Isometric embeddings of a class of separable metric spaces into Banach spaces

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Abstract. Let (M,d) be a bounded countable metric space and $c>0$ a constant, such that $d(x,y)+d(y,z)-d(x,z)\ge c$, for any pairwise distinct points x,y,z of M. For such metric spaces we prove that they can be isometrically embedded into any Banach space containing an isomorphic copy of ℓ_∞.

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Introduction

Let (M,d) be a metric space; following [4] we will call it *concave*, when the triangle inequality is strict, i.e., when $d(x,y)+d(y,z)>d(x,z)$ for any pairwise distinct points x,y,z of M.

In this note we are interested in (concave) metric spaces satisfying the stronger property: there is a constant $c>0$ such that $d(x,y)+d(y,z)-d(x,z)\ge c$ for any pairwise distinct points x,y,z. Let us call these spaces *strongly concave* metric spaces.

The main result we prove is an infinite dimensional version of Theorem 4.3 of [4], that is, if a Banach space X contains an isomorphic copy of ℓ_∞, then X contains isometrically any bounded countable strongly concave metric space (Theorem 2). An immediate consequence of this result is that any Banach space containing an isomorphic copy of c_0 admits an infinite equilateral set (Theorem 3). This result was first proved (by similar methods) in [5, Theorem 2].

A subset S of a metric space (M,d) is said to be equilateral, if there is a $\lambda>0$ such that for $x\neq y\in S$ we have $d(x,y)=\lambda$; we also call S a λ-equilateral set (see [8]).

If X is any (real) Banach space, then B_X and S_X denote its closed unit ball and unit sphere respectively. X is said to be strictly convex, if for any $x\neq y\in S_X$ we have $\|x+y\|<2$. The Banach-Mazur distance between two isomorphic Banach spaces X and Y is $d(X,Y)=\inf\{\|T\|\,\|T^{-1}\|\colon T \text{ is an isomorphism}\}$.

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Strongly concave metric spaces

We start by presenting some examples of concave metric spaces.

Examples 1. (1) a) Let (M, d) be a discrete metric space (i.e. $d(x, y) = 1$ when $x \neq y$). Clearly $1 = d(x, z) < d(x, y) + d(y, z) = 2$ for any pairwise distinct triplet $x, y, z \in M$. Therefore (M, d) is a concave metric space. In particular, every λ-equilateral subset of any metric space is a concave metric space.

b) More generally, every *ultrametric* space is concave. This holds since for any x, y, z pairwise distinct points we have $d(x, z) \leq \max\{d(x, y), d(y, z)\} < d(x, y) + d(y, z)$.

(2) Let $(X, \|\cdot\|)$ be a strictly convex Banach space. As is well known, if x, y, z are non collinear points of X then $\|x - z\| < \|x - y\| + \|y - z\|$.

It then follows that the unit sphere S_X and every affinely independent subset A of X with the norm metric are concave metric spaces (in any case no three pairwise distinct points are collinear).

(3) Let $(X, \|\cdot\|)$ be a Banach space and $A \subseteq B_X$ such that $x \neq y \in A \Rightarrow \|x - y\| > 1$ (see [3]). Then for any x, y, z pairwise distinct points of A we have $\|x - y\| + \|y - z\| - \|x - z\| > 1 + 1 - \|x - z\| \geq 1 + 1 - 2 = 0$. Hence A with the norm metric is concave.

(4) Let (M, d) be any metric space and $p \in (0, 1)$. Then it is rather easy to show that d^p is a concave metric on M. This follows from the fact that given $a, b, c > 0$ with $a \leq b + c$ then $a^p < b^p + c^p$. The metric d^p is then called the snowflaked version of d (see [6]).

We are interested in concave metric spaces (M, d) satisfying the stronger property: there is a constant $c > 0$ such that for any pairwise distinct points x, y, z of M we have $d(x, y) + d(y, z) - d(x, z) \geq c$, equivalently $d(x, z) + c \leq d(x, y) + d(y, z)$. Let us call these spaces *strongly concave* spaces.

Lemma 1. *Every strongly concave metric space is separated (or uniformly discrete).*

Proof: Assume that (M, d) is a c-strongly concave metric space. We claim that $x \neq y \in M \Rightarrow d(x, y) \geq c/2$. Assume for the purpose of contradiction that there is a pair $\{x, y\} \subseteq M$ with $d(x, y) < c/2$. Let also $z \in M \setminus \{x, y\}$. We then have $d(x, y) + d(y, z) \leq d(x, y) + (d(y, x) + d(x, z)) = 2d(x, y) + d(x, z) \Rightarrow d(x, y) + d(y, z) - d(x, z) \leq 2d(x, y) < 2c/2 = c$. The last inequality clearly contradicts the fact that M is c-strongly concave. □

The following are examples of strongly concave metric spaces.

Examples 2. (1) Every finite concave metric space is clearly strongly concave.

(2) Let A be a λ-equilateral subset of any metric space (M, d). For any pairwise distinct points x, y, z of A we have $d(x, y) + d(y, z) - d(x, z) = \lambda + \lambda - \lambda = \lambda$, so A is a λ-strongly concave metric subspace of (M, d).

(3) Let $(X, \|\cdot\|)$ be a Banach space. Also let $A \subseteq B_X$ with the property that $x \neq y \in A \Rightarrow \|x-y\| \geq 1+\varepsilon$, where $\varepsilon > 0$ is a constant. Then we have $\|x-y\| + \|y-z\| - \|x-z\| > (1+\varepsilon) + (1+\varepsilon) - 2 = 2\varepsilon$ (cf. Examples 1 (3)). Therefore A with the norm metric is a 2ε-strongly concave metric space.

Note that if $\dim X = \infty$, then by a result of J. Elton and E. Odell (see [2]) there is $A \subseteq S_X$ infinite and $\varepsilon > 0$ such that $x \neq y \in A \Rightarrow \|x-y\| \geq 1+\varepsilon$.

Remarks 1. (1) Clearly every separable strongly concave metric space M is at most countable (this is so because M is separated, hence it has the discrete topology).

(2) Every subspace of a concave (or strongly concave) space has the same property.

The following result is classical (see [6]).

Theorem 1 (Fréchet)**.** *Every separable metric space (M, d) embeds isometrically into ℓ_∞.*

Proof: Let $(x_n) \subseteq M$ be a dense sequence in M. Then the map

$$\varphi\colon x \in M \mapsto (d(x, x_n) - d(x_1, x_n))_{n \geq 1} \in \ell_\infty$$

satisfies our claim. $\square$

Remark 2. Let (M, d) be a separable metric space. We define a map

$$\sigma\colon M \to \mathbb{R}^{\mathbb{N}} \qquad \text{with } \sigma(x) = (d(x, x_n))_{n \geq 1}$$

where (x_n) is any dense sequence in M. Then the Fréchet embedding of M into ℓ_∞ is the map

$$\varphi(x) = \sigma(x) - \sigma(x_1), \qquad x \in X.$$

Note that if the space (M, d) is bounded (i.e., there is $k > 0$ such that $d(x, y) \leq k$ for all $x, y \in M$), then the map σ is already an isometric embedding of M into ℓ_∞, which we will still call the Fréchet embedding of M into ℓ_∞.

Proposition 1. *Let (M, d) be a bounded countable infinite metric space. Then there is an infinite subset N of M such that the Fréchet embedding of N into ℓ_∞ takes values into the space* $\mathbf{c}$.

Proof: Let $\{x_1, x_2, \dots, x_n, \dots\}$ be a one-to-one enumeration of M. Then $\sigma(x_k) = (d(x_k, x_n))_{n \geq 1} \in \ell_\infty$ for $k \in \mathbb{N}$, since d is a bounded metric. We construct by induction a subsequence $\{x'_1, x'_2, \dots, x'_n, \dots\}$ of (x_n) satisfying our claim.

Since $(d(x_1, x_n))_{n \geq 1}$ is a bounded sequence of real numbers, there is $A_1 \subseteq \mathbb{N}$ infinite, such that $d(x_1, x_n) \xrightarrow{n \in A_1} \alpha_1$. Set $n_1 = 1$.

Let $n_2 = \min A_1$ for which we may assume that $n_2 > n_1$. Then for the sequence $(d(x_{n_2}, x_n))_{n \in A_1}$, there is $A_2 \subseteq A_1$ infinite with $n_3 = \min A_2 > n_2$ such that $d(x_{n_2}, x_n) \xrightarrow{n \in A_2} \alpha_2$.

Then for the sequence $(d(x_{n_3}, x_n))_{n\in A_2}$, there is $A_3 \subseteq A_2$ infinite with $n_4 = \min A_3 > n_3$ such that $d(x_{n_3}, x_n) \xrightarrow{n\in A_3} \alpha_3$.

The inductive process should be clear. Now set a metric space $A = \{n_1 < n_2 < \cdots < n_k < \dots\}$. Clearly $\{n_k, n_{k+1}, \dots\} \subseteq A_k$ for $k \geq 1$ and hence $d(x_{n_k}, x_n) \xrightarrow{n\in A} \alpha_k$ for all $k \geq 1$. It is clear that the set $N = \{x'_k = x_{n_k} : k \geq 1\}$ satisfies our requirements. $\square$

The following theorem is the main result of this note; its proof resembles the proof of Theorem 4.3 of [4] and the proof of Theorem 2 of [5] (we use Schauder's fixed point theorem in the same way we did in [5]). The origins of these ideas can be traced in P. Braß (see [1] and [8]) and K. J. Swanepoel and R. Villa (see [9] and [10]).

Theorem 2. *Let X be any Banach space containing an isomorphic copy of ℓ_∞. Then X contains isometrically any bounded separable strongly concave metric space.*

Proof: We shall use a kind of non distortion property of ℓ_∞ proved independently by M. Talagrand (see [11]) and J.R. Partington (see [7]). Let us denote by $\|\cdot\|_\infty$ the usual norm of ℓ_∞.

Claim. *Let (M, d) be any bounded separable strongly concave metric space. There is $\delta > 0$, such that if $\|\cdot\|$ is any equivalent norm on ℓ_∞ with Banach Mazur distance*

$$d((\ell_\infty, \|\cdot\|_\infty), (\ell_\infty, \|\cdot\|)) \leq 1 + \delta$$

then the space (M, d) embeds isometrically into $(\ell_\infty, \|\cdot\|)$.

Proof of the Claim: Since (M, d) is strongly concave, there is $\eta > 0$ such that $d(x, y)+d(y, z)-d(x, z) \geq \eta$ for each triplet x, y, z of pairwise distinct points of M. We may assume that $\|x\| \leq \|x\|_\infty \leq (1 + \delta)\|x\|$ for $x \in \ell_\infty$, where $\delta > 0$ is to be determined.

Let $I = \{(m, n) : n < m,\ n, m \in \mathbb{N}\}$; denote by K the compact cube $[0, \eta]^I$. Since M is (strongly concave and) separable, it is at most countable, so let $M = \{x_1, x_2, \dots, x_n, \dots\}$. For $\varepsilon = (\varepsilon_{(m,n)}) \in K$ set

$$\begin{aligned}
p_1(\varepsilon) &= (d(x_1, x_1) - d(x_1, x_1), d(x_1, x_2) - d(x_1, x_2), \dots, d(x_1, x_n) \\
&\qquad - d(x_1, x_n), \dots) \\
&= (0, \dots, 0, \dots) \\
p_2(\varepsilon) &= (d(x_2, x_1) - d(x_1, x_1) + \varepsilon_{(2,1)}, d(x_2, x_2) - d(x_1, x_2), \dots, \\
&\qquad d(x_2, x_n) - d(x_1, x_n), \dots) \\
&\ \ \vdots
\end{aligned}$$

$$
\begin{aligned}
p_n(\varepsilon) = (d(x_n, x_1) - d(x_1, x_1) + \varepsilon_{(n,1)}, \dots, d(x_n, x_{n-1})\\
- d(x_1, x_{n-1}) + \varepsilon_{(n,n-1)}, d(x_n, x_n) - d(x_1, x_n), \dots)\\
\vdots
\end{aligned}
$$

(Note that $x_n \mapsto p_n(0)$ is the Fréchet embedding of M into $(\ell_\infty, \|\cdot\|_\infty)$).

For $n < m$ we have

$$\|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = \sup_k \left| d(x_n, x_k) + \varepsilon_{(n,k)} - (d(x_m, x_k) + \varepsilon_{(m,k)}) \right|$$

where we set $\varepsilon_{(k,l)} = 0$ for $l \geq k$. This supremum is equal to $d(x_n, x_m) + \varepsilon_{(m,n)}$ as for $k \neq n, m$ we have

$$d(x_n, x_k) - d(x_m, x_k) + \varepsilon_{(n,k)} - \varepsilon_{(m,k)} \leq d(x_n, x_m) - \eta + \varepsilon_{(n,k)} - \varepsilon_{(m,k)} \leq d(x_n, x_m).$$

We define a function

$$\varepsilon = (\varepsilon_{(m,n)}) \in K \overset{\varphi}{\longmapsto} \varphi(\varepsilon) = (\varphi_{(m,n)}(\varepsilon)) \in K,$$

by the rule $\varphi_{(m,n)}(\varepsilon) = d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|$. Note that $\varphi_{(m,n)}(\varepsilon) \geq d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = 0$ (using the computation above and the fact that the norm $\|\cdot\|_\infty$ dominates $\|\cdot\|$). We also have

$$
\begin{aligned}
d(x_n, x_m) + \varepsilon_{(m,n)} = \|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty \leq (1+\delta)\|p_n(\varepsilon) - p_m(\varepsilon)\|\\
\Rightarrow \frac{1}{1+\delta}(d(x_n, x_m) + \varepsilon_{(m,n)}) \leq \|p_n(\varepsilon) - p_m(\varepsilon)\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\varphi_{(m,n)}(\varepsilon) &= d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|\\
&\leq d(x_n, x_m) + \varepsilon_{(m,n)} - \frac{1}{1+\delta}(d(x_n, x_m) + \varepsilon_{(m,n)})\\
&= \frac{\delta}{1+\delta}(d(x_n, x_m) + \varepsilon_{(m,n)}).
\end{aligned}
$$

It then follows from (this inequality and) the fact that M is bounded that if δ is quite small, then $\varphi_{(m,n)}(\varepsilon) \leq \eta$ for $\varepsilon \in K$.

Since each coordinate function $\varphi_{(m,n)}$ is continuous (as dependent on finite coordinates, i.e., from the set $\{(k,l)\colon 1 \leq l < k \leq m\}$) it follows that φ is also continuous. By a classical result of Schauder, φ has a fixed point $\varepsilon' = (\varepsilon'_{(m,n)}) \in K$, that is $\varphi(\varepsilon') = \varepsilon'$, which implies $\|p_n(\varepsilon') - p_m(\varepsilon')\| = d(x_n, x_m)$ for all $n, m \in \mathbb{N}$. The proof of the Claim is complete. $\square$

Denote by $\|\cdot\|$ the norm of X and let Y be a subspace of X isomorphic to ℓ_∞. By the non distortion property of $(\ell_\infty, \|\cdot\|_\infty)$ there is a subspace $Z \subseteq Y$ (isomorphic to ℓ_∞) such that

$$d((Z,\|\cdot\|),(\ell_\infty,\|\cdot\|_\infty)) \le 1+\delta$$

(this is the $\delta > 0$ postulated in the Claim). It follows immediately from the Claim that the space $(Z,\|\cdot\|)$ contains an isometric copy of (M,d). $\square$

In the special case when (M,d) is the countable infinite discrete metric space we get the following result first proved in [5, Theorem 2], essentially with the same method.

Theorem 3. *Every Banach space X containing an isomorphic copy of c_0 admits an infinite equilateral set.*

Proof: Take in the proof of the previous theorem (M,d) to be the countable infinite discrete space. Then $\eta = 1$ and the resulting family $(p_n(\varepsilon))_{n\ge1}$, $\varepsilon \in K = [0,1]^I$ takes values in c_0 (remember that $x_n \mapsto p_n(0)$ is the Fréchet embedding of (M,d) into c_0). Since $(c_0,\|\cdot\|_\infty)$ is non distortable, we get the conclusion. $\square$

Theorem 2 can be improved in the following way.

Theorem 4. *Let (M,d) be an infinite bounded separable strongly concave metric space. Then there is $N \subseteq M$ infinite such that the metric space (N,d) can be isometrically embedded into any Banach space containing an isomorphic copy of the space c_0.*

Proof: By Proposition 1, there is $N \subseteq M$ infinite such that the Fréchet embedding $\sigma\colon N \to \ell_\infty$ takes values into $\mathbf{c}$. Then the proof of Theorem 2 gives us a family of embeddings $(p_n(\varepsilon))_{n\ge1}$, $\varepsilon \in K = [0,\eta]^I$ taking values into $\mathbf{c}$. Since $\mathbf{c}$ is isomorphic to c_0, we are done. $\square$

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