The universal Banach space with a K-suppression unconditional basis

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Abstract. Using the technique of Fraïssé theory, for every constant $K \geq 1$, we construct a universal object $\mathbb{U}_K$ in the class of Banach spaces possessing a normalized K-suppression unconditional Schauder basis.

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1. Introduction

A Banach space X is *complementably universal* for a given class of Banach spaces if X belongs to this class and every space from the class is isomorphic to a complemented subspace of X.

In 1969 A. Pełczyński in [11] constructed a complementably universal Banach space for the class of Banach spaces with a Schauder basis. In 1971 M. I. Kadec in [7] constructed a complementably universal Banach space for the class of spaces with the *bounded approximation property* (BAP). In the same year A. Pełczyński in [12] showed that every Banach space with BAP is complemented in a space with a basis. A. Pełczyński and P. Wojtaszczyk in [13] constructed a universal Banach space for the class of spaces with a finite-dimensional decomposition. Applying Pełczyński's decomposition argument, see [10], one immediately concludes that all three universal spaces are isomorphic. It is worth mentioning a negative result of W. B. Johnson and A. Szankowski, see [6], saying that no separable Banach space can be complementably universal for the class of all separable Banach spaces. In [4] the second author constructed an isometric version of the Kadec-Pełczyński-Wojtaszczyk space. The universal Banach space from [4] was constructed using the general categorical technique of Fraïssé limits, see [8]. This method was also applied by W. Kubiś and S. Solecki in [9] for constructing the Gurariĭ space, see [5], which possesses the property of extension of almost isometries, which implies the universality property that is stronger than the standard universality property of the Banach spaces l_∞ or $C[0,1]$.

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In this paper we apply the categorical method of Fraïssé limits for constructing a universal space $\mathbb{U}_K$ in the class of Banach spaces with a normalized K-suppression unconditional Schauder basis. The universal space constructed by this method has a nice property of extension of almost isometries, which is better than just the standard universality, established in the papers of A. Pełczyński, see [11], and G. Schechtman, see [14], (who gave a short alternative construction of universal space for class of Banach spaces with an unconditional bases). We also prove that the universal space $\mathbb{U}_K$ is isomorphic to the complementably universal space $\mathbb{U}$ for Banach spaces with unconditional basis, which was constructed by A. Pełczyński in [11].

2. Preliminaries

All Banach spaces considered in this paper are separable and over the field $\mathbb{R}$ of real numbers.

2.1 Definitions. Let X be a Banach space with a Schauder basis $(\mathsf{e}_n)_{n=1}^\infty$ and let $(\mathsf{e}_n^*)_{n=1}^\infty$ be the corresponding sequence of coordinate functionals. The basis $(\mathsf{e}_n)_{n=1}^\infty$ is called *K-supression* for a real constant K if for every finite subset $F \subset \mathbb{N}$ the projection $\mathrm{pr}_F\colon X \to X$, $\mathrm{pr}_F\colon x \mapsto \sum_{n\in F} \mathsf{e}_n^*(x)\cdot \mathsf{e}_n$, has norm $\|\mathrm{pr}_F\| \le K$. It is well-known, see [1, 3.1.5], that each K-suppression Schauder basis $(\mathsf{e}_n)_{n=1}^\infty$ is unconditional. So for any $x \in X$ and any permutation π of $\mathbb{N}$ the series $\sum_{n=1}^\infty \mathsf{e}_{\pi(n)}^*(x)\cdot \mathsf{e}_{\pi(n)}$ converges to x. This means that we can forget about the ordering and think of a K-suppression basis of a Banach space as a subset $ß \subset X$ such that for some bijection $\mathsf{e}\colon \mathbb{N} \to ß$ the sequence $(\mathsf{e}(n))_{n=1}^\infty$ is a K-suppression Schauder basis for X.

More precisely, by a *normalized K-suppression basis* for a Banach space X we shall understand a subset $ß \subset X$ for which there exists a family $\{\mathsf{e}_b^*\}_{b\in ß} \subset X$ of continuous functionals such that

- $\|b\| = 1 = \mathsf{e}_b^*(b)$ for any $b \in ß$;
- $\mathsf{e}_b^*(b') = 0$ for every $b \in ß$ and $b' \in ß \setminus \{b\}$;
- $x = \sum_{b\in ß} \mathsf{e}_b^*(x)\cdot b$ for every $x \in X$;
- for any finite subset $F \subset ß$ the projection $\mathrm{pr}_F\colon X \to X$, $\mathrm{pr}_F\colon x \mapsto \sum_{b\in F} \mathsf{e}_b^*(x)\cdot b$, has norm $\|\mathrm{pr}_F\| \le K$.

The equality $x = \sum_{b\in ß} \mathsf{e}_b^*(x)\cdot b$ in the third item means that for every $\varepsilon > 0$ there exists a finite subset $F \subset ß$ such that $\left\|x - \sum_{b\in E} \mathsf{e}_b^*(x)\cdot b\right\| < \varepsilon$ for every finite subset $E \subset ß$ containing F.

By a *K-based Banach space* we shall understand a pair $(X, ß_X)$ consisting of a Banach space X and a normalized K-suppression basis $ß_X$ for X. By a *based Banach space* we understand a K-based Banach space for some $K \ge 1$. We shall say that a based Banach space $(X, ß_X)$ is a *subspace* of a based Banach space $(Y, ß_Y)$ if $X \subseteq Y$ and $ß_X = X \cap ß_Y$.

For a Banach space X by $\|\cdot\|_X$ we denote the norm of X and by $B_X := \{x \in X\colon \|x\|_X \le 1\}$ the closed unit ball of X.

A finite dimensional based Banach space $(X, ß_X)$ is called *rational* if its unit ball B_X is a convex polyhedron spanned by finitely many vectors with rational coordinates in the basis $ß_X$. A based Banach space X is called *rational* if each finite-dimensional based subspace of X is rational.

2.2 Categories. Let $\mathfrak{K}$ be a category. For two objects A, B of the category $\mathfrak{K}$, by $\mathfrak{K}(A, B)$ we will denote the set of all $\mathfrak{K}$-morphisms from A to B. A *subcategory* of $\mathfrak{K}$ is a category $\mathfrak{L}$ such that each object of $\mathfrak{L}$ is an object of $\mathfrak{K}$ and each morphism of $\mathfrak{L}$ is a morphism of $\mathfrak{K}$. Morphisms and isomorphisms of a category $\mathfrak{K}$ will be called $\mathfrak{K}$-*morphisms* and $\mathfrak{K}$-*isomorphisms*, respectively.

A subcategory $\mathfrak{L}$ of a category $\mathfrak{K}$ is *full* if each $\mathfrak{K}$-morphism between objects of the category $\mathfrak{L}$ is an $\mathfrak{L}$-morphism. A subcategory $\mathfrak{L}$ of a category $\mathfrak{K}$ is *cofinal* in $\mathfrak{K}$ if for every object A of $\mathfrak{K}$ there exists a $\mathfrak{K}$-morphism $f\colon A \to B$ to an object B of $\mathfrak{L}$.

A category $\mathfrak{K}$ has the *amalgamation property* if for every objects A, B, C of $\mathfrak{K}$ and $\mathfrak{K}$-morphisms $f\colon A \to B$ and $g\colon A \to C$ there exist an object D of $\mathfrak{K}$ and $\mathfrak{K}$-morphisms $f'\colon K \to D$ and $g'\colon C \to D$ such that $f' \circ f = g' \circ g$.

In this paper we shall work in the category $\mathfrak{B}$, whose objects are based Banach spaces and morphisms are linear continuous operators $T\colon X \to Y$ between based Banach spaces $(X, ß_X)$ and $(Y, ß_Y)$ such that $T(ß_X) \subseteq ß_Y$.

A morphism $T\colon X \to Y$ of the category $\mathfrak{B}$ is called an *isometry* (or else an *isometry morphism*) if $\|T(x)\|_Y = \|x\|_X$ for any $x \in X$. By $\mathfrak{BI}$ we denote the category whose objects are based Banach spaces and morphisms are isometry morphisms of based Banach spaces. The category $\mathfrak{BI}$ is a subcategory of the category $\mathfrak{B}$.

For any real number $K \geq 1$ let $\mathfrak{B}_K$ (or $\mathfrak{BI}_K$) be the category whose objects are K-based Banach spaces and morphisms are (isometry) $\mathfrak{B}$-morphisms between K-based Banach spaces. So, $\mathfrak{B}_K$ and $\mathfrak{BI}_K$ are full subcategories of the categories $\mathfrak{B}$ and $\mathfrak{BI}$, respectively.

By $\mathfrak{FI}_K$ we denote the full subcategory of $\mathfrak{BI}_K$, whose objects are finite-dimensional K-based Banach spaces, and by $\mathfrak{RI}_K$ the full subcategory of $\mathfrak{FI}_K$ whose objects are rational finite-dimensional K-based Banach spaces. So, we have the inclusions $\mathfrak{RI}_K \subset \mathfrak{FI}_K \subset \mathfrak{BI}_K$ of categories.

From now on we assume that $K \geq 1$ is some fixed real number.

2.3 Amalgamation. In this section we prove that the categories $\mathfrak{FI}_K$ and $\mathfrak{RI}_K$ have the amalgamation property.

Lemma 2.1 (Amalgamation lemma)**.** *Let X, Y, Z be finite-dimensional K-based Banach spaces and $j\colon Z \to X$, $i\colon Z \to Y$ be $\mathfrak{BI}$-morphisms. Then there exist a finite-dimensional K-based Banach space W and $\mathfrak{BI}$-morphisms $j'\colon Y \to W$ and $i'\colon X \to W$ such that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{j'} & W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ Z & \xrightarrow{j} & X \end{array}$$

is commutative.

Moreover, if the K-based Banach spaces X, Y, Z are rational, then so is the K-based Banach space W.

Proof: We shall prove this lemma in the special case when the isometries i, j are identity inclusions; the general case is analogous but has more complicated notation. Our assumptions on i, j ensure that $Z = X \cap Y$ and $ß_Z = ß_X \cap ß_Y$, where $ß_X, ß_Y, ß_Z$ are the normalized K-suppression bases of the K-based Banach spaces X, Y, Z. It follows from $ß_Z = ß_X \cap ß_Y$ that the coordinate functionals of the bases $ß_X$ and $ß_Y$ agree on the intersection $Z = X \cap Y$.

Consider the direct sum $X \oplus Y$ of the Banach spaces X, Y endowed with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. Let $W = (X \oplus Y)/\Delta$ be the quotient space of $X \oplus Y$ by the subspace $\Delta = \{(z, -z)\colon z \in Z\}$.

We define linear operators $i'\colon X \to W$ and $j'\colon Y \to W$ by $i'(x) = (x, 0) + \Delta$ and $j'(y) = (0, y) + \Delta$.

Let us show i' and j' are isometries. Indeed, for every $x \in X$

$$\|i'(x)\|_W = \operatorname{dist}((x, 0), \Delta) \le \|(x, 0)\| = \|x\|_X + \|0\|_Y = \|x\|_X.$$

On the other hand, for every $z \in Z$

$$\begin{aligned}\|(x, 0) - (z, -z)\| &= \|(x - z, z)\| = \|x - z\|_X + \|z\|_Y \\ &= \|x - z\|_X + \|z\|_X \ge \|x - z + z\|_X = \|x\|_X\end{aligned}$$

and hence $\|x\|_X \le \inf_{z \in Z} \|(x, 0) - (z, -z)\| = \|i'(x)\|_W$. Therefore $\|i'(x)\|_W = \|x\|_X$. Similarly, we can show that j' is an isometry.

We shall identify X and Y with their images $i'(X)$ and $j'(Y)$ in W. In this case we can consider the union $ß_W := ß_X \cup ß_Y$ and can show that $ß_W$ is a normalized Schauder basis for the (finite-dimensional) Banach space W. Let $\{\mathsf{e}_b^*\}_{b \in ß_W} \subset W^*$ be the sequence of coordinate functionals of the basis $ß_W$.

Let us show that the basis $ß_W$ is K-suppression. Given any subset D of $ß_W$ we should prove that the projection $\mathrm{pr}_D\colon W \to W$, $\mathrm{pr}_D\colon w \mapsto \sum_{b \in D} \mathsf{e}_b^*(w) b$, has norm $\|\mathrm{pr}_D\| \le K$.

Write the set D as $D = D_Z \cup D_X \cup D_Y$, where $D_Z = D \cap ß_Z = D \cap ß_X \cap ß_Y$, $D_X = D \setminus ß_Y$ and $D_Y = D \setminus ß_X$.

Taking into account that the bases $ß_X$ and $ß_Y$ are K-suppression, for any $w \in W$ we obtain:

$$\begin{aligned}\|\mathrm{pr}_D(w)\|_W =\ & \inf\{\|x\|_X + \|y\|_Y \colon x \in X,\ y \in Y,\ x + y = \mathrm{pr}_D(w)\} \\ =\ & \inf\{\|\mathrm{pr}_{D_X}(w) + z'\|_X + \|z'' + \mathrm{pr}_{D_Y}(w)\|_Y \colon \\ & \qquad z', z'' \in Z,\ z' + z'' = \mathrm{pr}_{D_Z}(w)\} \\ \le\ & \inf\{\|\mathrm{pr}_{D_X}(w) + z'\|_X + \|z'' + \mathrm{pr}_{D_Y}(w)\|_Y \colon \\ & \qquad z', z'' \in \mathrm{pr}_{D_Z}(Z),\ z' + z'' = \mathrm{pr}_{D_Z}(w)\}\end{aligned}$$

$$
\begin{aligned}
&= \inf\{\|\mathrm{pr}_{D_X}(w) + \mathrm{pr}_{D_Z}(z')\|_X + \|\mathrm{pr}_{D_Z}(z'') + \mathrm{pr}_{D_Y}(w)\|_Y :\\
&\qquad\qquad\qquad\qquad z', z'' \in Z,\ z' + z'' = \mathrm{pr}_{ß_Z}(w)\}\\
&\le K \inf\{\|\mathrm{pr}_{ß_X\setminus ß_Z}(w) + z'\|_X + \|z' + \mathrm{pr}_{ß_Y\setminus ß_X}(w)\|_Y :\\
&\qquad\qquad\qquad\qquad z', z'' \in Z,\ z' + z'' = \mathrm{pr}_{ß_Z}(w)\}\\
&= K \inf\{\|x\|_X + \|y\|_Y : x + y = w\} = K\|w\|_W.
\end{aligned}
$$

If the finite-dimensional based Banach spaces X and Y are rational, then so is their sum $X \oplus Y$ and so is the quotient space W of $X \oplus Y$. □

3. $\mathfrak{B}$-universal based Banach spaces

Definition 3.1. A based Banach space U is defined to be $\mathfrak{B}$-*universal* if each based Banach space X is $\mathfrak{B}$-isomorphic to a based subspace of U.

Definition 3.1 implies that each $\mathfrak{B}$-universal based Banach space is complementably universal for the class of Banach spaces with unconditional basis. Reformulating Pełczyński's uniqueness Theorem 3, see [11], we obtain the following uniqueness result.

Theorem 3.2 (Pełczyński). *Any two $\mathfrak{B}$-universal based Banach spaces are $\mathfrak{B}$-isomorphic.*

A $\mathfrak{B}$-universal based Banach space $\mathbb{U}$ was constructed by A. Pełczyński in [11]. In the following sections we shall apply the technique of Fraïssé limits to construct many $\mathfrak{B}$-isomorphic copies of the Pełczyński's $\mathfrak{B}$-universal space $\mathbb{U}$.

4. $\mathfrak{RI}_K$-universal based Banach spaces

Definition 4.1. A based Banach space X is called $\mathfrak{RI}_K$-*universal* if for any rational finite-dimensional K-based Banach space A, any isometry morphism $f\colon \Lambda \to X$ defined on a based subspace Λ of A can be extended to an isometry morphism $\bar{f}\colon A \to X$.

We recall that $\mathfrak{RI}_K$ denotes the full subcategory of $\mathfrak{BI}$ whose objects are rational finite-dimensional K-based Banach spaces. Obviously, up to isomorphism the category $\mathfrak{RI}_K$ contains countably many objects. By Lemma 2.1, the category $\mathfrak{RI}_K$ has the amalgamation property. We now use the concepts from [8] for constructing a "generic" sequence in $\mathfrak{RI}_K$.

A sequence $(X_n)_{n\in\omega}$ of objects of the category $\mathfrak{BI}_K$ is called a *chain* if each K-based Banach space X_n is a subspace of the K-based Banach space X_{n+1}.

Definition 4.2. A chain of $(U_n)_{n\in\omega}$ of objects of the category $\mathfrak{RI}_K$ is *Fraïssé* if for any $n \in \omega$ and $\mathfrak{RI}_K$-morphism $f\colon U_n \to Y$ there exist $m > n$ and an $\mathfrak{RI}_K$-morphism $g\colon Y \to U_m$ such that $g \circ f\colon U_n \to U_m$ is the identity inclusion of U_n to U_m.

Definition 4.2 implies that the Fraïssé sequence $\{U_n\}_{n\in\omega}$ is cofinal in the category $\mathfrak{RI}_K$ in the sense that each object A of the category $\mathfrak{FI}_K$ admits an $\mathfrak{RI}_K$-morphism $A \to U_n$ for some $n \in \omega$. This means that the category $\mathfrak{RI}_K$ is countably cofinal.

The name "Fraïssé sequence", as in [8], is motivated by the model-theoretic theory of Fraïssé limits developed by R. Fraïssé in [3]. One of the results in [8] is that every countably cofinal category with amalgamation has a Fraïssé sequence. Applying this general result to our category $\mathfrak{RI}_K$ we get:

Theorem 4.3 ([8], Theorem 3.7)**.** *The category $\mathfrak{RI}_K$ has a Fraïssé sequence.*

From now on, we fix a Fraïssé sequence $(U_n)_{n\in\omega}$ in $\mathfrak{RI}_K$, which can be assumed to be a chain of finite-dimensional rational K-based Banach spaces. Let $\mathbb{U}_K$ be the completion of the union $\bigcup_{n\in\omega} U_n$ and $ß_{\mathbb{U}_K} = \bigcup_{n\in\omega} ß_{U_n} \subset \mathbb{U}_K$.

Theorem 4.4. *The pair $(\mathbb{U}_K, ß_{\mathbb{U}_K})$ is an $\mathfrak{RI}_K$-universal rational K-based Banach space.*

Proof: First we show that $ß_{\mathbb{U}_K} = \bigcup_{n\in\omega} ß_{U_n}$ is a normalized K-suppression basis for $\mathbb{U}_K$. The fact that $ß_{\mathbb{U}_K}$ is an unconditional Schauder basis with suppression constant K follows from Lemma 6.2 and Fact 6.3 in [2]. For each n the spaces U_n are K-based Banach spaces, so $\|b\| = 1$ for every $b \in ß_{U_n}$. This shows that the basis $ß_{\mathbb{U}_K}$ is normalized.

The based Banach space $(\mathbb{U}_K, ß_{\mathbb{U}_K})$ is rational, since each finite-dimensional based subspace of $(\mathbb{U}_K, ß_{\mathbb{U}_K})$ is contained in some rational based Banach space $(U_n, ß_{U_n})$ and hence is rational.

The $\mathfrak{RI}_K$-universality of the based Banach space $(\mathbb{U}_K, ß_{U_K})$ follows from the construction and [8, Proposition 3.1]. □

To shorten notation, let $\mathbb{U}_K$ is the $\mathfrak{RI}_K$-universal rational K-based Banach space $(\mathbb{U}_K, ß_{\mathbb{U}_K})$. The following theorem shows that such space is unique up to $\mathfrak{BI}$-isomorphism.

Theorem 4.5. *Any $\mathfrak{RI}_K$-universal rational K-based Banach spaces X, Y are $\mathfrak{BI}$-isomorphic, which means that there exists a linear bijective isometry $X \to Y$ preserving the bases of X and Y.*

Proof: By definition, the rational K-based Banach spaces X, Y can be written as the completions of unions $\bigcup_{n\in\omega} X_n$ and $\bigcup_{n\in\omega} Y_n$ of chains $(X_n)_{n\in\omega}$ and $(Y_n)_{n\in\omega}$ of rational finite-dimensional K-based Banach spaces such that $X_0 = \{0\}$ and $Y_0 = \{0\}$ are trivial K-based Banach spaces.

We define inductively sequences of $\mathfrak{RI}_K$-morphisms $\{f_k\}_{k\in\omega}$, $\{g_k\}_{k\in\omega}$ and increasing number sequences (n_k), (m_k) such that the following conditions are satisfied for every $k \in \omega$:

(1) $f_k\colon X_{n_{k-1}} \to Y_{m_k}$ and $g_k\colon Y_{m_k} \to X_{n_k}$ are morphisms of category $\mathfrak{RI}_K$;

(2) $f_{k+1} \circ g_k = \mathrm{id} \restriction Y_{m_k}$ and $g_{k+1} \circ f_{k+1} = \mathrm{id} \restriction X_{n_k}$.

We start the inductive construction letting $n_0 = 0 = m_0$ and $f_0\colon X_0 \to Y_0$, $g_0\colon Y_0 \to X_0$ be the unique isomorphisms of the trivial K-based Banach spaces X_0 and Y_0. To make an inductive step, assume that for some $k \in \omega$, the numbers n_k, m_k and $\mathfrak{RI}_K$-morphisms $f_k\colon X_{n_{k-1}} \to Y_{m_k}$, $g_k\colon Y_{m_k} \to X_{n_k}$ have been constructed. By Definition 4.1, the $\mathfrak{BI}$-morphism $g_k^{-1}\colon g_k(Y_{m_k}) \to Y$ defined on the based subspace $g_k(Y_{m_k})$ of the rational finite-dimensional K-based Banach space X_{n_k} extends to a $\mathfrak{BI}$-morphism $f_{k+1}\colon X_{n_k} \to Y$. So, $f_{k+1} \circ g_k = \mathrm{id} \restriction Y_{m_k}$. Since $f_{k+1}(ß_{X_{n_k}}) \subset ß_Y = \bigcup_{i\in\omega} ß_{Y_i}$, there exists a number m_{k+1} such that $f_{k+1}(ß_{X_{n_k}}) \subset ß_{Y_{m_{k+1}}}$ and hence $f_{k+1}(X_{n_k}) \subset Y_{m_{k+1}}$. Since the based space Y is rational, its based subspace $Y_{m_{k+1}}$ is an object of the category $\mathfrak{RI}_K$ and the morphism $f_{k+1}\colon X_{n_k} \to Y_{m_{k+1}}$ is an $\mathfrak{RI}_K$-morphism.

By analogy we can use the $\mathfrak{RI}_K$-universality of the based Banach space X to find a number $n_{k+1} > n_k$ and an $\mathfrak{RI}_K$-morphism $g_{k+1}\colon Y_{m_{k+1}} \to X_{n_{k+1}}$ such that $g_{k+1} \circ f_{k+1}$ is the identity inclusion X_{n_k} in $X_{n_{k+1}}$. This completes the inductive step.

After completing the inductive construction consider two isometries $f\colon \bigcup_{n\in\omega} X_n \to \bigcup_{m\in\omega} Y_m$ and $g\colon \bigcup_{m\in\omega} Y_m \to \bigcup_{n\in\omega} X_n$ such that $f \restriction X_{n_k} = f_{k+1}$ and $g \restriction Y_{m_k} = g_k$ for every $k \in \omega$.

By the uniform continuity, the isometries f, g extend to isometries $\bar f\colon X \to Y$ and $\bar g\colon Y \to X$.

The condition (2) of the inductive construction implies that $\bar f \circ \bar g = \mathrm{id}_Y$ and $\bar g \circ \bar f = \mathrm{id}_X$, so f and g are isometric isomorphisms of the Banach spaces X and Y. Since the isometries $g_k\colon Y_{m_k} \to X_{n_k}$ are morphisms of based Banach spaces, we get

$$g(ß_Y) = g\Big(\bigcup_{k\in\omega} ß_{Y_{m_k}}\Big) = \bigcup_{k\in\omega} g(ß_{Y_{m_k}}) = \bigcup_{k\in\omega} g_k(ß_{Y_{m_k}}) \subset \bigcup_{k\in\omega} ß_{X_{n_k}} = ß_X.$$

By analogy we can show that $f(ß_X) \subset ß_Y$. So, f and g are $\mathfrak{BI}$-isomorphisms. □

5. Almost $\mathfrak{FI}_K$-universality

By analogy with the $\mathfrak{RI}_K$-universal based Banach space, one can try to introduce a $\mathfrak{FI}_K$-universal based Banach space. However such notion is vacuous as each based Banach space has only countably many finite-dimensional based subspaces whereas the category $\mathfrak{FI}_K$ contains continuum many pairwise non $\mathfrak{BI}$-isomorphic 2-dimensional based Banach spaces. A "right" definition is that of an almost $\mathfrak{FI}_K$-universal based Banach space, introduced with the help of ε-isometries.

For a positive real number ε, a linear operator $f\colon X \to Y$ between Banach spaces X and Y is called an *ε-isometry* if

$$(1+\varepsilon)^{-1}\|x\|_X < \|f(x)\|_Y < (1+\varepsilon)\|x\|_X$$

for every $x \in X \setminus \{0\}$. This definition implies that each ε-isometry is an injective linear operator.

A morphism of the category $\mathfrak{B}$ of based Banach spaces is called an *ε-isometry $\mathfrak{B}$-morphism* if it is an ε-isometry of the underlying Banach spaces.

Definition 5.1. A based Banach space X called *almost $\mathfrak{FI}_K$-universal* if for any $\varepsilon > 0$ and finite dimensional K-based Banach space A, any ε-isometry $\mathfrak{B}$-morphism $f\colon \Lambda \to X$ defined on a based subspace Λ of A can be extended to an ε-isometry $\mathfrak{B}$-morphism $\bar{f}\colon A \to X$.

Theorem 5.2. *Any $\mathfrak{RI}_K$-universal rational K-based Banach space X is almost $\mathfrak{FI}_K$-universal.*

Proof: We shall use the fact that the norm of any finite-dimensional based Banach space can be approximated by a rational norm (which means that its unit ball coincides with the convex hull of finitely many points having rational coordinates in the basis).

To prove that X is almost $\mathfrak{FI}_K$-universal, take any $\varepsilon > 0$, any finite-dimensional K-based Banach space A and an ε-isometry $\mathfrak{B}$-morphism $f\colon \Lambda \to X$ defined on a based subspace Λ of A. We recall that by $\|\cdot\|_A$ and $\|\cdot\|_\Lambda$ we denote the norms of the Banach spaces A and Λ. The morphism f determines a new norm $\|\cdot\|'_\Lambda$ on Λ, defined by $\|a\|'_\Lambda = \|f(a)\|_X$ for $a \in \Lambda$. Since X is rational and K-based, $\|\cdot\|'_\Lambda$ is a rational norm on Λ such that $\|\mathrm{pr}_F(a)\|'_\Lambda \le K\|a\|'_\Lambda$ for every $a \in \Lambda$ and every subset $F \subset ß_\Lambda$. Taking into account that f is an ε-isometry, we conclude that $(1+\varepsilon)^{-1} < \|a\|'_\Lambda < (1+\varepsilon)$ for every $a \in \Lambda$ with $\|a\|_\Lambda = 1$. By the compactness of the unit sphere in Λ, there exists a positive $\delta < \varepsilon$ such that $(1+\delta)^{-1} < \|a\|'_\Lambda < (1+\delta)$ for every $a \in \Lambda$ with $\|a\|_\Lambda = 1$. This inequality implies $(1+\delta)^{-1}B_\Lambda \subset B'_\Lambda \subset (1+\delta)B_\Lambda$, where $B_\Lambda = \{a \in \Lambda\colon \|a\|_\Lambda \le 1\}$ and $B'_\Lambda = \{a \in \Lambda\colon \|a\|'_\Lambda \le 1\}$ are the closed unit balls of Λ in the norms $\|\cdot\|_\Lambda$ and $\|\cdot\|'_\Lambda$. Choose δ' such that $\delta < \delta' < \varepsilon$. Also choose a nonnegative real number $c \le K-1$ such that $K-c$ is rational and $K/(K-c) < 1+\delta$.

Let $B_A = \{a \in A\colon \|x\|_A \le 1\}$ be the closed unit ball of the Banach space A. Choose a rational polyhedron P in A such that $P = -P$ and $(1+\delta')^{-1}B_A \subset P \subset (1+\delta)^{-1}B_A$. Next consider the convex hull $B'_A := \mathrm{conv}(P')$ of the set $P' = B'_\Lambda \cup P \cup \bigcup_{F\subset ß_A}(K-c)^{-1}\mathrm{pr}_F(P)$ and observe that B'_A is a rational polyhedron in the based Banach space A. Taking into account that $P \subset (1+\delta)^{-1}B_A$, $B'_\Lambda \subset (1+\delta)B_\Lambda \subset (1+\delta)B_A$, and A is a K-based Banach space, we conclude that

$$P' \subset B'_\Lambda \cup \frac{1}{1+\delta}\Big(B_A \cup \bigcup_{F\subset ß_A} \frac{1}{K-c}\mathrm{pr}_F(B_A)\Big) = B'_\Lambda \cup \frac{1}{1+\delta}\Big(B_A \cup \frac{K}{K-c}B_A\Big)$$

$$= B'_\Lambda \cup \frac{1}{1+\delta}\frac{K}{K-c}B_A \subset (1+\delta)B_\Lambda \cup \frac{1}{1+\delta}(1+\delta)B_A \subset (1+\delta)B_A$$

and hence

$$\frac{1}{1+\delta'}B_A \subset P \subset B'_A := \mathrm{conv}(P') \subset (1+\delta)B_A.$$

The convex symmetric set $B'_A := \mathrm{conv}(P')$ determines a rational norm $\|\cdot\|'_A$ on A whose unit ball coincides with B'_A. We claim that the base ß_A of the Banach space $A' := (A, \|\cdot\|'_A)$ is K-suppression. Indeed, for any set $F \subset \text{ß}_A$ we have

$$\begin{aligned}\mathrm{pr}_F(P') = \mathrm{pr}_F(B'_\Lambda) \cup \mathrm{pr}_F(P) \cup \bigcup_{E \subset \text{ß}_A} \frac{1}{K-c}\mathrm{pr}_F \circ \mathrm{pr}_E(P)\\ \subset K\, B'_\Lambda \cup (K-c)P' \cup P' \subset K\, P'\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{pr}_F(B'_A) = \mathrm{pr}_F(\mathrm{conv}(P')) = \mathrm{conv}(\mathrm{pr}_F(P'))\\ \subset \mathrm{conv}(K\, P') = K\,\mathrm{conv}(P') = K\, B'_A,\end{aligned}$$

which means that the projection $\mathrm{pr}_F\colon A' \to A'$ has norm less than or equal to K and A' is a K-based Banach space.

It remains to check that $\|a\|'_A = \|a\|'_\Lambda$ for each $a \in \Lambda$, which is equivalent to the equality $B'_A \cap \Lambda = B'_\Lambda$. The inclusion $B'_\Lambda \subset B'_A \cap \Lambda$ is evident. To prove the reverse inclusion $B'_\Lambda \supset B'_A \cap \Lambda$ observe that

$$\begin{aligned}\Lambda \cap B'_A &= \Lambda \cap \mathrm{conv}(P') \subset \Lambda \cap \mathrm{conv}\Big(B'_\Lambda \cup \frac{1}{1+\delta}B_A\Big)\\ &= \Lambda \cap \Big\{t\lambda + (1-t)a\colon\ t \in [0,1],\ \lambda \in B'_\Lambda,\ a \in \frac{1}{1+\delta}B_A\Big\}\\ &= \Big\{t\lambda + (1-t)a\colon\ t \in [0,1],\ \lambda \in B'_\Lambda,\ a \in \frac{1}{1+\delta}(\Lambda \cap B_A)\Big\}\\ &\subset \mathrm{conv}(B'_\Lambda \cup B'_\Lambda) = B'_\Lambda.\end{aligned}$$

The inclusions $(1+\delta')^{-1}B_A \subset B'_A \subset (1+\delta)B_A$ imply the strict inequality

$$(1+\varepsilon)^{-1}\|a\|_A < \|a\|'_A < (1+\varepsilon)\|a\|_A \tag{1}$$

holding for all $a \in A\backslash\{0\}$.

Let Λ' and A' be the K-based Banach spaces Λ and A endowed with the new rational norms $\|\cdot\|'_\Lambda$ and $\|\cdot\|'_A$, respectively. It is clear that $\Lambda' \subset A'$. The definition of the norm $\|\cdot\|'_\Lambda$ ensures that $f\colon \Lambda' \to X$ is a $\mathfrak{BI}$-morphism. Using the $\mathfrak{RI}_K$-universality of X, extend the isometry morphism $f\colon \Lambda' \to X$ to an isometry morphism $\bar{f}\colon A' \to X$. The inequalities (1) ensure that $\bar{f}\colon A \to X$ is an ε-isometry $\mathfrak{B}$-morphism from A, extending the ε-isometry f. This completes the proof of the almost $\mathfrak{FI}_K$-universality of X. □

Theorem 5.3. *Let X and Y be almost $\mathfrak{FI}_K$-universal K-based Banach spaces and $\varepsilon > 0$. Each ε-isometry $\mathfrak{B}$-morphism $f\colon X_0 \to Y$ defined on a finite-dimensional based subspace X_0 of the K-based Banach space X can be extended to an ε-isometry $\mathfrak{B}$-isomorphism $\bar{f}\colon X \to Y$.*

Proof: Fix a positive real number ε. Using the compactness of the unit sphere of the finite dimensional Banach space X_0, we can find a positive $\delta < \varepsilon$ such that f is a δ-isometry. Write X and Y as the completions of the unions $\bigcup_{n\in\omega} X_n$ and $\bigcup_{n\in\omega} Y_n$ of chains of finite dimensional K-based Banach spaces such that $Y_0 = f(X_0)$. We define inductively sequences of $\mathfrak{B}$-morphisms $\{f_k\}_{k\in\omega}$, $\{g_k\}_{k\in\omega}$ and increasing number sequences (n_k), (m_k) such that $n_0 = m_0 = 0$, $f_0 = f$ and the following conditions are satisfied for every $k \in \omega$:

(1) $f_k\colon X_{n_{k-1}} \to Y_{m_k}$ and $g_k\colon Y_{m_k} \to X_{n_k}$ are δ-isometry $\mathfrak{B}$-morphisms;

(2) $f_{k+1} \circ g_k = \mathrm{id} \restriction Y_{m_k}$ and $g_{k+1} \circ f_{k+1} = \mathrm{id} \restriction X_{n_k}$.

To make the inductive step assume that for some $k \in \omega$, the numbers n_k, m_k and δ-isometries $f_k\colon X_{n_{k-1}} \to Y_{m_k}$, $g_k\colon Y_{m_k} \to X_{n_k}$ have been constructed. Definition 5.1 of almost $\mathfrak{FI}_K$-universality of the based Banach space Y yields a δ-isometry $\mathfrak{B}$-morphism $f_{k+1}\colon X_{n_k} \to Y$ such that $f_{k+1} \mid g_k(Y_{m_k}) = g_k^{-1} \mid g_k(Y_{m_k})$ and hence $f_{k+1} \circ g_k = \mathrm{id} \restriction Y_{m_k}$. Since $f_{k+1}(ß_{X_{n_k}})$ is a finite subset of the basis $ß_Y = \bigcup_{i\in\omega} ß_{Y_i}$ of Y, there exists a number $m_{k+1} > m_k$ such that $f_{k+1}(ß_{X_{n_k}}) \subset ß_{Y_{m_{k+1}}}$ and hence $f_{k+1}(X_{n_k}) \subset Y_{m_{k+1}}$.

By analogy, we can use the almost $\mathfrak{FI}_K$-universality of the based Banach space X and find a number $n_{k+1} > n_k$ and a δ-isometry $\mathfrak{B}$-morphism $g_{k+1}\colon Y_{m_{k+1}} \to X_{n_{k+1}}$ such that $g_{k+1} \circ f_{k+1} = \mathrm{id} \restriction X_{n_k}$. This completes the inductive step.

After completing the inductive construction consider two δ-isometries $\tilde{f}\colon \bigcup_{n\in\omega} X_n \to \bigcup_{m\in\omega} Y_m$ and $\tilde{g}\colon \bigcup_{m\in\omega} Y_m \to \bigcup_{n\in\omega} X_n$ such that for every $k \in \omega$, $\tilde{f} \restriction X_{n_k} = f_{k+1}$ and $\tilde{g} \restriction Y_{m_k} = g_k$. The condition (2) of the inductive construction implies that $\tilde{f} \circ \tilde{g}$ and $\tilde{g} \circ \tilde{f}$ are the identity maps of $\bigcup_{n\in\omega} X_n$ and $\bigcup_{m\in\omega} Y_m$, respectively.

Using the uniform continuity, the δ-isometries $\tilde{f}$, $\tilde{g}$ extend to ε-isometries $\bar{f}\colon X \to Y$ and $\bar{g}\colon Y \to X$ such that $\bar{f} \circ \bar{g} = \mathrm{id}_Y$ and $\bar{g} \circ \bar{f} = \mathrm{id}_X$. Taking into account that f_n and g_n are $\mathfrak{B}$-morphisms, we can show (repeating the argument from the proof of Theorem 4.5) that the operators $\tilde{f}$ and $\tilde{g}$ preserve the bases of the K-based Banach spaces X and Y and hence are $\mathfrak{B}$-isomorphisms. $\square$

Corollary 5.4. *For any almost $\mathfrak{FI}_K$-universal K-based Banach spaces X and Y and any $\varepsilon > 0$ there exists an ε-isometry $\mathfrak{B}$-isomorphism $f\colon X \to Y$.*

Theorem 5.5. *Let U be an almost $\mathfrak{FI}_K$-universal K-based Banach space. For any $\varepsilon > 0$ and any K-based Banach space X there exists an ε-isometry $\mathfrak{B}$-morphism $f\colon X \to U$.*

Proof: Write X as the completion of the union $\bigcup_{n\in\omega} X_n$ of a chain of finite dimensional K-based Banach subspaces X_n of X such that $X_0 = \{0\}$. Fix a positive real number ε and choose any $\delta < \varepsilon$. We shall define inductively a sequence of δ-isometry $\mathfrak{B}$-morphisms $(f_k\colon X_k \to U)_{k=0}^{\infty}$ such that $f_k \restriction X_{k-1} = f_{k-1}$ for every $k > 0$.

We set $f_0 = 0$. Suppose that for some $k \in \omega$ a δ-isometry $\mathfrak{B}$-morphism $f_k \colon X_k \to U$ has already been constructed. Using the definition of the almost $\mathfrak{F}\mathfrak{I}_K$-universality of the space U, we can find a δ-isometry $\mathfrak{B}$-morphism $f_{k+1} \colon X_{k+1} \to U$ such that $f_{k+1} \restriction X_k = f_k$. This completes the inductive step.

After completing the inductive construction consider the δ-isometry f such that $f \restriction X_k = f_k$ for every $k \in \omega$; $f \colon \bigcup_{k\in\omega} X_k \to U$.

By the uniform continuity, the δ-isometry f extends to an ε-isometry $\bar{f} \colon X \to U$ such that

$$f(\text{ß}_X) = f\big(\bigcup_{k\in\omega} B_{X_k}\big) = \bigcup_{k\in\omega} f(B_{X_k}) = \bigcup_{k\in\omega} f_k(B_{X_k}) \subset \text{ß}_{\mathbb{U}},$$

which means that f is a $\mathfrak{B}_K$-morphism. □

Corollary 5.6. *Each almost $\mathfrak{F}\mathfrak{I}_K$-universal K-based Banach space U is $\mathfrak{B}$-universal.*

Proof: Given a based Banach space X, we need to prove that X is $\mathfrak{B}$-isomorphic to a based subspace of U. Denote by X_1 the based Banach space X endowed with the equivalent norm

$$\|x\|_1 = \sup_{F\subset \text{ß}_X} \|\mathrm{pr}_F(x)\|.$$

It is easy to check that X_1 is a 1-based Banach space. By Theorem 5.5, for $\varepsilon = 1/2$ there exists an ε-isometry $\mathfrak{B}$-morphism $f \colon X_1 \to U$. Then f is a $\mathfrak{B}$-isomorphism between X and the based subspace $f(X) = f(X_1)$ of the based Banach space U. □

Corollary 5.6 combined with the Uniqueness Theorem 3.2 of Pełczyński implies

Corollary 5.7. *Each almost $\mathfrak{F}\mathfrak{I}_K$-universal K-based Banach space U_K is $\mathfrak{B}$-isomorphic to the $\mathfrak{B}$-universal space $\mathbb{U}$ of Pełczyński.*

Combining Corollary 5.7 with Theorem 5.2, we get another model of the $\mathfrak{B}$-universal Pełczyński's space $\mathbb{U}$.

Corollary 5.8. *Each $\mathfrak{R}\mathfrak{I}_K$-universal rational K-based Banach space $\mathbb{U}_K$ is $\mathfrak{B}$-isomorphic to the $\mathfrak{B}$-universal Pełczyński's space $\mathbb{U}$.*

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