A note on Dunford-Pettis like properties and complemented spaces of operators

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Abstract. Equivalent formulations of the Dunford-Pettis property of order p (DPP_p), $1 < p < \infty$, are studied. Let $\mathrm{L}(X,Y)$, $\mathrm{W}(X,Y)$, $\mathrm{K}(X,Y)$, $\mathrm{U}(X,Y)$, and $\mathrm{C}_p(X,Y)$ denote respectively the sets of all bounded linear, weakly compact, compact, unconditionally converging, and p-convergent operators from X to Y. Classical results of Kalton are used to study the complementability of the spaces $\mathrm{W}(X,Y)$ and $\mathrm{K}(X,Y)$ in the space $\mathrm{C}_p(X,Y)$, and of $\mathrm{C}_p(X,Y)$ in $\mathrm{U}(X,Y)$ and $\mathrm{L}(X,Y)$.

Keywords: Dunford-Pettis property of order p; p-convergent operator; complemented spaces of operators

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1. Introduction

In this paper we study equivalent formulations of the DPP_p, $1 < p < \infty$. We give a characterization of dual Banach spaces with the DPP_p. We show that X^* has the DPP_p if and only if every operator $T\colon X \to Y$ with weakly p-compact adjoint has a completely continuous bitranspose, $1 < p < \infty$. Our results are motivated by results in [3].

For many years mathematicians have been interested in the problem of whether an operator ideal is complemented in the space $\mathrm{L}(X,Y)$ of all bounded linear operators between X and Y, e.g. see [10], [9], [17], [12], and [11]. In [1] the authors studied the complementability of the space $\mathrm{W}(X,\ell_\infty)$ in $\mathrm{L}(X,\ell_\infty)$. It was shown that if X is not reflexive, then $\mathrm{W}(X,\ell_\infty)$ is not complemented in $\mathrm{L}(X,\ell_\infty)$, see [1, Theorem 3]. Let $\mathrm{CC}(X,Y)$, $\mathrm{Lcc}(X,Y)$, or $\mathrm{LC}_p(X,Y)$ denote the set of all completely continuous, limited completely continuous, or limited p-convergent, respectively, operators from X to Y. We use classical results of Kalton to study the complementability of $\mathrm{W}(X,Y)$, $\mathrm{K}(X,Y)$, and $\mathrm{CC}(X,Y)$ in $\mathrm{C}_p(X,Y)$, and of $\mathrm{C}_p(X,Y)$ in $\mathrm{U}(X,Y)$. Further, we study the complementability of $\mathrm{C}_p(X,Y)$, $\mathrm{Lcc}(X,Y)$, and $\mathrm{LC}_p(X,Y)$ in $\mathrm{L}(X,Y)$.

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2. Definitions and notation

Throughout this paper, X, Y, E and F denote Banach spaces. The unit ball of X is denoted by B_X and X^* denotes the continuous linear dual of X. The space X embeds in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An operator $T\colon X \to Y$ is a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y is denoted by $\mathrm{L}(X,Y)$, $\mathrm{W}(X,Y)$, and $\mathrm{K}(X,Y)$.

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. An operator $T\colon X \to Y$ is called *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact.

An operator $T\colon X \to Y$ is called *completely continuous* (or *Dunford-Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

For $1 \le p < \infty$, p^* denotes the conjugate of p. If $p = 1$, ℓ_{p^*} plays the role of c_0. The unit vector basis of ℓ_p is denoted by (e_n).

Let $1 \le p \le \infty$. A sequence (x_n) in X is called *weakly p-summable sequence* if $(x^*(x_n)) \in \ell_p$ for each $x^* \in X^*$ [6, page 32]. Let $\ell_p^w(X)$ denote the set of all weakly p-summable sequences in X. The space $\ell_p^w(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

We recall the following isometries: $\mathrm{L}(\ell_{p^*}, X) \simeq \ell_p^w(X)$ for $1 < p < \infty$; $\mathrm{L}(c_0, X) \simeq \ell_p^w(X)$ for $p = 1$; $T \to (T(e_n))$, see [6, Proposition 2.2, page 36].

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* if for every $x^* \in X^*$ the series $\sum |x^*(x_n)|$ is convergent. An operator $T\colon X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

Let $1 \le p \le \infty$. An operator $T\colon X \to Y$ is called *p-convergent* if T maps weakly p-summable sequences into norm null sequences, see [3]. The set of all p-convergent operators is denoted by $\mathrm{C}_p(X,Y)$.

The 1-convergent operators are precisely the unconditionally converging operators and the ∞-convergent operators are precisely the completely continuous operators. If $p < q$, then $C_q(X,Y) \subseteq \mathrm{C}_p(X,Y)$.

A sequence (x_n) in X is called *weakly p-convergent* to $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable, see [3]. Weakly ∞-convergent sequences are precisely the weakly convergent sequences.

Let $1 \le p \le \infty$. A bounded subset K of X is *relatively weakly p-compact* if every sequence in K has a weakly p-convergent subsequence. An operator $T\colon X \to Y$ is *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact, see [3].

The set of weakly p-compact operators $T\colon X \to Y$ is denoted by $\mathrm{W}_p(X,Y)$. If $p < q$, then $\mathrm{W}_p(X,Y) \subseteq \mathrm{W}_q(X,Y)$. A Banach space $X \in \mathrm{C}_p$ (or $X \in \mathrm{W}_p$) if $\mathrm{id}(X) \in \mathrm{C}_p(X,X)$ (or $\mathrm{id}(X) \in \mathrm{W}_p(X,X)$, respectively), see [3], where $\mathrm{id}(X)$ is the identity map on X.

A sequence (x_n) in X is called *weakly p-Cauchy* if $(x_{n_k} - x_{m_k})$ is weakly p-summable for any increasing sequences (n_k) and (m_k) of positive integers.

Every weakly p-convergent sequence is weakly p-Cauchy, and the weakly ∞-Cauchy sequences are precisely the weakly Cauchy sequences.

Let $1 \le p \le \infty$. We say that a subset S of X is called *weakly p-precompact* if every sequence from S has a weakly p-Cauchy subsequence. The weakly ∞-precompact sets are precisely the weakly precompact sets.

Let $1 \le p \le \infty$. An operator $T\colon X \to Y$ is called *weakly p-precompact* (or *almost weakly p-compact*) if $T(B_X)$ is weakly p-precompact. The set of all weakly p-precompact operators is denoted by $\mathrm{WPC}_p(X,Y)$. We say that $X \in \mathrm{WPC}_p$ if $\mathrm{id}(X) \in \mathrm{WPC}_p(X,X)$.

The weakly ∞-precompact operators are precisely the weakly precompact operators. If $p < q$, then $\ell_p^w(X) \subseteq \ell_q^w(X)$, thus $\mathrm{WPC}_p(X,Y) \subseteq \mathrm{WPC}_q(X,Y)$.

A Banach space X has the *Dunford-Pettis property* (DPP) if every weakly compact operator $T\colon X \to Y$ is completely continuous for any Banach space Y. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ whenever (x_n^*) is weakly null in X^* and (x_n) is weakly null in X, see [4, Theorem 1]. If X is a $C(K)$-space or an L_1-space, then X has the DPP. The reader can check [5], [4], and [7] for results related to the DPP.

The bounded subset A of X is called a *Dunford-Pettis* (or *limited*) subset of X if each weakly null (or w*-null, respectively) sequence (x_n^*) in X^* tends to 0 uniformly on A; i.e.

$$\sup_{x \in A} |x_n^*(x)| \to 0.$$

The bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A; i.e.

$$\sup_{x^* \in A} |x^*(x_n)| \to 0.$$

A bounded subset A of X^* is called a *V-subset* of X^* provided that

$$\sup_{x^* \in A} |x^*(x_n)| \to 0$$

for each weakly unconditionally convergent series $\sum x_n$ in X.

The Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact. The following results were established in [21]: $C(K)$ spaces and reflexive spaces have property (V); X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact.

Let $1 \le p \le \infty$. A Banach space X has the *Dunford-Pettis property of order p* (DPP_p) if every weakly compact operator $T\colon X \to Y$ is p-convergent for any Banach space Y, see [3].

If X has the DPP_p, then it has the DPP_q, if $q < p$. Also, the DPP_∞ is precisely the DPP, and every Banach space X has the DPP_1. $C(K)$ spaces and L_1 have the DPP, and thus the DPP_p for all p. If $1 < r < \infty$, then ℓ_r has the DPP_p for $p < r^*$. If $1 < r < \infty$, then $L_r(\mu)$ has the DPP_p for $p < \min(2, r^*)$. Tsirelson's space T has the DPP_p for all $p < \infty$. Since T is reflexive, it does not have the DPP. Tsirelson's dual space T^* does not have the DPP_p, if $p > 1$, see [3].

Let $1 \le p < \infty$. We say that a bounded subset A of X^* is called a *weakly p-L-set* if for all weakly p-summable sequences (x_n) in X,

$$\sup_{x^* \in A} |x^*(x_n)| \to 0.$$

The weakly 1-L-subsets of X^* are precisely the V-subsets. If $p < q$, then a weakly q-L-subset is a weakly p-L-subset, since $\ell_p^w(X) \subseteq \ell_q^w(X)$.

The Banach space X has the *reciprocal Dunford-Pettis* (RDP) property if every completely continuous operator T from X to any Banach space Y is weakly compact, see [16, page 153]. The space X has the RDP property if and only if every L-subset of X^* is relatively weakly compact, see [15]. Banach spaces with property (V) of Pełczyński, in particular reflexive spaces and $C(K)$ spaces, have the RDP property, see [21].

Let $1 \le p < \infty$. We say that the space X has the *reciprocal Dunford-Pettis of order p* or RDP_p property if every weakly p-L-subset of X^* is relatively weakly compact.

If X has the RDP_p property, then X has the RDP property (since any L-subset of X^* is a weakly p-L-set). If $p < q$ and X has the RDP_p property, then X has the RDP_q property.

The space X has the *Gelfand-Phillips (GP) property* (or is a *Gelfand-Phillips space*) if every limited subset of X is relatively compact. Schur spaces and separable spaces have the Gelfand-Phillips property, see [2].

The sequence (x_n) in X is called limited if the corresponding set of its terms is a limited set. If the sequence (x_n) is also weakly null (or weakly p-summable), then (x_n) is called a limited weakly null (or limited weakly p-summable, respectively) sequence in X.

An operator $T\colon X \to Y$ is called *limited completely continuous* (lcc) if it maps limited weakly null sequences to norm null sequences, see [22].

Let $1 \le p < \infty$. A Banach space X has the *p-Gelfand-Phillips(p-GP) property* (or is a *p-Gelfand-Phillips space*) if every limited weakly p-summable sequence in X is norm null, see [13]. If X has the GP property, then X has the p-GP property for any $1 \le p < \infty$.

3. The Dunford-Pettis property of order p

The following theorem gives equivalent conditions for a Banach space X to have the DPP_p. We note that an operator $T\colon X \to Y$ is p-convergent if and only if T takes weakly p-compact subsets of X into norm compact subsets of Y.

Theorem 1. *Let $1 < p < \infty$. The following statements are equivalent about a Banach space X.*

(1) *X has the* DPP_p.
(2) *If (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly null sequence in X^*, then $x_n^*(x_n) \to 0$.*
(3) *For all Banach spaces Y, every weakly compact operator $T\colon X \to Y$ is p-convergent.*
(4) *Every weakly compact operator $T\colon X \to c_0$ is p-convergent.*
(5) *If (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly Cauchy sequence in X^*, then $x_n^*(x_n) \to 0$.*
(6) *For all Banach spaces Y, every operator $T\colon X \to Y$ with weakly precompact adjoint is p-convergent.*
(7) *Every operator $T\colon X \to c_0$ with weakly precompact adjoint is p-convergent.*
(8) *If (x_n) is a weakly p-Cauchy sequence in X and (x_n^*) is a weakly null sequence in X^*, then $x_n^*(x_n) \to 0$.*
(9) *If $T\colon Y \to X$ is a weakly p-precompact operator, then $T^*\colon X^* \to Y^*$ is completely continuous for all Banach spaces Y.*
(10) *If $T\colon \ell_{p^*} \to X$ is an operator, then $T^*\colon X^* \to \ell_p$ is completely continuous.*

Proof: The statements (1), (2), and (3) are equivalent by [3, Proposition 3.2].

(2) $\Rightarrow$ (5) Suppose (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly Cauchy sequence in X^*, but $x_n^*(x_n) \not\to 0$. By passing to a subsequence if necessary, assume that $|x_n^*(x_n)| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$. Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|x_{n_1}^*(x_{n_2})| < \epsilon/2$. We can do this since (x_n) is weakly null. Continue inductively. Choose $n_{k+1} > n_k$ so that $|x_{n_k}^*(x_{n_{k+1}})| < \epsilon/2$. By hypothesis, $(x_{n_{k+1}}^* - x_{n_k}^*)(x_{n_{k+1}}) \to 0$. Since

$$|(x_{n_{k+1}}^* - x_{n_k}^*)(x_{n_{k+1}})| \geq |x_{n_{k+1}}^*(x_{n_{k+1}})| - |x_{n_k}^*(x_{n_{k+1}})| > \frac{\epsilon}{2},$$

we have a contradiction.

(5) $\Rightarrow$ (6) Let $T\colon X \to Y$ be an operator with weakly precompact adjoint such that T is not p-convergent. Let (x_n) be a weakly p-summable sequence in X so that $\|T(x_n)\| > \epsilon$. Let (y_n^*) be a sequence in B_{Y^*} such that $y_n^*(T(x_n)) > \epsilon$ and let $x_n^* = T^*(y_n^*)$. Since T^* is weakly precompact, we can assume that (x_n^*) is weakly Cauchy. By assumption, $x_n^*(x_n) = T^*(y_n^*)(x_n) \to 0$, a contradiction.

(3) $\Rightarrow$ (4), (6) $\Rightarrow$ (7), and (7) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2) Let (x_n) be a weakly p-summable sequence in X and (x_n^*) be a weakly null sequence in X^*. Define $T\colon X \to c_0$, $T(x) = (x_i^*(x))$. Note that $T^*\colon \ell_1 \to X^*$, $T^*(b) = \sum b_i x_i^*$, $b = (b_i) \in \ell_1$. Note that T^* takes B_{ℓ_1} into the closed and absolutely convex hull of $\{x_i^*\colon i \in \mathbb{N}\}$, which is a relatively weakly compact set, see [7, page 51]. Then T^*, hence T, is weakly compact. By assumption, T is p-convergent. Thus $|x_n^*(x_n)| \leq \|T(x_n)\| = \sup_i |x_i^*(x_n)| \to 0$.

Thus (1)–(7) are equivalent.

(2) $\Rightarrow$ (8) Let (x_n) be weakly p-Cauchy in X and (x_n^*) be weakly null in X^*. Suppose by contradiction that $x_n^*(x_n) \not\to 0$. Without loss of generality assume that $|x_n^*(x_n)| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$. Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|x_{n_2}^*(x_{n_1})| < \epsilon/2$. We can do this since (x_n^*) is w^*-null. Continue inductively. Choose $n_k > n_{k-1}$ so that $|x_{n_k}^*(x_{n_{k-1}})| < \epsilon/2$. By hypothesis, $x_{n_k}^*(x_{n_k} - x_{n_{k-1}}) \to 0$. However,

$$|x_{n_k}^*(x_{n_k} - x_{n_{k-1}})| \geq |x_{n_k}^*(x_{n_k})| - |x_{n_k}^*(x_{n_{k-1}})| > \frac{\epsilon}{2},$$

a contradiction.

(8) $\Rightarrow$ (2) is obvious, since every weakly p-summable sequence in X is weakly p-Cauchy.

(8) $\Rightarrow$ (9) Suppose $T\colon Y \to X$ is a weakly p-precompact operator. Suppose $T^*\colon X^* \to Y^*$ is not completely continuous. Let (x_n^*) be a weakly null sequence in X^* such that $\|T^*(x_n^*)\| > \epsilon$ for some $\epsilon > 0$. Choose (y_n) in B_Y such that $\langle T^*(x_n^*), y_n\rangle > \epsilon$. Without loss of generality we can assume that $(T(y_n))$ is weakly p-Cauchy. Hence $\langle T(y_n), x_n^*\rangle \to 0$, a contradiction.

(9) $\Rightarrow$ (10) Suppose $T\colon \ell_{p^*} \to X$ is an operator. Since $1 < p^* < \infty$, $\ell_{p^*} \in W_p$, see [3, Proposition 1.4]. Then T is weakly p-compact. Thus T^* is completely continuous.

(10) $\Rightarrow$ (2) Suppose (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly null sequence in X^*. Define $T\colon \ell_{p^*} \to X$ by $T(b) = \sum b_i x_i$, $b = (b_i) \in \ell_{p^*}$. Note that $T^*\colon X^* \to \ell_p$, $T^*(x^*) = (x^*(x_i))$. Since T^* is completely continuous, $|x_n^*(x_n)|^p \leq \|T^*(x_n^*)\|^p = \sum_i |x_n^*(x_i)|^p \to 0$. $\square$

Corollary 2. *Let $1 < p < \infty$. If X has the* DPP_p *and Y is complemented in X, then Y has the* DPP_p.

Proof: Suppose X has the DPP_p and let $P\colon X \to Y$ be a projection. Let (y_n) be a weakly p-summable sequence in Y and (y_n^*) be a weakly null sequence in Y^*. Since $(P^* y_n^*)$ is weakly null in X^*, by Theorem 1, $\langle y_n^*, P(y_n)\rangle = \langle P^* y_n^*, y_n\rangle \to 0$. Thus Y has the DPP_p. $\square$

Corollary 3. *Let $1 < p < \infty$. Then the following are equivalent:*

(i) *X has the* DPP_p*;*
(ii) *every weakly precompact subset of X^* is a weakly p-L-set;*
(iii) *every weakly p-precompact subset of X is a DP set.*

Proof: (i) $\Rightarrow$ (ii) Suppose X has the DPP_p. Let A be weakly precompact subset of X^* and let (x_n^*) be a sequence in A. By passing to a subsequence, we may suppose that (x_n^*) is weakly Cauchy. Let (x_n) be a weakly p-summable sequence in X. By Theorem 1, $x_n^*(x_n) \to 0$. Hence A is a weakly p-L-set.

(ii) $\Rightarrow$ (i) Let (x_n^*) be a weakly Cauchy sequence in X^* and (x_n) be a weakly p-summable sequence in X. Since $\{x_n^*\colon n \in \mathbb{N}\}$ is a weakly p-L-subset of X^*, $x_n^*(x_n) \to 0$. By Theorem 1, X has the DPP_p.

(i) $\Rightarrow$ (iii) Suppose X has the DPP_p. Let A be a weakly p-precompact subset of X and let (x_n) be a sequence in A. By passing to a subsequence, we may suppose that (x_n) is weakly p-Cauchy. Suppose (x_n^*) is a weakly null sequence in X^*. By Theorem 1, $x_n^*(x_n) \to 0$. Hence A is a DP set.

(iii) $\Rightarrow$ (i) Let (x_n) be a weakly p-summable sequence in X and (x_n^*) be a weakly null sequence in X^*. Since $\{x_n : n \in \mathbb{N}\}$ is a weakly p-precompact subset of X, it is a DP set. Then $x_n^*(x_n) \to 0$ and X has the DPP_p by Theorem 1. $\square$

We note that an operator $T\colon X \to Y$ is p-convergent if and only if T takes weakly p-precompact subsets of X into norm compact subsets of Y.

Corollary 4. *Let $1 < p < \infty$.*

(i) *Suppose $S\colon X \to Y$ is weakly p-precompact and $T\colon Y \to Z$ is weakly compact. If Y has the DPP_p, then TS is compact.*
(ii) *Suppose X has the DPP_p. If $T\colon X \to X$ is a weakly p-compact operator, then T^2 is compact.*

Proof: (i) Suppose $S\colon X \to Y$ is weakly p-precompact and $T\colon Y \to Z$ is weakly compact. Since Y has the DPP_p, T is p-convergent. Then TS is compact.

(ii) Suppose X has the DPP_p and $T\colon X \to X$ is a weakly p-compact operator. Since T is weakly compact, T^2 is compact by (i). $\square$

Corollary 5. *Let $1 < p < \infty$.*

(i) *Suppose X has the DPP_p. If $Y \in W_p$ and Y is complemented in X, then Y is finite dimensional.*
(ii) *If Y is infinite dimensional and $Y \in W_p$, then Y does not have the DPP_p.*

Proof: (i) Let $P\colon X \to Y$ be a projection of X onto Y. Since $Y \in W_p$, P is weakly p-compact. By Corollary 4, $P = P^2$ is compact. Since $B_Y \subset P(B_X)$, B_Y is relatively compact. Thus Y is finite dimensional.

(ii) Apply (i). $\square$

The following result gives a characterization of dual spaces with the DPP_p.

Theorem 6. *Let $1 < p < \infty$. Let X be a Banach space. The following are equivalent.*

(i) *X^* has the DPP_p.*
(ii) *If $S\colon Y \to X^*$ is a weakly p-precompact operator, then $S^*\colon X^{**} \to Y^*$ is completely continuous for all Banach spaces Y.*
(iii) *If $S\colon \ell_{p^*} \to X^*$ is an operator, then $S^*\colon X^{**} \to \ell_p$ is completely continuous.*
(iv) *If $T\colon X \to Y$ is an operator such that $T^*\colon Y^* \to X^*$ is weakly p-precompact, then $T^{**}\colon X^{**} \to Y^{**}$ is completely continuous for all Banach spaces Y.*
(v) *If $T\colon X \to Y$ is an operator such that $T^*\colon Y^* \to X^*$ is weakly p-compact, then $T^{**}\colon X^{**} \to Y$ is completely continuous for all Banach spaces Y.*

(vi) *If $T\colon X \to \ell_p$ is an operator, then $T^{**}\colon X^{**} \to \ell_p$ is completely continuous.*

PROOF: (i), (ii), and (iii) are equivalent by Theorem 1.

(ii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (v) is clear. We note that since T^* is weakly p-compact, T^*, thus T, is weakly compact. Hence $T^{**}(X^{**}) \subseteq Y$.

(v) $\Rightarrow$ (vi) Suppose $T\colon X \to \ell_p$ is an operator. Since $1 < p^* < \infty$, $\ell_{p^*} \in W_p$, see [3, Proposition 1.4]. Then T^* is weakly p-compact. Thus T^{**} is completely continuous.

(vi) $\Rightarrow$ (i) Suppose (x_n^*) is weakly p-summable in X^* and (x_n^{**}) is weakly null in X^{**}. Define $T\colon X \to \ell_p$ by $T(x) = (x_n^*(x))$. Then $T^*\colon \ell_{p^*} \to X^*$, $T^*(b) = \sum b_i x_i^*$, $b = (b_i) \in \ell_{p^*}$. If $x^{**} \in X^{**}$, then $T^{**}(x^{**}) = (x^{**}(x_i^*))$. Since T^{**} is completely continuous,

$$|x_n^{**}(x_n^*)|^p \le \|T^{**}(x_n^{**})\|^p = \sum_i |x_n^{**}(x_i^*)|^p \to 0,$$

and thus X^* has the DPP_p. $\square$

In the following theorem we use a lifting result of Lohman.

Lemma 7 ([19]). *Let X be a Banach space, Y a subspace not containing copies of ℓ_1, and $Q\colon X \to X/Y$ the quotient map. Let (x_n) be a bounded sequence in X such that $(Q(x_n))$ is weakly Cauchy. Then (x_n) has a weakly Cauchy subsequence.*

Let E be a Banach space and F be a subspace of E^*. Let

$${}^{\perp}F = \{x \in E\colon y^*(x) = 0 \text{ for all } y^* \in F\}.$$

The space $C[0,1]$ has the DPP, and thus the DPP_p for all p. The space ℓ_2 embeds in $C[0,1]$, but ℓ_2 fails the DPP_p for $p \ge 2$ (since $\ell_2 \in W_2$ by [3, Proposition 1.4], it fails the DPP_2, and thus the DPP_p for $p \ge 2$). Hence the DPP_p is not inherited by closed subspaces.

Theorem 8. *Let $1 \le p < \infty$. Suppose E has the DPP_p and F is a w^*-closed subspace of E^* not containing ℓ_1. Then ${}^{\perp}F$ has the DPP_p.*

PROOF: Suppose (x_n) is weakly p-summable in ${}^{\perp}F$ and (z_n^*) is weakly Cauchy in $({}^{\perp}F)^* \simeq E^*/F$. Let $Q\colon E^* \to E^*/F$ be the quotient map. By Lemma 7, we can assume that $z_n^* = Q(x_n^*)$, where (x_n^*) is weakly Cauchy in E^*. Let $i\colon {}^{\perp}F \to E$ be the isometric embedding. By [20, Theorem 1.10.16],

$$\langle x_n, Q(x_n^*)\rangle = \langle i(x_n), x_n^*\rangle.$$

Since E has the DPP_p, $\langle i(x_n), x_n^*\rangle \to 0$ by Theorem 1. Hence $\langle x_n, z_n^*\rangle \to 0$, and ${}^{\perp}F$ has the DPP_p. $\square$

4. Complementability of spaces of operators

We begin by studying the complementability of $\mathrm{W}(X,\ell_\infty)$ and $\mathrm{K}(X,\ell_\infty)$ in $\mathrm{C}_p(X,\ell_\infty)$.

Lemma 9 ([18, Proposition 5])**.** *Let X be a separable Banach space, and $\varphi\colon \ell_\infty \to L(X,\ell_\infty)$ be a bounded linear operator so that $\varphi(e_n)=0$ for all n. Then there is an infinite subset M of $\mathbb{N}$ such that for each $b\in\ell_\infty(M)$, $\varphi(b)=0$, where $\ell_\infty(M)$ is the set of all $b=(b_n)\in\ell_\infty$ with $b_n=0$ for each $n\notin M$.*

Observation 1 ([1, Lemma 2.4])**.** If $T\colon Y\to X^*$ is an operator such that $T^*|_X$ is weakly compact (or compact), then T is weakly compact (or compact, respectively). To see this, let $T\colon Y\to X^*$ be an operator such that $T^*|_X$ is weakly compact (or compact). Let $S=T^*|_X$. Suppose $x^{**}\in B_{X^{**}}$ and choose a net (x_α) in B_X which is w*-convergent to x^{**}. Then $(T^*(x_\alpha))\xrightarrow{\mathrm{w}^*} T^*(x^{**})$. Now, $(T^*(x_\alpha))\subseteq S(B_X)$, which is a relatively weakly compact (or relatively compact) set. Then $(T^*(x_\alpha))\xrightarrow{\mathrm{w}} T^*(x^{**})$ (or $(T^*(x_\alpha))\to T^*(x^{**})$). Hence $T^*(B_{X^{**}})\subseteq\overline{S(B_X)}$, which is relatively weakly compact (or relatively compact). Therefore $T^*(B_{X^{**}})$ is relatively weakly compact (or relatively compact), and thus T is weakly compact (or compact, respectively).

Theorem 10. *Let $1<p<\infty$. If X has the* DPP_p *and X does not have the* RDP_p *property, then $\mathrm{W}(X,\ell_\infty)$ is not complemented in $\mathrm{C}_p(X,\ell_\infty)$.*

Proof: Since X has the DPP_p, every weakly compact operator $T\colon X\to\ell_\infty$ is p-convergent. Let A be a weakly p-L-subset of X^* which is not relatively weakly compact. Let (x_n^*) be a sequence in A with no weakly convergent subsequence. Define $S\colon X\to\ell_\infty$ by $S(x)=(x_n^*(x))_n$, $x\in X$. Since $S^*(e_n^*)=x_n^*$, S^*, thus S, is not weakly compact. Let (y_n) be a sequence in B_X such that $(S(y_n))$ has no weakly convergent subsequence. Let $X_0=[y_n]$ be the closed linear span of $\{y_n\colon n\in\mathbb{N}\}$. Note that X_0 is a separable subspace of X and $L=S|_{X_0}$ is not weakly compact. If $y_n^*=x_n^*|_{X_0}$, then $(y_n^*)\subseteq X_0^*$ is bounded and has no weakly convergent subsequence. (If (y_n^*) is weakly convergent, then $L^*|_{\ell_1}$ is weakly compact, since $L^*(e_n^*)=y_n^*$. By Observation 1, L is weakly compact. This is a contradiction.)

Define $T\colon\ell_\infty\to L(X,\ell_\infty)$ by $T(b)(x)=(b_n\,x_n^*(x))_n$, $b=(b_n)\in\ell_\infty$, $x\in X$. Note that the operator T is well-defined and $T(e_n)=x_n^*\otimes e_n$ for each $n\in\mathbb{N}$. Let $b\in\ell_\infty$ and suppose that (x_m) is a weakly p-summable sequence in X. Since (x_n^*) is a weakly p-L-set,

$$\lim_m \|T(b)(x_m)\|=\lim_m \sup_n |b_n\,x_n^*(x_m)|=0,$$

and thus $T(b)$ is p-convergent.

Suppose that $W(X,\ell_\infty)$ is complemented in $\mathrm{C}_p(X,\ell_\infty)$. Let $P\colon \mathrm{C}_p(X,\ell_\infty)\to \mathrm{W}(X,\ell_\infty)$ be a projection, and let $R\colon \mathrm{L}(X,\ell_\infty)\to\mathrm{L}(X_0,\ell_\infty)$ be the natural restriction map. Define $\varphi\colon\ell_\infty\to\mathrm{C}_p(X_0,\ell_\infty)$ by $\varphi(b)=RT(b)$ and $\psi\colon\ell_\infty\to$

$\mathrm{W}(X_0, \ell_\infty)$ by $\psi(b) = RPT(b)$. Since $T(e_n)$ is a rank one operator, it is compact, hence weakly compact. Thus

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \varphi(e_n)$$

for each $n \in \mathbb{N}$.

By Lemma 9, there is an infinite subset M of $\mathbb{N}$ such that $\psi(\chi_M) = \varphi(\chi_M)$. Hence $\varphi(\chi_M)$ is weakly compact. However, $\varphi(\chi_M)^*(e_n^*) = y_n^*$, $n \in M$. This contradiction concludes the proof. □

We note that every compact operator is p-convergent.

Theorem 11. *Let $1 < p < \infty$. If X is a Banach space such that X^* contains a weakly p-L-subset which is not relatively compact, then $\mathrm{K}(X, \ell_\infty)$ is not complemented in $\mathrm{C}_p(X, \ell_\infty)$.*

Proof: The proof is similar to the proof of Theorem 10. □

Corollary 12. *Let $1 < p < \infty$. Suppose $\ell_\infty \hookrightarrow Y$. Then the following assertions hold.*

(i) *If X has the* DPP_p *and does not have the* RDP_p *property, then $\mathrm{W}(X, Y)$ is not complemented in $\mathrm{C}_p(X, Y)$.*

(ii) *If X^* contains a weakly p-L-subset which is not relatively compact, then $\mathrm{K}(X, Y)$ is not complemented in $\mathrm{C}_p(X, Y)$.*

Proof: We only prove (i). The other proof is similar. Suppose that $\mathrm{W}(X, Y)$ is complemented in $\mathrm{C}_p(X, Y)$. Since ℓ_∞ is injective and $\ell_\infty \hookrightarrow Y$, $\ell_\infty \overset{c}{\hookrightarrow} Y$, see [5, page 71]. Then $\mathrm{W}(X, \ell_\infty)$ is complemented in $\mathrm{W}(X, Y)$, and thus in $\mathrm{C}_p(X, Y)$. Since $\mathrm{W}(X, \ell_\infty) \subseteq \mathrm{C}_p(X, \ell_\infty) \subseteq \mathrm{C}_p(X, Y)$, it follows that $\mathrm{W}(X, \ell_\infty)$ is complemented in $\mathrm{C}_p(X, \ell_\infty)$, a contradiction with Theorem 10. Hence $\mathrm{W}(X, Y)$ is not complemented in $\mathrm{C}_p(X, Y)$. □

In the next corollary we need the following result.

Theorem 13 ([14, Theorem 21])**.** *Let $1 \le p < \infty$. Suppose that X is a Banach space. The following are equivalent.*

(i) *For every Banach space Y, if $T\colon X \to Y$ is a p-convergent operator, then $T^*\colon Y^* \to X^*$ is weakly compact (or compact).*

(ii) *The same as* (i) *with $Y = \ell_\infty$.*

(iii) *Every weakly p-L-subset of X^* is relatively weakly compact (or relatively compact).*

Corollary 14. *Let $1 < p < \infty$. Suppose X and Y are Banach spaces.*

1. *If X has the* DPP_p *property and $\ell_\infty \hookrightarrow Y$, then the following are equivalent:*
   (i) *X has the* RDP_p *property;*
   (ii) *$\mathrm{C}_p(X, Y) = \mathrm{W}(X, Y)$;*
   (iii) *$\mathrm{W}(X, Y)$ is complemented in $\mathrm{C}_p(X, Y)$.*

2. *If $\ell_\infty \hookrightarrow Y$, then the following are equivalent:*
   (i) $\mathrm{C}_p(X,Y) = \mathrm{K}(X,Y)$;
   (ii) $\mathrm{K}(X,Y)$ *is complemented in* $\mathrm{C}_p(X,Y)$.

PROOF: 1. (i) $\Rightarrow$ (ii) Since X has the RDP$_p$ property, $\mathrm{C}_p(X,Y) \subseteq \mathrm{W}(X,Y)$ (by Theorem 13). Since X also has the DPP$_p$, $\mathrm{C}_p(X,Y) = \mathrm{W}(X,Y)$.

(iii) $\Rightarrow$ (i) by Corollary 12.

2. (ii) $\Rightarrow$ (i) Suppose there is a p-convergent operator $T: X \to Y$ which is not compact. By Theorem 13, X^* contains a weakly p-L-subset which is not relatively compact. Hence $\mathrm{K}(X,Y)$ is not complemented in $\mathrm{C}_p(X,Y)$ by Corollary 12. $\square$

Theorem 15. *Let $1 < p < \infty$. Suppose that U has an unconditional and seminormalized basis (u_i) with biorthogonal coefficients (u_i^*), $U \overset{c}{\hookrightarrow} X$, and $T: U \to Y$ is an operator such that $(T(u_i))$ is not relatively weakly p-compact in Y. Let $S(X,Y)$ be a closed linear subspace of $\mathrm{L}(X,Y)$ which properly contains $\mathrm{W}_p(X,Y)$ such that $\varphi(b) \in S(U,Y)$ for all $b \in \ell_\infty$, where $\varphi(b)(u) = \sum b_i u_i^*(u) T(u_i)$, $u \in U$. Then $W_p(X,Y)$ is not complemented in $S(X,Y)$.*

PROOF: The proof is similar to the proof of [1, Theorem 20], replacing "relatively weakly p-compact" with "relatively compact". $\square$

Corollary 16. *Let $1 < p < \infty$. If $\ell_1 \overset{c}{\hookrightarrow} X$ and $Y \notin W_p$, then $W_p(X,Y)$ is not complemented in* $\mathrm{L}(X,Y)$.

PROOF: Let (y_n) be a sequence in B_Y with no weakly p-convergent subsequence and $S(X,Y) = \mathrm{L}(X,Y)$. Define $T: \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. Let $\varphi: \ell_\infty \to L(\ell_1, Y)$, $\varphi(b)(x) = \sum b_n x_n y_n$, $x = (x_n) \in \ell_1$. Apply Theorem 15. $\square$

We use the following notation. Let $A: X \to \ell_\infty$ be an operator and M be a nonempty subset of $\mathbb{N}$. We define $A_M: X \to \ell_\infty$ by

$$A_M(x) = \sum_{n \in M} e_n^*(A(x)) e_n, \qquad x \in X.$$

A closed operator ideal $\mathcal{O}$ *has property* $(*)$ if whenever X is a Banach space and $A \notin \mathcal{O}(X, \ell_\infty)$, then there is an infinite subset M_0 of $\mathbb{N}$ such that $A_M \notin \mathcal{O}(X, \ell_\infty)$ for all infinite subsets M of M_0, see [1].

Theorem 17. *Let $1 < p < \infty$. The ideal of p-convergent operators has property $(*)$.*

PROOF: The idea for the proof comes from Theorem 2.17 in [1]. Let $A: X \to \ell_\infty$ be an operator which is not p-convergent. Let (x_n) be a weakly p-summable sequence in X and $\delta > 0$ such that $\|A(x_n)\| > \delta$ for each $n \in \mathbb{N}$. Let $n_1 = 1$ and choose $N_1 \in \mathbb{N}$ such that $|e_{N_1}^*(A(x_{n_1}))| > \delta$. Since $(A(x_n))$ is weakly null, $\lim_n e_{N_1}^*(A(x_n)) = 0$. Choose $n_2 > n_1$ so that $|e_k^*(A(x_n))| < \delta$ for $n \geq n_2$ and $1 \leq k \leq N_1$. Choose $N_2 > N_1$ such that $|e_{N_2}^*(A(x_{n_2}))| > \delta$. Continuing this process we obtain a subsequence (x_{n_i}) of (x_n) and an increasing sequence (N_i) of natural numbers such that $|e^*_{N_i}(A(x_{n_i}))| > \delta$ for each $i \in \mathbb{N}$. Let $M_0 = \{N_i : i = 1, 2, \ldots\}$. Note that M_0 is an infinite subset of $\mathbb{N}$ and $\|A_{M_0}(x_{n_i})\| \geq \delta$ for each $i \in \mathbb{N}$. If M is an infinite subset of M_0, then A_M is not p-convergent. Therefore the operator ideal of p-convergent operators has property $(*)$. $\square$

We note that every p-convergent operator is unconditionally converging.

Theorem 18. *Let $1 < p < \infty$. If X^* contains a V-set which is not a weakly p-L-set, then $\mathrm{C}_p(X, \ell_\infty)$ is not complemented in $\mathrm{U}(X, \ell_\infty)$.*

Proof: Let A be a V-subset of X^* which is not a weakly p-L-set. Let (x^*_n) be a sequence in A and (x_n) be a weakly p-summable sequence in X such that $|x^*_n(x_n)| \not\to 0$. Without loss of generality assume that for some $\epsilon > 0$, $|x^*_n(x_n)| > \epsilon$ for all n. Define $S\colon X \to \ell_\infty$ by $S(x) = (x^*_n(x))_n$, $x \in X$. Since $\|S(x_n)\| > \epsilon$, S is not p-convergent. Let $X_0 = [x_n]$ be the closed linear span of $\{x_n : n \in \mathbb{N}\}$. Note that X_0 is a separable subspace of X and $S|_{X_0}$ is not p-convergent. By Theorem 17, there is an infinite subset M_0 of $\mathbb{N}$ so that $S_M \notin \mathrm{C}_p(X_0, \ell_\infty)$ for all infinite subsets M of M_0.

Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by $T(b)(x) = (b_n\, x^*_n(x))_n$, $b = (b_n) \in \ell_\infty$, $x \in X$. Note that the operator T is well-defined and $T(e_n) = x^*_n \otimes e_n$ for each $n \in \mathbb{N}$. Let $b \in \ell_\infty$ and suppose that $\sum x_m$ is weakly unconditionally convergent in X. Since (x^*_n) is a V-set,

$$\lim_m \|T(b)(x_m)\| = \lim_m \sup_n |b_n\, x^*_n(x_m)| = 0,$$

and thus $T(b)$ is unconditionally converging.

Suppose that $\mathrm{C}_p(X, \ell_\infty)$ is complemented in $\mathrm{U}(X, \ell_\infty)$. Let $P\colon \mathrm{U}(X, \ell_\infty) \to \mathrm{C}_p(X, \ell_\infty)$ be a projection, and let $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\varphi\colon \ell_\infty \to \mathrm{U}(X_0, \ell_\infty)$ by $\varphi(b) = RT(b)$ and $\psi\colon \ell_\infty \to \mathrm{C}_p(X_0, \ell_\infty)$ by $\psi(b) = RPT(b)$. Since $T(e_n)$ is a rank one operator,

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \varphi(e_n)$$

for each $n \in \mathbb{N}$. By Lemma 9, there is an infinite subset M of M_0 such that $\psi(\chi_M) = \varphi(\chi_M)$. Therefore $\varphi(\chi_M)$ is p-convergent. Nevertheless, $\varphi(\chi_M) = T(\chi_M)|_{X_0} = S_M$. This contradiction proves that $\mathrm{C}_p(X, \ell_\infty)$ is not complemented in $\mathrm{U}(X, \ell_\infty)$. $\square$

Corollary 19. *Let $1 < p < \infty$. If X does not have the DPP_p, then $\mathrm{C}_p(X, \ell_\infty)$ is not complemented in $\mathrm{U}(X, \ell_\infty)$.*

Proof: Since X does not have the DPP_p, there exist a weakly p-summable sequence (x_n) in X and a weakly null sequence (x^*_n) in X^* such that $x^*_n(x_n) \not\to 0$. Since (x^*_n) is weakly null, it is a V-set, see [21]. Thus (x^*_n) is a V-subset of X^* which is not a weakly p-L-set. Apply Theorem 18. $\square$

Corollary 20. *Let $1 < p < \infty$. If X does not have the* DPP_p *and $\ell_\infty \hookrightarrow Y$, then* $\mathrm{C}_p(X, Y)$ *is not complemented in* $\mathrm{U}(X, Y)$.

Corollary 21. *Let $1 < p < \infty$. Suppose X has property (V). If $\ell_\infty \hookrightarrow Y$, then the following are equivalent:*

(i) *X has the* DPP_p;
(ii) $\mathrm{C}_p(X, Y) = \mathrm{U}(X, Y)$;
(iii) $\mathrm{C}_p(X, Y)$ *is complemented in* $\mathrm{U}(X, Y)$.

PROOF: (i) $\Rightarrow$ (ii) Every unconditionally converging operator $T\colon X \to Y$ is p-convergent, since X has property (V) and the DPP_p. Since $\mathrm{C}_p(X, Y) \subseteq \mathrm{U}(X, Y)$, it follows that $\mathrm{C}_p(X, Y) = \mathrm{U}(X, Y)$.

(iii) $\Rightarrow$ (i) by Corollary 20. □

Theorem 22. *Let $1 < p < \infty$. If X^* contains a weakly p-L-set which is not an L-set, then* $\mathrm{CC}(X, \ell_\infty)$ *is not complemented in* $\mathrm{C}_p(X, \ell_\infty)$.

PROOF: Let A be a weakly p-L-set which is not an L-set. Let (x_n^*) be a sequence in A and (x_n) be a weakly null sequence in X such that for some $\epsilon > 0$, $|x_n^*(x_n)| > \epsilon$ for all n. Define $S\colon X \to \ell_\infty$ by $S(x) = (x_n^*(x))_n$, $x \in X$. Since $\|S(x_n)\| > \epsilon$, S is not completely continuous. Let $X_0 = [x_n]$ be the closed linear span of $\{x_n : n \in \mathbb{N}\}$. Note that X_0 is a separable subspace of X and $S|_{X_0}$ is not completely continuous. By [1, Theorem 2.17], the ideal of completely continuous operators has property $(*)$. Let M_0 be an infinite subset of $\mathbb{N}$ so that $S_M \notin \mathrm{CC}(X_0, \ell_\infty)$ for all infinite subsets M of M_0.

Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by $T(b)(x) = (b_n\, x_n^*(x))_n$, $b = (b_n) \in \ell_\infty$, $x \in X$. Note that the operator T is well-defined and $T(e_n) = x_n^* \otimes e_n$ for each $n \in \mathbb{N}$. Since (x_n^*) is a weakly p-L-set, $T(b)$ is p-convergent.

Suppose that $\mathrm{CC}(X, \ell_\infty)$ is complemented in $\mathrm{C}_p(X, \ell_\infty)$. Let $P\colon \mathrm{C}_p(X, \ell_\infty) \to \mathrm{CC}(X, \ell_\infty)$ be a projection, and let $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\varphi\colon \ell_\infty \to \mathrm{C}_p(X_0, \ell_\infty)$ by $\varphi(b) = RT(b)$ and $\psi\colon \ell_\infty \to \mathrm{CC}(X_0, \ell_\infty)$ by $\psi(b) = RPT(b)$. Since $T(e_n)$ is a rank one operator,

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \varphi(e_n)$$

for each $n \in \mathbb{N}$. By Lemma 9, there is an infinite subset M of M_0 such that $\psi(\chi_M) = \varphi(\chi_M)$. Hence $\varphi(\chi_M)$ is completely continuous. However, $\varphi(\chi_M) = T(\chi_M)|_{X_0} = S_M$. This is a contradiction that completes the proof. □

Corollary 23. *Let $1 < p < \infty$. If $X \in \mathrm{C}_p$ and X does not have the Schur property, then* $\mathrm{CC}(X, \ell_\infty)$ *is not complemented in* $\mathrm{C}_p(X, \ell_\infty)$.

PROOF: Since $X \in \mathrm{C}_p$ and X does not have the Schur property, B_{X^*} is a weakly p-L-set which is not an L-set. Apply Theorem 22. □

Tsirelson's space T is reflexive and $T \in \mathrm{C}_p$ for all $p < \infty$, see [3]. Thus T satisfies the hypothesis of the previous corollary.

Corollary 24. *Let $1 < p < \infty$. Suppose $X \in \mathrm{C}_p$ and X does not have the Schur property. If $\ell_\infty \hookrightarrow Y$, then $\mathrm{CC}(X, Y)$ is not complemented in $\mathrm{C}_p(X, Y)$.*

Theorem 25. *Let $1 < p < \infty$. If $X \notin \mathrm{C}_p$, then $\mathrm{C}_p(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$.*

Proof: Suppose that (x_n) is a weakly p-summable normalized sequence in X. Since (x_n) is weakly null and normalized, we can assume that it is a basic sequence (by the Bessaga-Pełczynski selection principle, see [5]). Let $X_0 = [x_n]$ and let (x_n^*) be the associated sequence of coefficient functionals. For each $n \in \mathbb{N}$, let $f_n^* \in X^*$ be a Hahn-Banach extension of x_n^*. Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by $T(b)(x) = (b_n f_n^*(x))$, $b = (b_n) \in \ell_\infty$, $x \in X$. Note that the operator T is well-defined and $T(e_n) = f_n^* \otimes e_n$ for each $n \in \mathbb{N}$.

Suppose that $\mathrm{C}_p(X, \ell_\infty)$ is complemented in $L(X, \ell_\infty)$. Let $P\colon L(X, \ell_\infty) \to \mathrm{C}_p(X, \ell_\infty)$ be a projection, and let $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\varphi\colon \ell_\infty \to L(X_0, \ell_\infty)$ by $\varphi(b) = RT(b)$ and $\psi\colon \ell_\infty \to \mathrm{C}_p(X_0, \ell_\infty)$ by $\psi(b) = RPT(b)$. Note that $\varphi(e_n) = x_n^* \otimes e_n = \psi(e_n)$ for each $n \in \mathbb{N}$. By Lemma 9, there is an infinite subset M of $\mathbb{N}$ such that $\psi(\chi_M) = \varphi(\chi_M)$. Hence $\varphi(\chi_M)$ is p-convergent. However, $\varphi(\chi_M)(x_n) = e_n$, $n \in M$, a contradiction. □

A Banach space X has the Gelfand-Phillips property if and only if every limited weakly null sequence in X is norm null, see [8].

Theorem 26. (i) *If X does not have the Gelfand-Phillips property, then $\mathrm{Lcc}(X, \ell_\infty)$ is not complemented in $\mathrm{L}(X, \ell_\infty)$.*

(ii) *Let $1 < p < \infty$. If X does not have the p-GP property, then $\mathrm{LC}_p(X, \ell_\infty)$ is not complemented in $\mathrm{L}(X, \ell_\infty)$.*

Proof: (i) Let (x_n) be a limited weakly null sequence in X of norm one. The proof is similar to that of Theorem 25.

(ii) Let (x_n) be a limited weakly p-summable sequence in X of norm one. The proof is similar to that of Theorem 25. □

Corollary 27. *Let $1 < p < \infty$. Suppose $\ell_\infty \hookrightarrow Y$.*

(i) *If $X \notin \mathrm{C}_p$, then $\mathrm{C}_p(X, Y)$ is not complemented in $\mathrm{L}(X, Y)$.*
(ii) *If X does not have the Gelfand-Phillips property, then $\mathrm{Lcc}(X, Y)$ is not complemented in $\mathrm{L}(X, Y)$.*
(iii) *If X does not have the p-GP property, then $\mathrm{LC}_p(X, Y)$ is not complemented in $\mathrm{L}(X, Y)$.*

Corollary 28. *Suppose X and Y are Banach spaces, $\ell_\infty \hookrightarrow Y$, and $1 < p < \infty$. Then the following are equivalent:*

(1) (i) *$X \in \mathrm{C}_p$;*
(ii) *$\mathrm{C}_p(X, Y) = \mathrm{L}(X, Y)$;*
(iii) *$\mathrm{C}_p(X, Y)$ is complemented in $\mathrm{L}(X, Y)$.*

(2) (i) *X has the Gelfand-Phillips property;*
(ii) $\mathrm{Lcc}(X, Y) = \mathrm{L}(X, Y)$*;*
(iii) $\mathrm{Lcc}(X, Y)$ *is complemented in* $L(X, Y)$*.*

(3) (i) *X has the p-GP property;*
(ii) $\mathrm{LC}_p(X, Y) = \mathrm{L}(X, Y)$*;*
(iii) $\mathrm{LC}_p(X, Y)$ *is complemented in* $\mathrm{L}(X, Y)$*.*

Proof: (i) $\Rightarrow$ (ii) (1) If $X \in \mathrm{C}_p$, then every operator $T\colon X \to Y$ is p-convergent. (2) If X has the Gelfand-Phillips property, then every operator $T\colon X \to Y$ is limited completely continuous. (3) If X has the p-GP property, then every operator $T\colon X \to Y$ is limited p-convergent.

(iii) $\Rightarrow$ (i) by Corollary 27. □

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