CF-modules over commutative rings

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Abstract. Let R be a commutative ring with unit. We give some criterions for determining when a direct sum of two CF-modules over R is a CF-module. When R is local, we characterize the CF-modules over R whose tensor product is a CF-module.

Keywords: CF-couple; CF-module; commutative ring; local ring

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1. Introduction

Finitely generated modules over particular commutative rings have been extensively studied (see, for example, [1], [4], [7], [9], [12], [13]). Many works have been done on modules that have a decomposition as a direct sum of cyclic modules. Particularly, many studies have been made on commutative rings which have the property that every module is a direct sum of cyclic modules (see, for example, [2], [5], [6], [8], [15]). Here we are interested in a category of modules that are both finitely generated and having quite particular decompositions into direct sums of cyclic modules. Our work concerns the modules and not the underlying rings.

All rings considered in this paper are supposed to be with unit. Let R be a commutative ring. A canonical form for a module M is a decomposition $M \cong \bigoplus_{i=1}^{n} R/I_i$, where the I_i are ideals of R such that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \neq R$. If M has a canonical form, the ideals I_i are uniquely determined (see [3, Lemma 15.13]). In this case M is called a CF-module of type $(I_1, I_2, \ldots, I_n)$. This notion of CF-module was introduced by Shores and Wiegand in [10], [11] under the designation "canonical form for a module". Note that in [10] and [11] a complete structure theory is developed for those rings for which every module that is finitely generated direct sum of cyclic modules is a CF-module. These rings are called CF-rings.

Our work focuses on some operations on the CF-modules, especially the sum. We show that the direct sum of two CF-modules over R is not necessarily a CF-module even in the case where R is local. We give some criterions for determining when a direct sum of two CF-modules over R is a CF-module, before showing that the tensor product of two CF-modules, a submodule of a CF-module and the quotient of a CF-module are not necessarily CF-modules. In the case R is local, we show that a direct factor of a CF-module over R is also a CF-module, and we characterize the CF-modules over R whose tensor product is a CF-module.

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2. The results

Let R be a commutative ring. The minimal number of generators of a finitely generated R-module M, which is denoted by $\mu_R(M)$, is the smallest cardinal of the generating families of M. If $M = (0)$, then we put $\mu_R(M) = 0$.

We will need the following two lemmas.

Lemma 2.1 ([3, Lemma 15.12]). *Let R be a commutative ring. Suppose I_1, $I_2, \ldots, I_n$ are ideals in R such that $I_1 + I_2 + \cdots + I_n \neq R$. Then, $\mu_R(R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n) = n$.*

Lemma 2.2 ([3, Lemma 15.13]). *Let R be a commutative ring. Suppose $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ and $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_m$ are two sequences of ideals in R. We assume $I_n \neq R \neq J_m$. If $\bigoplus_{i=1}^{n} R/I_i \cong \bigoplus_{j=1}^{m} R/J_j$ as R-modules, then $n = m$ and $I_i = J_i$ for all $i \in \{1, 2, \ldots, n\}$.*

There are rings R for which the direct sum of two CF-modules is a CF-module. That is the case of CF-rings. But in the general case, this result is not true, even in the case where the ring R is local, as shown in the following example.

Example 2.3. Let K be a commutative field of characteristic $p > 2$. In this example, we show that there exists a local finitely dimensional K-algebra R that is not serial. Then, there are two incomparable ideals I_1 and I_2, and $M = R/I_1 \oplus R/I_2$ is not a CF-module, since this decomposition into indecomposables is unique by Krull-Schmidt theorem.

Let $R = K[G_1 \times G_2]$, where G_1 and G_2 are cyclic p-groups generated respectively by σ_1 and σ_2. By theorem of Wallace (cf. [16, Theorem 7.1.5]), the Jacobson radical of R corresponds to its augmentation ideal. In particular, the ring R is local and all cyclic R-modules are indecomposable. The two ideals $(\sigma_1 - 1)R$ and $(\sigma_2 - 1)R$ of R are such that $(\sigma_1 - 1)R \nsubseteq (\sigma_2 - 1)R$ and $(\sigma_2 - 1)R \nsubseteq (\sigma_1 - 1)R$. Indeed, if $(\sigma_1 - 1)R \subseteq (\sigma_2 - 1)R$, then there exists $x \in K[G_1 \times G_2]$ such that $\sigma_1 - 1 = (\sigma_2 - 1)x$. Let p^n be the order of σ_2. Since $p > 2$ is necessarily prime, it is odd. Since p is the characteristic of K, we get that

$$(\sigma_2 - 1)^{p^n} = \sum_{i=0}^{p^n} \binom{p^n}{i} (-1)^i {\sigma_2}^{p^n - i} = {\sigma_2}^{p^n} + (-1)^{p^n} = 0.$$

It follows that

$$(\sigma_2 - 1)^{p^n - 1}(\sigma_1 - 1) = (\sigma_2 - 1)^{p^n - 1}(\sigma_2 - 1)x = 0.$$

As

$$(\sigma_2 - 1)^{p^n - 1}(\sigma_1 - 1) = \left(\sum_{i=0}^{p^n - 2} \binom{p^n - 1}{i} (-1)^i {\sigma_2}^{p^n - 1 - i} \right) \sigma_1$$
$$+ \sum_{i=0}^{p^n - 2} \binom{p^n - 1}{i} (-1)^{i+1} {\sigma_2}^{p^n - 1 - i} + (-1)^{p^n - 1}\sigma_1 + (-1)^{p^n},$$

then $-1 = (-1)^{p^n} = 0$ in K, which is impossible. Similarly, we show that $(\sigma_2 - 1)R \nsubseteq (\sigma_1 - 1)R$. So, $R = K[G_1 \times G_2]$ is a local finitely dimensional K-algebra that is not serial. Now, if we take $I_1 = (\sigma_1 - 1)R$ and $I_2 = (\sigma_2 - 1)R$, then M is not a CF-module.

For two ideals I and J of a commutative ring R, $(I : J)$ will denote the quotient of I and J, i.e., $(I : J) = \{x \in R\colon xJ \subseteq I\}$.

To show our first interesting theorem, we give the following lemma.

Lemma 2.4. *Let R be a commutative ring. Let M and N be R-modules. Suppose that M and N are CF-modules of respective types $(I_1, I_2, \dots, I_m)$ and $(J_1, J_2, \dots, J_n)$ such that*

$$(I_i : xR) + (J_n : xR) \neq R \quad \text{for all } \; x \in R\backslash(I_i \cup J_n),\; i \in \{1, 2, \dots, m\}.$$

If $M \oplus N$ is a CF-module, then the set $\{I_i\colon 1 \leq i \leq m\} \cup \{J_n\}$ is totally ordered by inclusion.

Proof: Assume that $M \oplus N$ is a CF-module of type $(L_1, L_2, \dots, L_r)$. Then, we have

$$\begin{aligned}\mu_R(M \oplus N) &= \mu_R\left(\left(\bigoplus_{i=1}^{m} R/I_i\right) \oplus \left(\bigoplus_{j=1}^{n} R/J_j\right)\right)\\ &= \mu_R\left(\bigoplus_{k=1}^{r} R/L_k\right).\end{aligned}$$

As $I_1 + I_2 + \dots + I_m + J_1 + J_2 + \dots + J_n = I_m + J_n = (I_m : R) + (J_n : R) \subsetneq R$ and $L_1 + L_2 + \dots + L_r = L_r \neq R$, then by Lemma 2.1, $\mu_R(M \oplus N) = r = m + n = \mu_R(M) + \mu_R(N)$. Let $I_{m+1} = R$ and let k_0 be the smallest integer such that $J_n \subseteq I_{k_0}$. If $k_0 = 1$, then we are done (since $J_n \subseteq I_1 \subseteq I_2 \subseteq \dots \subseteq I_m$). If $k_0 > 1$, then we will show that $I_{k_0 - 1} \subseteq J_n$. Let $x \in R$. As $x(R/I) \cong R/(I : xR)$ for all ideal I in R (see [3, page 191]), then

$$\begin{aligned}x(M \oplus N) &\cong \left(\bigoplus_{i=1}^{m} R/(I_i : xR)\right) \oplus \left(\bigoplus_{j=1}^{n} R/(J_j : xR)\right)\\ &\cong \bigoplus_{k=1}^{m+n} R/(L_k : xR).\end{aligned}$$

So,

$$\begin{aligned}&\left(\mu_R\left(\bigoplus_{k=1}^{m+n} R/(L_k : xR)\right) < n + k_0 - 1\right)\\ &\qquad\Leftrightarrow \left(\mu_R\left(\left(\bigoplus_{i=1}^{m} R/(I_i : xR)\right) \oplus \left(\bigoplus_{j=1}^{n} R/(J_j : xR)\right)\right) < n + k_0 - 1\right).\end{aligned}$$

We have $(L_1 : xR) \subseteq (L_2 : xR) \subseteq \cdots \subseteq (L_{m+n} : xR)$. So, if $x \in L_0$, then $\mu_R\big(\bigoplus_{k=1}^{m+n} R/(L_k : xR)\big)$ is equal to 0, and if $x \notin L_0$, then by Lemma 2.1, $\mu_R\big(\bigoplus_{k=1}^{m+n} R/(L_k : xR)\big)$ is equal to the largest k such that $x \notin L_k$. Therefore,

$$x \in L_{n+k_0-1} \text{ if and only if } \mu_R(x(M \oplus N)) = \mu_R\left(\bigoplus_{k=1}^{m+n} R/(L_k : xR)\right) < n+k_0-1.$$

We also have

$$\mu_R(x(M \oplus N)) \leq \mu_R\left(\bigoplus_{i=1}^{m} R/(I_i : xR)\right) + \mu_R\left(\bigoplus_{j=1}^{n} R/(J_j : xR)\right).$$

Suppose that $x \in I_{k_0-1} \cup J_n$. As $J_n \subseteq I_{k_0}$, $(I_1 : xR) \subseteq (I_2 : xR) \subseteq \cdots \subseteq (I_m : xR)$ and $(J_1 : xR) \subseteq (J_2 : xR) \subseteq \cdots \subseteq (J_n : xR)$, then

$$\mu_R(x(M \oplus N)) < n + k_0 - 1.$$

Now, suppose that $x \notin I_{k_0-1} \cup J_n$. Let m_0 be the largest i such that $x \notin I_i$. We have $k_0 - 1 \leq m_0$. As $(I_{m_0} : xR) + (J_n : xR) \neq R$ (by hypothesis), then by Lemma 2.2, $\mu_R(x(M \oplus N)) = m_0 + n$. So,

$$n + k_0 - 1 \leq \mu_R(x(M \oplus N)).$$

Therefore, $x \in I_{k_0-1} \cup J_n$ if and only if

$$\begin{aligned}\mu_R(x(M \oplus N)) &= \mu_R\left(\left(\bigoplus_{i=1}^{m} R/(I_i : xR)\right) \oplus \left(\bigoplus_{j=1}^{n} R/(J_j : xR)\right)\right)\\ &< n + k_0 - 1.\end{aligned}$$

So,

$$x \in L_{n+k_0-1} \;\Leftrightarrow\; x \in I_{k_0-1} \cup J_n,$$

i.e.,

$$L_{n+k_0-1} = I_{k_0-1} \cup J_n.$$

Hence, $I_{k_0-1} \subseteq J_n$ or $J_n \subseteq I_{k_0-1}$. As k_0 is minimal, then $I_{k_0-1} \subseteq J_n$. This ends this proof (since $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq J_n$ or $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{k_0-1} \subseteq J_n \subseteq I_{k_0} \subseteq \cdots \subseteq I_m$). □

Now we can prove the following theorem.

Theorem 2.5. *Let R be a commutative ring. Let M and N be R-modules. If M and N are CF-modules of respective types $(I_1, I_2, \ldots, I_m)$ and $(J_1, J_2, \ldots, J_n)$ such that*

$$(I_i : xR) + (J_j : xR) \neq R$$

for all $x \in R \backslash (I_i \cup J_j)$, $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, then the following conditions are equivalent:

(1) *$M \oplus N$ is a CF-module;*
(2) *the set $\{I_i \colon 1 \leq i \leq m\} \cup \{J_j \colon 1 \leq j \leq n\}$ is totally ordered by inclusion.*

Proof: (1) $\Rightarrow$ (2) We use induction on n. By Lemma 2.4, the statement holds for $n = 1$. We assume that the statement holds for $n = k \geq 1$. Let M and N be R-modules which are CF-modules of respective types $(I_1, I_2, \dots, I_m)$ and $(J_1, J_2, \dots, J_{k+1})$ such that

$$(I_i : xR) + (J_j : xR) \neq R$$

for all $x \in R \backslash (I_i \cup J_j)$, $(i,j) \in \{1,2,\dots,m\} \times \{1,2,\dots,k+1\}$ and $M \oplus N$ is a CF-module. By Lemma 2.4, $\{I_i \colon 1 \leq i \leq m\} \cup \{J_{k+1}\}$ is totally ordered by inclusion. We put $I'_i = I_i$ for all $i \in \{1,2,\dots,m\}$, $I'_{m+1} = J_{k+1}$, $M' = \bigoplus_{i=1}^{m+1} R/I'_i$ and $N' = \bigoplus_{j=1}^{k} R/J_j$. Then, M' and N' are CF-modules of respective types $(I'_1, I'_2, \dots, I'_{m+1})$ and $(J_1, J_2, \dots, J_k)$ such that $M' \oplus N' \cong M \oplus N$ is a CF-module and

$$(I'_i : xR) + (J_j : xR) \neq R$$

for all $x \in R \backslash (I'_i \cup J_j)$, $(i,j) \in \{1,2,\dots,m+1\} \times \{1,2,\dots,k\}$. By virtue of the induction hypothesis the set $\{I'_i \colon 1 \leq i \leq m+1\} \cup \{J_j \colon 1 \leq j \leq k\}$ is totally ordered by inclusion, i.e., $\{I_i \colon 1 \leq i \leq m\} \cup \{J_j \colon 1 \leq j \leq k+1\}$ is totally ordered by inclusion. So, the statement holds for $n = k+1$.

(2) $\Rightarrow$ (1) Obvious. □

In the following, $\mathbb{Z}$ denotes the ring of rational integers.

Let $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$. We take $x = 2$. We have $(I : x\mathbb{Z}) = 2\mathbb{Z}$ and $(J : x\mathbb{Z}) = 3\mathbb{Z}$. So, we have $I + J \neq \mathbb{Z}$, but $(I : x\mathbb{Z}) + (J : x\mathbb{Z}) = \mathbb{Z}$.

Corollary 2.6. *Let R be a commutative local ring. Let M and N be R-modules. If M and N are CF-modules of respective types $(I_1, I_2, \dots, I_m)$ and $(J_1, J_2, \dots, J_n)$, then the following conditions are equivalent:*

(1) *$M \oplus N$ is a CF-module;*
(2) *the set $\{I_i \colon 1 \leq i \leq m\} \cup \{J_j \colon 1 \leq j \leq n\}$ is totally ordered by inclusion.*

Proof: It suffice to see that for R local the condition

$$(I_i : xR) + (J_j : xR) \neq R$$

for all $x \in R \backslash (I_i \cup J_j)$, $(i,j) \in \{1,2,\dots,m\} \times \{1,2,\dots,n\}$, is always satisfied. □

The result shown in Corollary 2.6 is not true if the ring R is not local, as shown in the following example.

Example 2.7. We have

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z} \;\cong\; \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/210\mathbb{Z}.$$

Let $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ and $N = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}$. Then, M and N are two CF-modules and $M \oplus N$ is a CF-module, but the set $\{2\mathbb{Z}, 10\mathbb{Z}, 3\mathbb{Z}, 21\mathbb{Z}\}$ is not totally ordered by inclusion.

Corollary 2.8. *Let R be a CF-ring. Let I_1 and I_2 be two ideals of R. Then,*

$$((I_1 : xR) + (I_2 : xR) \neq R \quad \text{for all} \;\; x \in R\backslash(I_1 \cup I_2)) \Leftrightarrow (I_1 \subseteq I_2 \;\; \text{or} \;\; I_2 \subseteq I_1).$$

PROOF: Obvious from Theorem 2.5. □

Definition 2.9. Let R be a commutative ring. We say that a pair (I, J) of ideals in R is a CF-couple, if $R/I \oplus R/J \cong R/(I \cap J) \oplus R/(I + J)$.

To show our second interesting theorem, we give the following lemmas and remarks.

Lemma 2.10. *Let R be a commutative ring. Let $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$ be ideals of R. If (I_2, J_2) is a CF-couple, then $(I_2 + J_1, I_1 + J_2)$ is a CF-couple.*

PROOF: Assume that (I_2, J_2) is a CF-couple. We have

$$R/I_2 \oplus R/J_2 \cong R/(I_2 \cap J_2) \oplus R/(I_2 + J_2).$$

So,

$$R/I_1 \otimes_R (R/I_2 \oplus R/J_2) \cong R/I_1 \otimes_R (R/(I_2 \cap J_2) \oplus R/(I_2 + J_2)),$$

which gives

$$R/I_2 \oplus R/(I_1 + J_2) \cong R/(I_1 + (I_2 \cap J_2)) \oplus R/(I_2 + J_2).$$

As $I_1 + (I_2 \cap J_2) = I_2 \cap (I_1 + J_2)$ (by the modular law) and $I_2 + J_2 = I_2 + (I_1 + J_2)$, then

$$R/I_2 \oplus R/(I_1 + J_2) \cong R/(I_2 \cap (I_1 + J_2)) \oplus R/(I_2 + (I_1 + J_2)),$$

i.e., $(I_2, I_1 + J_2)$ is a CF-couple.

Now, $J_1 \subseteq I_1 + J_2$ and $I_1 \subseteq I_2$ are ideals of R such that $(I_1 + J_2, I_2)$ is a CF-couple. From the above $(I_1 + J_2, J_1 + I_2)$ is a CF-couple, i.e., $(I_2 + J_1, I_1 + J_2)$ is a CF-couple. □

In Lemma 2.10, if we take $J_1 = 0$, then $(I_2, I_1 + J_2)$ is a CF-couple, and if we suppose in addition that (I_1, J_2) is a CF-couple, then $(I_1 + I_2 \cap J_1, J_2)$ is a CF-couple (it suffice to see that the couples $(I_1 \cap J_1, I_1)$ and $(I_2 \cap J_1, J_2)$ satisfy the conditions of Lemma 2.10).

Remark 2.11. Keeping the assumptions of Lemma 2.10, we have

$$R/I_2 \oplus R/(I_1 + J_2) \cong R/(I_1 + (I_2 \cap J_2)) \oplus R/(I_2 + J_2),$$

and if in addition (I_1, J_2) is a CF-couple, then

$$R/I_1 \oplus R/I_2 \oplus R/J_2 \cong R/I_2 \oplus R/(I_1 \cap J_2) \oplus R/(I_1 + J_2),$$

which gives

$$R/I_1 \oplus R/I_2 \oplus R/J_2 \cong R/(I_1 \cap J_2) \oplus R/(I_1 + (I_2 \cap J_2)) \oplus R/(I_2 + J_2).$$

So, $R/I_1 \oplus R/I_2 \oplus R/J_2$ is a CF-module of type $(I_1 \cap J_2, I_1 + I_2 \cap J_2, I_2 + J_2)$.

Lemma 2.12. *Let R be a commutative ring. Let M be a CF-module of type $(I_1, I_2, \dots, I_n)$, where n is a nonzero natural number and let $N \cong R/J$ be a cyclic module. If (I_i, J) is a CF-couple for all $i \in \{1, 2, \dots, n\}$ with $I_n + J \neq R$, then $M \oplus N$ is a CF-module of type $(I_1 \cap J, I_1 + I_2 \cap J, I_2 + I_3 \cap J, \dots, I_{n-1} + I_n \cap J, I_n + J)$.*

Proof: We prove the result by induction on n. The statement holds for $n = 1, 2$ (see Remark 2.11). We assume that the statement holds for $n = k \geq 1$. Let I_1, I_2, $\dots$, I_k, I_{k+1} be ideals of R such that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{k+1}$, and (I_i, J) is a CF-couple for all $i \in \{1, 2, \dots, k+1\}$ with $I_{k+1} + J \neq R$. We have (by virtue of the induction hypothesis)

$$\begin{aligned} &R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_k \oplus R/I_{k+1} \oplus R/J \\ &\quad \cong R/(I_1 \cap J) \oplus R/(I_1 + I_2 \cap J) \oplus R/(I_2 + I_3 \cap J) \oplus \cdots \oplus R/(I_{k-1} + I_k \cap J) \\ &\qquad \oplus R/(I_k + J) \oplus R/I_{k+1}. \end{aligned}$$

By Lemma 2.10, $(I_{k+1}, I_k + J)$ is a CF-couple. So,

$$R/(I_k + J) \oplus R/I_{k+1} \cong R/(I_{k+1} \cap (I_k + J)) \oplus R/(I_{k+1} + (I_k + J)).$$

As $I_{k+1} \cap (I_k + J) = I_k + I_{k+1} \cap J$ (by the modular law) and $I_{k+1} + (I_k + J) = I_{k+1} + J$, then

$$R/(I_k + J) \oplus R/I_{k+1} \cong R/(I_k + I_{k+1} \cap J) \oplus R/(I_{k+1} + J).$$

Consequently,

$$\begin{aligned} &R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_k \oplus R/I_{k+1} \\ &\quad \cong R/(I_1 \cap J) \oplus R/(I_1 + I_2 \cap J) \oplus R/(I_2 + I_3 \cap J) \oplus \cdots \oplus R/(I_{k-1} + I_k \cap J) \\ &\qquad \oplus R/(I_k + I_{k+1} \cap J) \oplus R/(I_{k+1} + J). \end{aligned}$$

So, the statement holds for $n = k + 1$. □

Remark 2.13. For a commutative ring R and for any ideal J of R, $(\{0\}, J)$ and (R, J) are CF-couples.

Now we can prove the following theorem.

Theorem 2.14. *Let R be a commutative ring. Let M and N be two CF-modules of respective types $(I_1, I_2, \dots, I_m)$ and $(J_1, J_2, \dots, J_n)$, where m and n are two nonzero natural numbers. If for all $(i, j) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$, (I_i, J_j) is a CF-couple, then $M \oplus N$ is a CF-module.*

Proof: We prove the result by induction on n. For $n=1$, by Lemma 2.12, $M \oplus R/J_1$ is a CF-module of type which we denote by $(I_{1,1}, I_{1,2}, \dots, I_{1,m+1})$. Moreover, Lemma 2.12 shows that $I_{1,i} = I_{i-1} + I_i \cap J_1$ (where $I_0 = 0$ and $I_{m+1} = R$) for all $i \in \{1, 2, \dots, m+1\}$. By Lemmas 2.10 and 2.12 and Remark 2.13, $(I_{1,i}, J_2)$ is a CF-couple for all $i \in \{1, 2, \dots, m, m+1\}$. Let $n > 1$. We assume that $M \oplus \left(\bigoplus_{j=1}^{n-1} R/J_j\right)$ is a CF-module of type $(I_{n-1,1}, I_{n-1,2}, \dots, I_{n-1,m+n-1})$ and $(I_{n-1,i}, J_n)$ is a CF-couple for all $i \in \{1, 2, \dots, m, m+n-1\}$. Then,

$$\begin{aligned} M \oplus \left(\bigoplus_{j=1}^{n} R/J_j\right) &= \left(M \oplus \left(\bigoplus_{j=1}^{n-1} R/J_j\right)\right) \oplus R/J_n \\ &\cong \left(\bigoplus_{i=1}^{m+n-1} R/I_{n-1,i}\right) \oplus R/J_n. \end{aligned}$$

As, by Lemma 2.12, $\left(\bigoplus_{i=1}^{m+n-1} R/I_{n-1,i}\right) \oplus R/J_n$ is a CF-module, then $M \oplus \bigoplus_{j=1}^{n} R/J_j$ is a CF-module. □

By [3, Exercice 13, page 202], $\mathfrak{U} = (3, X+1) \subseteq \mathbb{Z}[X]$ is not a direct sum of cyclic $\mathbb{Z}[X]$-modules. So, a submodule of a CF-module is not necessarily a CF-module.

Let R be a commutative ring in which there exist two ideals I and J such that (I, J) is not a CF-couple and $I \cap J \neq \{0\}$. We have

$$(R/(I \cap J) \oplus R/(I \cap J))/(I/(I \cap J) \oplus J/(I \cap J)) \cong R/I \oplus R/J.$$

As $R/(I \cap J) \oplus R/(I \cap J)$ is a CF-module, then the quotient of CF-module is not necessarily a CF-module.

The tensor product of two CF-modules is not necessarily a CF-module as shown in the following example.

Example 2.15. Let F be a field, and let $R = F[x, y]$ be the polynomial ring in two variables. We consider the following two R-modules

$$M = R/(xy) \oplus R/(x) \quad \text{and} \quad N = R/(xy) \oplus R/(y).$$

M and N are CF-modules, and we have

$$M \otimes_R N \cong R/(xy) \oplus R/(x) \oplus R/(y) \oplus R/((x) + (y)).$$

Let $M' = R/(xy) \oplus R/(x)$ and $N' = R/(y) \oplus R/((x) + (y))$. Then, M' and N' are two CF-modules, and we have $M \otimes_R N = M' \oplus N'$. We also have $(y) \neq R$, and $(xy) \subset (y)$. Let $f \in R \backslash ((xy) \cup (y))$, i.e., $f \in R \backslash (y)$. Therefore, $((y) : fR) \neq R$. As $((xy) : fR) \subset ((y) : fR)$ (since $(xy) \subset (y)$), then

$$((xy) : fR) + ((y) : fR) = ((y) : fR) \neq R.$$

Similarly we see that

$$((xy) : fR) + (((x) + (y)) : fR) = (((x) + (y)) : fR) \neq R$$

for all $f \in R\backslash((xy) \cup ((x) + (y)))$, and

$$((x) : fR) + (((x) + (y)) : fR) = (((x) + (y)) : fR) \neq R$$

for all $f \in R\backslash((x) \cup ((x) + (y)))$. Now, let $f \in R\backslash((x) \cup (y))$. We have

$$((x) : fR) + ((y) : fR) \neq R.$$

Supposing otherwise, there are polynomials $g, h \in R$ such that x divides fg, y divides fh and $g + h = 1$. As R is a gaussian domain and $f \notin (x) \cup (y)$, x divides g and y divides h. Thus $1 = g + h \in (x) + (y) \neq R$, which is not the case. So, by Theorem 2.5, $M' \oplus N'$ is CF-module if and only if the set $\{(xy)\} \cup \{(x)\} \cup \{(y)\} \cup \{((x) + (y))\}$ is totally ordered by inclusion. Or $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$. In conclusion $M \otimes_R N$ is not a CF-module.

In Example 2.15 we have seen that the two R-modules $R/(xy) \oplus R/(x)$ and $R/(y) \oplus R/((x) + (y))$ are two CF-modules while the direct sum $R/(xy) \oplus R/(x) \oplus R/(y) \oplus R/((x) + (y))$ is not a CF-module. So, here we have another example of a direct sum of two CF-modules that is not a CF-module.

The case of a commutative local ring is quite interesting as shown by Corollary 2.6 and the following results.

Lemma 2.16 ([14, Proposition 3]). *Let R be a commutative local ring and M an R-module. If $M = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$, where Λ is a set of index, and each I_λ is an ideal of R, then every summand of M is also a direct sum of cyclic R-modules, each isomorphic to one of the R/I_λ.*

Proposition 2.17. *Let R be a commutative local ring. Then, a summand of a CF-module is also a CF-module.*

Proof: Obvious from Lemma 2.16. □

Corollary 2.6 can be easily deduced from Lemmas 2.2 and 2.16.

Proposition 2.18. *Let R be a commutative local ring. Let M and N be R-modules. If M and N are CF-modules of respective types $(I_1, I_2, \ldots, I_m)$ and $(J_1, J_2, \ldots, J_n)$, then the following conditions are equivalent:*

(1) *$M \otimes_R N$ is a CF-module;*
(2) *the set $\{I_i + J_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is totally ordered by inclusion.*

Proof: (1) ⇒ (2) Assume that $M \otimes_R N$ is a CF-module of type $(L_1, L_2, \ldots, L_r)$. So, we have

$$M \otimes_R N \cong \bigoplus_{k=1}^{r} R/L_k \cong \bigoplus_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} R/(I_i + J_j).$$

For each $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, $R/(I_i + J_j)$ is isomorphic to a summand of $\bigoplus_{k=1}^{r} R/L_k$. By Lemma 2.16, every summand of $\bigoplus_{k=1}^{r} R/L_k$ is also a direct sum of cyclic R-modules, each isomorphic to one of the R/L_k. As every cyclic R-module is indecomposable (since R is local), then there exists $k \in \{1, 2, \ldots, r\}$ such that $R/(I_i + J_j) \cong R/L_k$. So, $I_i + J_j = L_k$ and therefore, the set $\{I_i + J_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is totally ordered by inclusion.

(2) $\Rightarrow$ (1) Obvious. $\square$

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