Owing to [12, Theorem 3.5], (a1) and (9), we have that:

(b3) $\left\| \frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x \right\| = O(t^{\gamma(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]})$, $t > 0$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$ and $\left\| \frac{T_{\gamma,\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x \right\| = O\big(t^{\gamma(\beta+\theta-1)}\|x\|_{E^\theta_{\mathcal{A}}}\big)$, $t > 0$, provided $1 > \theta > 1-\beta$, $x \in E^\theta_{\mathcal{A}}$, $\nu > -\beta$.

Set $\xi := \min((1/\gamma - 1)\pi/2, \pi)$. It is worth noting that the proof of [2, Theorem 3.3 (i)–(ii)] implies that for every $\nu > -\beta$ the mapping $t \mapsto T_{\gamma,\nu}(t)x$, $t > 0$ can be analytically extended to the sector Σ_ξ (we will denote this extension by the same symbol) and that for every $\theta \in (0,1)$, $\epsilon \in (0,\xi)$ and $\nu > -\beta$,

(b4) $\|T_{\gamma,\nu}(z)\| = O(|z|^{\gamma(\nu+\beta-1)})$, $z \in \Sigma_{\xi-\epsilon}$,

as well as that, for every $\theta \in (0,1)$, $\epsilon \in (0,\xi)$ and $\nu > \theta - \beta$,

(b5) $\|T_{\gamma,\nu}(z)\|_{L(E,E^\theta_{\mathcal{A}})} = O(|z|^{\gamma(\nu+\beta-1-\theta)})$, $z \in \Sigma_{\xi-\epsilon}$.

Keeping in mind (b4)–(b5) and the Cauchy integral formula, it is very simple to prove that for every $\theta \in (0,1)$, $\epsilon \in (0,\xi)$, $\nu > -\beta$ and $n \in \mathbb{N}$,

(b4)' $\|(\mathrm{d}^n/\mathrm{d}z^n)T_{\gamma,\nu}(z)\| = O(|z|^{\gamma(\nu+\beta-1)-n})$, $z \in \Sigma_{\xi-\epsilon}$,

as well as that for every $\theta \in (0,1)$, $\epsilon \in (0,\xi)$, $\nu > \theta - \beta$ and $n \in \mathbb{N}$,

(b5)' $\|(\mathrm{d}^n/\mathrm{d}z^n)T_{\gamma,\nu}(z)\|_{L(E,E^\theta_{\mathcal{A}})} = O(|z|^{\gamma(\nu+\beta-1-\theta)-n})$, $z \in \Sigma_{\xi-\epsilon}$.

In the case that $\epsilon \in (0,\xi)$ and $z \in \Sigma_{\xi-\epsilon}$, then the uniqueness theorem for analytic functions, (a1) and the asymptotic expansion formula for the Wright functions (1) (cf. also the first part of proof of [2, Theorem 3.3]) together imply that $\int_0^\infty z^{-\gamma(1+\nu)} s^\nu \Phi_\gamma(sz^{-\gamma})\,\mathrm{d}s = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$; hence,

$$\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^\infty s^\nu \Phi_\gamma(s\mathrm{e}^{\mathrm{i}\varphi})[\,T(s|z|^\gamma)x - x]\,\mathrm{d}s,$$

where $\varphi = -\gamma\arg(z)$. Keeping in mind this identity, (C), [12, Theorem 3.5] and the proof of [2, Theorem 3.3], we can deduce the following extension of [38, Theorem 3.4 (i)] and the properties (b1), (b3):

(b1)' Suppose that $\epsilon \in (0,\xi)$ and $\delta = \xi - \epsilon$. Then $\lim_{z\to 0, z\in\Sigma_\delta} \frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x = \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, provided that $\theta > 1-\beta$ and $x \in D((-\mathcal{A})^\theta)$, or that $1 > \theta > 1-\beta$ and $x \in E^\theta_{\mathcal{A}}$, $\nu > -\beta$.

(b3)' Suppose that $\epsilon \in (0,\xi)$ and $\delta = \xi - \epsilon$. Then $\left\|\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\right\| = O\big(|z|^{\gamma(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^\theta)]}\big)$, $z \in \Sigma_\delta$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$, and $\left\|\frac{T_{\gamma,\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\right\| = O\big(|z|^{\gamma(\beta+\theta-1)}\|x\|_{E^\theta_{\mathcal{A}}}\big)$, $z \in \Sigma_\delta$, provided $1 > \theta > 1-\beta$, $x \in D((-\mathcal{A})^\theta)$.

Remark. In some cases, the angle of analyticity of considered operator families can be increased depending on the concrete value of constant $c > 0$ from the condition (P). Here we will not discuss this question in more detail.

Following E. Bazhlekova in [2] and R.-N. Wang, D.-H. Chen and T.-J. Xiao in [38], we define

$$S_\gamma(z) := T_{\gamma,0}(z) \quad \text{and} \quad P_\gamma(z) := \gamma T_{\gamma,1}(z)/z^\gamma, \qquad z \in \Sigma_\xi;$$

cf. the proof of [38, Theorem 3.1], where the corresponding operators have been denoted by $\mathcal{S}_\gamma(z)$ and $\mathcal{P}_\gamma(z)$. The analysis contained in the proof of property (b4) enables one to see that the estimate [38, (3.1)] holds on closed subsectors of $\Sigma_{\pi/2-\omega}$ (cf. the formulation of [38, Theorem 3.1]). It is clear that the first statement of [38, Theorem 3.2] holds since the operators $\mathcal{S}_\gamma(z)$ and $\mathcal{P}_\gamma(z)$ depend analytically, in the uniform operator topology, on the complex parameter z belonging to an appropriate sector containing $(0,\infty)$. Furthermore, due to (b4)'–(b5)', we have that for each $\epsilon\in(0,\xi)$ the following holds: $\|(\mathrm{d}/\mathrm{d}z)T_{\gamma,\nu}(z)\|=O(|z|^{\gamma(\nu+\beta-1)-1})$, $z\in\Sigma_{\xi-\epsilon}$ and $\|(\mathrm{d}/\mathrm{d}z)P_\gamma(z)\|=O(|z|^{\gamma(\nu+\beta-1)-1})$, $z\in\Sigma_{\xi-\epsilon}$. By the Darboux inequality and an analyticity argument, it readily follows that for every $R>0$ the mappings $z\mapsto S_\gamma(z)\in L(E)$, $z\in\Sigma_{\xi-\epsilon}\setminus B_R$, and $z\mapsto P_\gamma(z)\in L(E)$, $z\in\Sigma_{\xi-\epsilon}\setminus B_R$, are uniformly continuous. Arguing in such a way, we have proved an extension of the second statement in [38, Theorem 3.2] for degenerate fractional differential equations.

It is clear that $T_{\gamma,\nu}(z)x=z^{-\gamma}\int_0^\infty s^\nu\Phi_\gamma(sz^{-\gamma})T(s)x\,\mathrm{d}s$, $z\in\Sigma_\xi$, $x\in E$ and $s\mapsto(2\pi\mathrm{i})^{-1}\int_\Gamma(-\lambda)^\theta \mathrm{e}^{s\lambda}(\lambda-\mathcal{A})^{-1}\,\mathrm{d}\lambda$ is a bounded linear section of the operator $(-\mathcal{A})^\theta T(s)$ for $\theta>1-\beta$ and $s\geq 0$ (cf. [12, Proposition 3.2, pages 48–49]). Along with Lemma 2.1, definition of $P_\gamma(\cdot)$ and above-mentioned proposition, the above implies:

$$\begin{aligned}P_{\gamma,\theta}(z)x:=\frac{\gamma z^{-2\gamma}}{2\pi\mathrm{i}}\int_0^\infty & s\Phi_\gamma(sz^{-\gamma})\\ &\times\left[\int_\Gamma(-\lambda)^\theta\mathrm{e}^{s\lambda}(\lambda-\mathcal{A})^{-1}x\,\mathrm{d}\lambda\right]\mathrm{d}s\in(-\mathcal{A})^\theta P_\gamma(z)x\end{aligned}\tag{10}$$

for all $z\in\Sigma_\xi$ and $x\in E$, as well as that $(P_{\gamma,\theta}(z))_{z\in\Sigma_\xi}\subseteq L(E)$ for $\theta>1-\beta$. By the foregoing, we have that:

$$\|P_{\gamma,\theta}(z)\|=O(|z|^{\gamma(\beta-\theta-1)}),\qquad z\in\Sigma_{\xi-\epsilon},\ \epsilon\in(0,\xi).\tag{11}$$

Differentiating (7), it is not difficult to prove that

$$\frac{\mathrm{d}}{\mathrm{d}z}S_\gamma(z)x=\frac{\gamma z^{-\gamma-1}}{2\pi\mathrm{i}}\int_0^\infty s\,\Phi_\gamma(sz^{-\gamma})T'(s)x\,\mathrm{d}s,\qquad z\in\Sigma_\xi,\ x\in E.\tag{12}$$

Applying (10) with $\theta=1$, and (12), we get that:

$$\frac{\mathrm{d}}{\mathrm{d}z}S_\gamma(z)x=-z^{\gamma-1}P_{\gamma,1}(z)x\in z^{\gamma-1}\mathcal{A}P_\gamma(z)x,\qquad z\in\Sigma_\xi,\ x\in E.\tag{13}$$

Further on, Lemma 2.1 and Lemma 2.2 can serve one to prove that the assumption $y\in\mathcal{A}x$ implies $S_\gamma(z)y\in\mathcal{A}S_\gamma(z)x$ and $P_\gamma(z)y\in\mathcal{A}P_\gamma(z)x$, $z\in\Sigma_\xi$, so that the mapping $t\mapsto(\mathrm{d}/\mathrm{d}t)S_\gamma(t)x$, $t>0$ is locally integrable for any $x\in D(\mathcal{A})$ by (13). Keeping in mind (b4), we have proved an extension of [38, Theorem 3.3] to degenerate fractional differential equations. Before proceeding further, we would like to point out that for every $x\in D((-\mathcal{A})^\theta)\cap E^\theta_{\mathcal{A}}$ the mapping $z\mapsto(\mathrm{d}/\mathrm{d}z)S_\gamma(z)x=(\mathrm{d}/\mathrm{d}z)[S_\gamma(z)x-x]$ is bounded by $|z|^{\gamma(\beta+\theta-1)-1}$ on subsectors of Σ_ξ, $1>\theta>1-\beta$; this follows from the Cauchy integral formula and the property (b3)' with $\nu = 0$. In particular, the mapping $t \mapsto (\mathrm{d}/\mathrm{d}t)S_\gamma(t)x$, $t > 0$, is locally integrable for any $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$, where $1 > \theta > 1 - \beta$.

Suppose that $(x, y) \in \mathcal{A}$. Then an elementary application of Cauchy formula, combined with Lemma 2.2 and definition of $T(\cdot)$, implies that $T(s)y = T'(s)x$, $s > 0$. Having in mind (6) with $\nu = 1$, and definition of $P_\gamma(\cdot)$, it readily follows that $P_\gamma(z)y = -P_{\gamma,1}(z)x$, $z \in \Sigma_\xi$; therefore, $(\mathrm{d}/\mathrm{d}z)S_\gamma(z)x = z^{\gamma-1}P_\gamma(z)y$, provided $z \in \Sigma_\xi$ and $(x, y) \in \mathcal{A}$. After integration, we obtain that, under the same conditions, $S_\gamma(z)x - x = \int_0^z \lambda^{\gamma-1}P_\gamma(\lambda)y\,\mathrm{d}\lambda$. This extends the assertion of [38, Theorem 3.4 (ii)].

Suppose again that $(x, y) \in \mathcal{A}$. Performing the Laplace transform, we obtain with the help of (b2) and Lemma 2.2 that $(g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t) = \int_0^t S_\gamma(s)y\,\mathrm{d}s$, $t \geq 0$. This immediately implies that $\mathbf{D}_t^\gamma S_\gamma(t)x = S_\gamma(t)y \in \mathcal{A}S_\gamma(t)x$, $t > 0$, which extends the assertion of [38, Theorem 3.4 (iii)]. The original proof of this result, much more complicated than ours, is based on the use of functional calculus for almost sectorial operators established by F. Periago and B. Straub in [30] (it is very difficult to develop a similar calculus for almost sectorial multivalued linear operators). Furthermore, we want to observe that this result is not optimal. Speaking matter-of-factly, let $1 > \theta > 1 - \beta$ and let $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$ be fixed. Then the mapping $t \mapsto F(t) := (g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t)$, $t \geq 0$, is continuous and its restriction on $(0, \infty)$ can be analytically extended to the sector Σ_ξ, with the estimate $\|F(z)\| = O(|z|^{\gamma(\beta+\theta-2)+1})$ on any proper subsector of Σ_ξ (cf. (b3)'). By the Cauchy integral formula, we obtain that $\|F'(z)\| = O(|z|^{\gamma(\beta+\theta-2)})$ on proper subsectors of Σ_ξ. In particular, the Caputo fractional derivative $\mathbf{D}_t^\gamma S_\gamma(t)x$ is defined. On the other hand, Lemma 2.1 in combination with [12, Proposition 3.2, 3.4] implies that

$$t \mapsto F_\gamma(t)x := \frac{1}{2\pi\mathrm{i}} \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma}) \Big[\int_\Gamma \lambda \mathrm{e}^{\lambda s} R(\lambda : \mathcal{A})x\,\mathrm{d}\lambda \Big]\,\mathrm{d}s, \qquad t > 0,$$

is a continuous section of the multivalued mapping $\mathcal{A}S_\gamma(t)x$, $t > 0$, with the clear meaning. Then $T(t)x - x = \int_0^t T'(s)x\,\mathrm{d}s$, $t \geq 0$, and $T'(t)x = (2\pi\mathrm{i})^{-1}\int_\Gamma \lambda \mathrm{e}^{\lambda t} R \times (\lambda : \mathcal{A})x\,\mathrm{d}\lambda$, $t > 0$, which simply implies by (5) that

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-zt}T'(t)x\,\mathrm{d}t &= zR(z : \mathcal{A})x - x \\ &= \int_0^\infty \mathrm{e}^{-zt}\Big[\frac{1}{2\pi\mathrm{i}}\int_\Gamma \lambda \mathrm{e}^{\lambda t}R(\lambda : \mathcal{A})x\,\mathrm{d}\lambda\Big]\,\mathrm{d}z, \qquad z > 0. \end{aligned} \tag{14}$$

Using Fubini theorem, definition of $F_\gamma(\cdot)$ and the identity [2, (3.10)], we get that

$$\int_0^\infty \mathrm{e}^{-zt}F_\gamma(t)x\,\mathrm{d}z = z^{\gamma-1}\int_0^\infty \mathrm{e}^{-z^\gamma t}\Big[\frac{1}{2\pi\mathrm{i}}\int_\Gamma \lambda \mathrm{e}^{\lambda t}R(\lambda : \mathcal{A})x\,\mathrm{d}\lambda\Big]\,\mathrm{d}z, \qquad z > 0,$$

which clearly implies with the help of (14) that:

$$\int_0^\infty \mathrm{e}^{-zt}\int_0^t F_\gamma(s)x\,\mathrm{d}s\,\mathrm{d}z = z^{\gamma-2}\int_0^\infty \mathrm{e}^{-z^\gamma t}\left[\frac{1}{2\pi \mathrm{i}}\int_\Gamma \lambda \mathrm{e}^{\lambda t}R(\lambda:\mathcal{A})x\,\mathrm{d}\lambda\right]\mathrm{d}z$$
$$= z^{\gamma-2}[z^\gamma R(z^\gamma:\mathcal{A})x - x], \qquad z>0.$$

Using this equation, (b2) and the uniqueness theorem for Laplace transform, it readily follows that

$$(g_{1-\gamma} * [S_\gamma(\cdot)x - x])(t) = \int_0^t F_\gamma(s)x\,\mathrm{d}s, \qquad t\geq 0.$$

Now it becomes clear that:

$$\text{(15)}\qquad \mathbf{D}_t^\gamma S_\gamma(t)x = F_\gamma(t)x \in \mathcal{A}S_\gamma(t)x,$$
$$t>0,\ x\in D((-\mathcal{A})^\theta)\cap E_{\mathcal{A}}^\theta,\ 1>\theta>1-\beta.$$

The identity [2, (3.10)] almost immediately implies that

$$\int_0^\infty \mathrm{e}^{-\lambda t}t^{-\gamma-1}\Phi_\gamma(st^{-\gamma})\,\mathrm{d}t = \frac{1}{\gamma s}\mathrm{e}^{-\lambda^\gamma s}, \qquad s>0,\ \lambda>0.$$

Keeping in mind this equality and (5), we get that

$$\int_0^\infty\int_0^\infty \gamma s T(s)x[\mathrm{e}^{-\lambda t}t^{-\gamma-1}\Phi_\gamma(st^{-\gamma})\,\mathrm{d}t]\,\mathrm{d}s = (\lambda^\gamma - \mathcal{A})^{-1}x, \qquad \lambda>0,\ x\in E.$$

Using Fubini theorem and definition of $T_{\gamma,1}(\cdot)$, the above yields

$$\int_0^\infty \mathrm{e}^{-\lambda t}t^{\gamma-1}P_\gamma(t)x\,\mathrm{d}t = (\lambda^\gamma - \mathcal{A})^{-1}x, \qquad \lambda>0,\ x\in E.$$

By (b2) and the uniqueness theorem for Laplace transform, we obtain finally the following generalization of [38, Theorem 3.4 (iv)]:

$$\text{(16)}\qquad S_\gamma(t)x = (g_{1-\gamma} * [\cdot^{\gamma-1}P_\gamma(\cdot)x])(t), \qquad t>0,\ x\in E.$$

This identity continues to hold on sector Σ_ξ.

Arguing as in non-degenerate case (cf. [38, Lemma 3.1, Theorem 3.5]), we can prove that the compactness of $R(\lambda:\mathcal{A})$ for some $\lambda\in\rho(\mathcal{A})$ implies the compactness of operators $S_\gamma(t)$ and $P_\gamma(t)$ for all $t>0$.

The consideration carried out in [38, Lemma 4.1] is completely meaningful for abstract degenerate fractional differential equations and gives rise us to introduce the following definition (cf. [38, Definition 4.1, Definition 4.2] and compare to [12, Definition, page 53]):

Definition 3.1. Let $T \in (0, \infty)$ and $f \in L^1((0,T) : E)$. Consider the following abstract degenerate fractional inclusion:

$$(\mathrm{DFP})_f : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), & t \in (0,T], \\ u(0) = u_0. \end{cases}$$

(i) By a mild solution of $(\mathrm{DFP})_f$, we mean a function

$$u(t) = S_\gamma(t)u_0 + \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\,\mathrm{d}s, \qquad t \in (0,T].$$

(ii) By a classical solution of $(\mathrm{DFP})_f$, we mean any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^{\gamma} u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^{\gamma} u(t) - f(t) \in \mathcal{A}u(t)$ for $t \in (0,T]$.

A mild solution $u(t)$ of problem $(\mathrm{DFP})_f$ is automatically continuous on $(0,T]$. If $x \in D((-\mathcal{A})^\theta) \cap E^\theta_{\mathcal{A}}$, where $1 > \theta > 1 - \beta$, then (15) implies that the mapping $u(t) = S_\gamma(t)x$ is a classical solution of $(\mathrm{DFP})_f$ with $f \equiv 0$.

The following theorem is an important extension of [38, Theorem 4.1], even for non-degenerate fractional differential equations with almost sectorial operators.

Theorem 3.1. *Suppose $T \in (0,\infty)$, $1 > \theta > 1 - \beta$, $u_0 \in D((-\mathcal{A})^\theta)$, or $u_0 \in E^\theta_{\mathcal{A}}$, there exist constants $\sigma > \gamma(1-\beta)$ and $M \geq 1$ such that*

$$\|f(t) - f(s)\| \leq M|t-s|^\sigma, \qquad 0 < t,\ s \leq T, \tag{17}$$

and

$$f \in L^\infty((0,T) : [D((-\mathcal{A})^\theta)]), \qquad \text{or } f \in L^\infty((0,T) : E^\theta_{\mathcal{A}}).$$

Then there exists a unique classical solution of problem $(\mathrm{DFP})_f$.

Proof: We will prove the theorem only in the case that $u_0 \in D((-\mathcal{A})^\theta)$. The uniqueness of classical solutions of problem $(\mathrm{DFP})_f$ is an immediate consequence of Ljubich uniqueness type theorem [20, Theorem 3.1.6] and, by the foregoing arguments, it suffices to show that the function

$$\omega(t) := \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\,\mathrm{d}s, \qquad 0 \leq t \leq T,$$

enjoys the following properties:

(i) $\omega(t)$ is continuous at the point $t = 0$;

(ii) $\mathbf{D}_t^{\gamma}\omega(t) = \omega_1(t) := \int_0^t S'_\gamma(t-s)f(s)\,\mathrm{d}s + f(t)$, $0 < t \leq T$, and $\omega_1(t)$ is continuous on $(0,T]$;

(iii) $\omega_2(t) := \omega_1(t) - f(t) = \int_0^t S'_\gamma(t-s)f(s)\,\mathrm{d}s \in \mathcal{A}\omega(t)$, $0 < t \leq T$.

The statement (i) follows from the Hölder continuity of $f(\cdot)$ (cf. (17)) and a simple computation involving the estimate $\|P_\gamma(t)\| = O(t^{\gamma(\beta-1)})$, $t > 0$. For the proof of (ii), we will have to observe that (12) in combination with contour representation of $T'(\cdot)$ and [12, Proposition 3.2, Proposition 3.4] implies that there exist constants $C_\theta > 0$ and $C'_\theta > 0$ such that for every $0 < s \leq T$ and $0 < \omega \leq T$,

$$
\begin{aligned}
\|S'_\gamma(\omega)f(s)\| &= \left\| \frac{\gamma}{2\pi \mathrm{i}} \int_0^\infty v\omega^{\gamma-1}\Phi_\gamma(v)T'(v\omega^\gamma)f(s)\,\mathrm{d}v \right\| \\
&\leq C_\theta \frac{\gamma}{2\pi} \|f(s)\|_{[D((-\mathcal{A})^\theta)]} \int_0^\infty v\omega^{\gamma-1}\Phi_\gamma(v)(v\omega^\gamma)^{\beta+\theta-2}\,\mathrm{d}v \\
&= C_\theta \frac{\gamma}{2\pi} \|f(s)\|_{[D((-\mathcal{A})^\theta)]}\omega^{\gamma(\beta+\theta-1)-1} \int_0^\infty v^{\beta+\theta-1}\Phi_\gamma(v)\,\mathrm{d}v \\
&= C'_\theta \|f(s)\|_{[D((-\mathcal{A})^\theta)]}\omega^{\gamma(\beta+\theta-1)-1}.
\end{aligned} \tag{18}
$$

Using this estimate with $\omega = t - s$, where $0 < s < t \leq T$, and integrating the obtained estimate along the interval $[0, T]$ in variable s, we get that there exists a constant $C''_\theta > 0$ such that for every $0 < t \leq T$

$$
\begin{aligned}
\left\| \int_0^t S'_\gamma(t-s)f(s)\,\mathrm{d}s \right\| &\leq C''_\theta \int_0^t (t-s)^{\gamma(\beta+\theta-1)-1}\|f(s)\|_{[D((-\mathcal{A})^\theta)]}\,\mathrm{d}s \\
&\leq \frac{C''_\theta}{\gamma(\beta+\theta-1)} t^{\gamma(\beta+\theta-1)}\|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}.
\end{aligned} \tag{19}
$$

Let $h > 0$ and let $h \leq T - t$ for some fixed $0 < t < T$. Making use of (18) and dominated convergence theorem, we obtain that

$$
\lim_{h\to 0+} \int_0^t \frac{S_\gamma(t+h-s) - S_\gamma(t-s)}{h} f(s)\,\mathrm{d}s = \int_0^t S'_\gamma(t-s)f(s)\,\mathrm{d}s; \tag{20}
$$

here it is only worth noting that (18) and the mean value theorem together imply that for every $s \in (0, t)$

$$
\begin{aligned}
\left\| \frac{S_\gamma(t+h-s) - S_\gamma(t-s)}{h} f(s) \right\| &\leq \frac{1}{h} \int_{t-s}^{t+h-s} \|S'_\gamma(r)f(s)\|\,\mathrm{d}r \\
&\leq \frac{\|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}}{h} \int_{t-s}^{t+h-s} r^{\gamma(\beta+\theta-1)-1}\,\mathrm{d}r \\
&\leq \mathrm{Const.}[(t-s)^{\gamma(\beta+\theta-1)-1} + (t+1-s)^{\gamma(\beta+\theta-1)-1}].
\end{aligned}
$$

Having in mind the estimate $\|S_\gamma(t)\| = O(t^{\gamma(\beta-1)})$, $t > 0$, the strong continuity of operator family $S_\gamma(\cdot)$ on $D((-\mathcal{A})^\theta)$ and the Hölder continuity of $f(\cdot)$, we can repeat almost literally the arguments from the corresponding part of proof of [38, Theorem 4.1] in order to see that

$$\lim_{h\to 0_+}\frac{1}{h}\int_t^{t+h} S_\gamma(t+h-s)f(s)\,\mathrm{d}s = f(t). \tag{21}$$

Due to (20)–(21), we have that the mapping $t\mapsto \int_0^t S_\gamma(t-s)f(s)\,\mathrm{d}s$, $0<t<T$, is differentiable from the right; we can similarly prove the differentiability of this mapping from the left for $0<t\leq T$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S_\gamma(t-s)f(s)\,\mathrm{d}s = \int_0^t S_\gamma'(t-s)f(s)\,\mathrm{d}s + f(t), \qquad 0<t\leq T.$$

Now it is not difficult to prove with the help of (16) and (19) that $\mathbf{D}_t^\gamma\omega(t)$ exists and equals to $\omega_1(t)$, as claimed. Now we will prove that the mapping $t\mapsto\int_0^t S_\gamma'\times (t-s)f(s)\,\mathrm{d}s$, $0<t\leq T$, is continuous (observe here that this mapping is continuous for $t=0$; cf. (19)). As in the proof of [38, Theorem 4.1], we have $\int_0^t S_\gamma'(t-s)f(s)\,\mathrm{d}s = I_1(t)+I_2(t)$, where $I_1(t):=\int_0^t S_\gamma'(t-s)[f(s)-f(t)]\,\mathrm{d}s$ and $I_2(t):=\int_0^t S_\gamma'(t-s)f(t)\,\mathrm{d}s$. By (b3), we have that $I_2(t+h)\to I_2(t)$ as $h\to 0$ for $0<t\leq T$ and the meaning is clear. To complete the whole proof, it suffices to show that the mapping $I_1(t):=\int_0^t S_\gamma'(t-s)[f(s)-f(t)]\,\mathrm{d}s$, $0<t\leq T$, is continuous. For the sake of brevity, we will only prove that the above mapping is continuous from the right for $0<t<T$. Suppose, as above, $h>0$ and $h\leq T-t$. Then

$$I_1(t+h)-I_1(t) = h_1(t)+h_2(t)+h_3(t),$$

where

$$h_1(t) := \int_0^t (S_\gamma'(t+h-s)-S_\gamma'(t-s))[f(s)-f(t)]\,\mathrm{d}s,$$
$$h_2(t) := \int_0^t S_\gamma'(t+h-s)[f(t)-f(t+h)]\,\mathrm{d}s$$

and

$$h_3(t) := \int_t^{t+h} S_\gamma'(t+h-s)[f(s)-f(t+h)]\,\mathrm{d}s.$$

We can prove that $h_1(t)\to 0$ as $h\to 0_+$ by using the dominated convergence theorem and the following estimates (cf. (18)–(19)):

$$\begin{aligned}&\left\|\int_0^t S_\gamma'(t+h-s)[f(s)-f(t)]\,\mathrm{d}s\right\|\\ &\quad\leq \text{Const.}(t+h-s)^{\gamma(\beta+\theta-1)-1}\|f(s)-f(t)\|_{[D((-\mathcal{A})^\theta)]}\\ &\quad\leq \text{Const.}\ \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}\big[(t-s)^{\gamma(\beta+\theta-1)-1}+(t+1-s)^{\gamma(\beta+\theta-1)-1}\big]\end{aligned}$$

and

$$\Big\| \int_0^t S'_\gamma(t-s)[f(s)-f(t)]\,\mathrm{d}s \Big\| \leq \frac{2C''_\theta}{\gamma(\beta+\theta-1)} t^{\gamma(\beta+\theta-1)} \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])}.$$

On the other hand, we may conclude that $h_2(t) \to 0$ as $h \to 0_+$ by using the estimate $\|S'_\gamma(t)\| = O(t^{\gamma(\beta-1)-1})$, $t > 0$, the Hölder continuity of $f(\cdot)$ and our standing assumption $\sigma > \gamma(1-\beta)$:

$$\begin{aligned}\|h_2(t)\| &\leq \text{Const.} \int_0^t (t+h-s)^{\gamma(\beta-1)-1} h^\sigma \,\mathrm{d}s \\ &\leq \text{Const.}\ h^\sigma[(t+h)^{\gamma(\beta-1)} - h^{\gamma(\beta-1)}] \to 0 \qquad \text{as } \ h \to 0_+.\end{aligned}$$

Finally, an application of (18) yields:

$$\begin{aligned}\|h_3(t)\| &\leq \text{Const.} \int_t^{t+h} (t+h-s)^{\gamma(\beta+\theta-1)-1} \|f(s)-f(t+h)\|_{[D((-\mathcal{A})^\theta)]} \,\mathrm{d}s \\ &\leq \text{Const.}\ \|f(\cdot)\|_{L^\infty((0,T):[D((-\mathcal{A})^\theta)])} h^{\gamma(\beta+\theta-1)} \to 0 \qquad \text{as } \ h \to 0_+.\end{aligned}$$

This completes the proof of (ii). The proof of (iii) follows by applying (13), (19) and Lemma 2.1. □

Remark. It is clear that the validity of condition (17) implies that the sequence $(f_n(t))_{n\in\mathbb{N}} \subseteq C([0,T]:E)$, where $f_n(t) := f(t)$ for $t \in [1/n, T]$ and $f_n(t) := f(1/n)$ for $t \in [0, 1/n]$, is a Cauchy sequence in $C([0,T]:E)$ and therefore convergent. Hence, there exists $\lim_{t\to 0+} f(t)$ in E and $f(t)$ can be extended to a Hölder continuous function from the space $C^\sigma([0,T]:E)$. This implies that the Caputo fractional derivative $\mathbf{D}_t^\gamma \omega(t)$ (cf. (ii)) is defined in the strong sense [2, page 11, line −3] and that (ii) holds, in fact, for $0 \leq t \leq T$.

Theorem 3.1 can be applied in the analysis of a large class of relaxation degenerate differential equations that are subordinated to degenerate differential equations of first order that have been already considered in [12, Section 3.7]. For example, Theorem 3.1 can be almost straightforwardly applied in the analysis of the following inhomogeneous fractional Poisson heat equation in the space $L^p(\Omega)$:

$$(\mathrm{P})_\gamma: \begin{cases} \mathbf{D}_t^\gamma[m(x)v(t,x)] = \Delta v - bv + f(t,x), & t \geq 0,\ x \in \Omega, \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $0 < \gamma < 1$. As already mentioned, Theorem 3.1 is new even for non-degenerate fractional differential equations with almost sectorial operators, so that we can consider the wellposedness results for abstract relaxation degenerate differential equations with higher order elliptic differential operators with variable coefficients in Hölder spaces (cf. [37] for more details).

4. Semilinear degenerate Cauchy inclusions

Assume that the condition (P) holds. In this section, we consider the following semilinear degenerate fractional Cauchy inclusion:

$$(\mathrm{DFP})_{f,s,\gamma}\colon \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t,u(t)), & t \in (0,T], \\ u(0) = u_0, \end{cases}$$

where $T \in (0,\infty)$ and $0 < \gamma \leq 1$ (for more details about abstract semilinear degenerate Cauchy problems with integer order derivatives, we refer the reader to [3]–[5], [11]–[15], [19], [33] and [35]). We will use the terminology from the previous section.

Suppose first that $0 < \gamma < 1$. By a mild solution $u(t) := u(t;u_0)$ of problem $(\mathrm{DFP})_{f,s,\gamma}$, we mean any function $u \in C((0,T] : E)$ such that

$$u(t) = S_\gamma(t)u_0 + \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s,u(s))\,\mathrm{d}s, \qquad t \in (0,T].$$

As in linear case, a classical solution of $(\mathrm{DFP})_f$ is any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^{\gamma} u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^{\gamma} u(t) - f(t,u(t)) \in \mathcal{A}u(t)$ for $t \in (0,T]$. In [38, Theorem 5.1, Theorem 5.3, Corollary 5.1], the authors have applied various types of fixed point theorems in the study of existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{f,s,\gamma}$, provided that the operator $\mathcal{A}$ is single-valued, linear and almost sectorial. In contrast to the assertions of [38, Theorem 5.2, Theorem 5.4], the above-mentioned results can be immediately extended to semilinear degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\gamma}$. Situation is the same with the assertions of [18, Theorem 2.1] and [23, Theorem 3.1] (in the last mentioned theorem, F. Li has considered the existence of mild solutions for a class of delay semilinear fractional differential equations).

Following T. Dlotko [9], we can similarly define the notions of mild and classical solutions of semilinear degenerate Cauchy inclusion $(\mathrm{DFP})_{f,s,1}$ of first order $\gamma = 1$, that is, any function $u \in C((0,T] : E)$ such that

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s,u(s))\,\mathrm{d}s, \qquad t \in (0,T]$$

is said to be a mild solution of problem $(\mathrm{DFP})_{f,s,1}$. By a classical solution, we mean any function $u \in C([0,T] : E) \cap C^1((0,T] : E)$ such that $u(t) \in D(\mathcal{A})$, $t \in (0,T]$, $u(0) = u_0$ and $u'(t) \in \mathcal{A}u(t) + f(t,u(t))$, $t \in (0,T]$. The extensions of [9, Theorem 1, Proposition 2] for semilinear degenerate Cauchy inclusions of first order can be simply proved.

In the remaining part of paper, we will reconsider the assertions of [29, Theorem 3.1, Theorem 3.2] for semilinear degenerate Cauchy inclusions; cf. [28, Theorem 1.4, page 185] for a pioneering result in this direction.

As the next two theorems show, Theorem 3.1 and Theorem 3.2 of [29] can be fully generalized to semilinear degenerate Cauchy inclusions of first order.

Theorem 4.1. *Let $T > 0$, and let $\gamma = 1$. Suppose that the following condition holds.*

(H): *the mapping $f \colon [0,T] \times E \to E$ is continuous in t on $[0,T]$ and for each $t_0 > 0$ and $K > 0$ there exists a constant $L(t_0,K) > 0$ such that $\|f(t,x) - f(t,y)\| \le L(t_0,K)\|x-y\|$, provided $0 < t < t_0$, $x,y \in E$ and $\|x\|$, $\|y\| \le K$.*

Denote by $\mathbf{\Omega}$ the domain of continuity of semigroup $(T(t))_{t\ge 0}$; that is, $\mathbf{\Omega} = \{x \in E \colon \lim_{t\to 0_+} T(t)x = x\}$. Then, for every $u_0 \in \mathbf{\Omega}$, there exist a number $\tau_{\max} = \tau_{\max}(u_0) > 0$ and a unique mild solution $u \in C([0,\tau_{\max}) : E)$ of problem $(\mathrm{DFP})_{f,s,1}$. If

(i) *$f(t,x) \in D(\mathcal{A})$ for all $t > 0$ and $x \in \mathbf{\Omega}$;*

(ii) *for each $t_0 > 0$ and $K > 0$ there exists a constant $C = C(t_0,K) > 0$ such that*

$$\|f(t,x)\|_{[D(\mathcal{A})]} \le C \quad \text{for all } x \in \mathbf{\Omega} \text{ with } \|x\| \le C \text{ and } 0 < t < t_0; \tag{22}$$

(iii) *there exists a function $g \in L^\infty_{\mathrm{loc}}((0,\infty) : \mathbb{R})$ such that*

$$\|f(t,x)\| \le g(t)\|x\| \quad \text{a.e. } t \ge 0 \text{ and } x \in \mathbf{\Omega},$$

then $\tau_{\max} = \infty$.

Proof: The proof is almost the same as that of [29, Theorem 3.1]; here we only want to observe that the term

$$\int_0^t \big\|[T(\tau_{\max}-s) - T(\tau_{\max}-s)]f(s,u(s))\big\|\,\mathrm{d}s,$$

appearing on [29, page 418, l. 11], can be estimated with the help of mean value theorem, (22) and [12, Proposition 3.2, 3.4], as follows:

$$\int_0^t \big\|[T(\tau_{\max}-s) - T(\tau_{\max}-s)]f(s,u(s))\big\|\,\mathrm{d}s \le (\tau_{\max}-t)C_1Ct^\beta/\beta,$$

where C is the constant from (22) and C_1 is the constant from the formulation of [12, Proposition 3.4]. □

For the sequel, we need the following. Suppose that $y \in (-\mathcal{A})^\theta x$, where $1 > \theta > 1-\beta$. Then (2) and the obvious equality $(s-\mathcal{A})^{-1}T(t)y = T(t)(s-\mathcal{A})^{-1}y$, $t,s > 0$ together imply

$$\begin{aligned}(-\mathcal{A})^{-\theta}T(t)y &= \frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(s-\mathcal{A})^{-1}T(t)y\,\mathrm{d}s\\ &= T(t)\frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(s-\mathcal{A})^{-1}y\,\mathrm{d}s\end{aligned}$$

$$= T(t)(-\mathcal{A})^{-\theta}y = T(t)x, \qquad t > 0.$$

Hence,

$$T(t)(-\mathcal{A})^{\theta} \subseteq (-\mathcal{A})^{\theta}T(t), \qquad t > 0,\ 1 > \theta > 1-\beta. \tag{23}$$

Owing to (23), we can estimate the term $\|u(t;u_0) - u(t;u_1)\|_{[D((-\mathcal{A})^{\theta})]}$ (cf. line 11 of Step 2, page 420, the proof of [29, Theorem 3.2]) as in non-degenerate case; furthermore, on the same page of proof, we can use [12, Theorem 3.5] ([12, Proposition 3.2]) in place of [30, Theorem 3.9 (vii)] ([30, Theorem 3.9 (iii)]). Keeping in mind these observations, we can formulate the following extension of [29, Theorem 3.2] for abstract degenerate Cauchy inclusions of first order.

Theorem 4.2. *Let $\gamma = 1$, and let condition* (H) *hold. Suppose that $\beta > \theta > 1-\beta$ and $0 < t < \tau_{\max}(u_0)$. Then there exist $r > 0$ and $K > 0$ such that the assumption $u_1 \in B_{\theta,r}(u_0) := \{u \in D((-\mathcal{A})^{\theta})\colon \|u-u_0\|_{[D((-\mathcal{A})^{\theta})]} \leq r\}$ implies that there exists a unique mild solution $u(t;u_1) \in C([0,\tau_{\max}(u_1)) : E)$ of problem* $(\mathrm{DFP})_{f,s,1}$ *with $\tau_{\max}(u_1) \geq \tau$. Moreover,*

$$\|u(t;u_0) - u(t;u_1)\| \leq K\|u_0 - u_1\|_{[D((-\mathcal{A})^{\theta})]}, \qquad 0 \leq t \leq \tau,$$

and for every $\epsilon \in (0,\tau)$ there exists a constant $C_\epsilon > 0$ such that

$$\|u(t;u_0) - u(t;u_1)\|_{[D((-\mathcal{A})^{\theta})]} \leq C_\epsilon\|u_0 - u_1\|_{[D((-\mathcal{A})^{\theta})]}, \qquad \epsilon \leq t \leq \tau.$$

The situation is much more complicated if we consider abstract degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\gamma}$ of order $\gamma \in (0,1)$. Concerning [29, Theorem 3.1], we would like to point out that we cannot use, in fractional case, the well-known procedure for construction of a mild solution of problem $(\mathrm{DFP})_{f,s,\gamma}$ defined in a maximal time interval (see e.g. the integral equation [30, (8), page 417]). The best we can do is to prove the local existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{f,s,\gamma}$, as it has been explained in [38, Remark 4.1].

Concerning [29, Theorem 3.2], we can prove the following:

Theorem 4.3. *Let $\gamma \in (0,1)$, and let condition* (H) *hold. Suppose that $\beta > \theta > 1-\beta$. Then there exist $r > 0$, $\tau > 0$ and $K > 0$ such that, for every $u_1 \in B_{\theta,r}(u_0)$, there exists a unique mild solution $u(t;u_1) \in C([0,\tau] : E)$ of problem* $(\mathrm{DFP})_{f,s,\gamma}$. *Moreover,*

$$\|u(t;u_0) - u(t;u_1)\| \leq K\|u_0 - u_1\|_{[D((-\mathcal{A})^{\theta})]}, \qquad 0 \leq t \leq \tau,$$

and there exists a constant $C > 0$ such that for every $\epsilon \in (0,\tau)$ we have

$$\|u(t;u_0) - u(t;u_1)\|_{[D((-\mathcal{A})^{\theta})]} \leq C\epsilon^{\gamma(\beta-1)}\|u_0 - u_1\|_{[D((-\mathcal{A})^{\theta})]}, \qquad \epsilon \leq t \leq \tau.$$

Proof: We will only outline the most relevant details of proof in degenerate case.

(1) Line 6 of Step 1, page 419, the proof of [29, Theorem 3.2]: Due to (18), we have that

$$\begin{aligned} \|S_\gamma(t)x - x\| &= \left\| \int_0^t S'_\gamma(s)x \, \mathrm{d}s \right\| \\ &\le C'_\theta t^{\gamma(\beta+\theta-1)} \|x\|_{[D((-\mathcal{A})^\theta)]}, \qquad t > 0; \end{aligned}$$

therefore, $\lim_{t\to 0_+} \|S_\gamma(t)u_1 - u_1\| = 0$ uniformly on the ball $B_{\theta,r}(u_0)$.

(2) Line 1, [29, page 420]: Here we may apply (b3) in order to get the existence of a constant $c_\theta > 0$ such that

$$\|S_\gamma(t)(u_1 - u_0)\| \le c_\theta t^{\gamma(\beta+\theta-1)} \|u_1 - u_0\|_{[D((-\mathcal{A})^\theta)]}, \qquad t > 0.$$

(3) By (23), Lemma 2.1 and definition of $S_\gamma(\cdot)$, we have that

$$S_\gamma(t)(-\mathcal{A})^\theta \subseteq (-\mathcal{A})^\theta S_\gamma(t), \qquad t > 0,\ 1 > \theta > 1 - \beta. \tag{24}$$

From this, we may conclude that

$$\begin{aligned} \|S_\gamma(t)x\|_{[D((-\mathcal{A})^\theta)]} &\le \|S_\gamma(t)\| \|x\|_{[D((-\mathcal{A})^\theta)]} \\ &= O(t^{\gamma(\beta-1)} \|x\|_{[D((-\mathcal{A})^\theta)]}), \qquad t > 0. \end{aligned} \tag{25}$$

On the other hand, (10) and Lemma 2.1 imply that

$$\begin{aligned} &\frac{\gamma}{2\pi \mathrm{i}} \int_0^t (t-s)^{-\gamma-1} \int_0^\infty r\Phi_\gamma(r(t-s)^{-\gamma}) \\ &\quad \times \left[\int_\Gamma (-\lambda)^\theta \mathrm{e}^{r\lambda} (\lambda - \mathcal{A})^{-1} f(s, u(s; u_1)) \, \mathrm{d}\lambda \right] \mathrm{d}r \, \mathrm{d}s \\ &\quad \in (-\mathcal{A})^\theta \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s, u(s; u_1)) \, \mathrm{d}s, \qquad t > 0,\ 0 < s \le \tau, \end{aligned} \tag{26}$$

since $\beta > \theta$ and the norm of integrand in the first line by (11) does not exceed $(t-s)^{\gamma(\beta-\theta-1)} \|f(s, u(s; u_1))\|$. Hence, $u(t; u_1) \in D((-\mathcal{A})^\theta)$ for all $t \in [0, \tau]$ and $u_1 \in B_{\theta,r}(u_0)$. For a fixed element $u_1 \in B_{\theta,r}(u_0)$, the continuity of mapping $t \mapsto u(t; u_1) \in [D((-\mathcal{A})^\theta)]$, $t \in (0, \tau]$, follows from (25), the analyticity of $S_\gamma(\cdot)$, the expression (26) and the dominated convergence theorem. The remaining part of proof of [29, Theorem 3.2] can be repeated verbatim. □

Observe that Theorem 4.2 and Theorem 4.3 continue to hold if we consider the space $E^\theta_{\mathcal{A}}$ in place of $[D((-\mathcal{A})^\theta)]$. These theorems, as well some theorems previously considered in this section, require the condition $\beta > 1/2$, which seems to be restrictive in degenerate case (in a great number of examples from [12, Chapter III], the condition (P) holds with $\beta = 1/2$). For example, in the case of consideration of semilinear analogons of problem $(P)_\gamma$, Theorem 4.2 and Theorem 4.3 can be applied provided the additional condition [12, (3.42)] on the function $m(x)$, which ensures us to get the better exponent $\beta = 1/(2-\rho)$ in (P), with $0 < \rho \leq 1$.

During the peer-review process, the author has reconsidered and slightly improved the results of this paper for abstract degenerate fractional differential inclusions with multivalued linear operators satisfying the following condition (cf. [21] for more details):

(QP): There exist finite numbers $0 < \beta \leq 1$, $0 < d \leq 1$, $M > 0$ and $0 < \eta' < \eta'' < 1$ such that

$$\Psi := \{\lambda \in \mathbb{C} : |\lambda| \leq d \text{ or } \lambda \in \overline{\Sigma_{\pi\eta''/2}}\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda\colon \mathcal{A})\| \leq M(1+|\lambda|)^{-\beta}, \qquad \lambda \in \Psi.$$

We close the paper with the observation that we have not been able to improve the assertion of Theorem 4.3 for multivalued linear operators satisfying the condition (QP).

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