On dihedral 2-groups as inner mapping groups of finite commutative inverse property loops

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Abstract. We show that finite commutative inverse property loops may not have nonabelian dihedral 2-groups as their inner mapping group.

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1. Introduction

If Q is a loop then the mappings $L_a(x) = ax$ and $R_a(x) = xa$ are called the left and right translation. These two mappings are permutations on Q for every $a \in Q$ and the permutation group $M(Q) = \langle L_a, R_a \colon a \in Q\rangle$ is called the multiplication group of Q. The stabilizer of the neutral element of Q is the inner mapping group of Q and we denote it by $I(Q)$. If Q is a group then $I(Q) = \mathrm{Inn}(Q)$, the group of inner automorphisms of Q. We know that in the case of commutative groups $\mathrm{Inn}(Q)$ is always trivial whereas noncommutative groups have nontrivial inner automorphism groups. For example, there are three nonisomorphic groups of order 16, whose inner automorphism group is the dihedral group of order 8. We also know that there exist nonassociative and noncommutative loops of order 16 whose inner mapping group is the dihedral group of order 8 (private communication from Michael Kinyon). What about the situation in finite commutative loops? In [2, pages 357–358], Drápal introduces a construction of finite commutative loops of order $2m$, $m \geq 3$, such that the inner mapping group $I(Q)$ is the dihedral group of order $2m$.

A loop Q is an inverse property loop if Q has a unique left and right inverse x^{-1} and $x^{-1}(xy) = y = (yx)x^{-1}$ for every $x, y \in Q$. The purpose of this paper is to show that in the case of finite commutative inverse property loops, nonabelian dihedral 2-groups do not appear as inner mapping groups of these loops.

We consider only finite loops and groups in this paper. The proofs of our main theorems rely on the use of connected transversals in finite groups and this notion and some basic results about these transversals are explained in the following section. For basic facts about loop theory and its connections to group theory the reader is advised to consult [1] and [5].

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2. Loops and groups: some results

Our proofs are based on the use of connected transversals in a group (this group turns out to be the multiplication group $M(Q)$ mentioned in the introduction) and therefore we shall start with a brief discussion about connected transversals and try to give some insight into the relationship between loops and groups given by this notion.

Let G be a group and $H \leq G$. If A and B are two left transversals to H in G and $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and for every $b \in B$, then we say that the two transversals are H-connected in G. If $A = B$, then we say that A is a selfconnected transversal to H in G. In the following lemmas and theorems we consider some basic properties of H-connected transversals A and B. We denote by H_G the core of H in G (it is the largest normal subgroup of G contained in H).

Lemma 2.1. *If $H_G = 1$, then $1 \in A \cap B$ and $AZ(G) \subseteq A$ and $BZ(G) \subseteq B$.*

For the proof, see [5, page 113] and [4, Lemma 2.3].

From Lemma 2.1 it follows that if $H_G = 1$, then $Z(G) \subseteq A \cap B$.

Lemma 2.2. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

For the proof, see [5, Proposition 2.7].

Theorem 2.3. *Let H be a nilpotent subgroup of G. If $G = \langle A, B \rangle$ and $H_G = 1$, then H is subnormal in G and $Z(G) > 1$.*

For the proof, see [3, Theorem 2.8].

Theorem 2.4. *If H is cyclic and $G = \langle A, B \rangle$, then $G' \leq H$.*

For the proof, see [5, Theorem 3.5].

Theorem 2.5. *Let p be a prime number. If $H \cong C_p \times C_p$ and $G = \langle A, B \rangle$, then $G' \leq N_G(H)$.*

For the proof, see [6, Lemma 4.2].

Next we prove

Lemma 2.6. *Let $G = \langle A, B \rangle$. If H is nilpotent and $H_G = 1$, then the core of $HZ(G)$ in G properly contains $Z(G)$.*

Proof: By Lemma 2.2 and Theorem 2.3, $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. If the core of $HZ(G)$ in G equals $Z(G)$, then from Lemma 2.2 it follows that

$$N_{G/Z(G)}(HZ(G)/Z(G)) = HZ(G)/Z(G) \times Z(G/Z(G)).$$

Now we write $Z(G/Z(G)) = M/Z(G)$ and thus $N_G(HZ(G)) = HM$, where M is normal in G, $Z(G)$ is a proper subgroup of M and $H \cap M = 1$. Then $HM = CH = DH$, where $C \subseteq A$ and $D \subseteq B$. By Lemma 2.1,

$$Z(G/Z(G)) \subseteq AZ(G)/Z(G) \cap BZ(G)/Z(G).$$

We conclude that $M \subseteq CZ(G) \cap DZ(G)$. If $m \in M$, then $m = cz = dr$, where $c \in A$, $d \in B$ and z and r are elements from $Z(G)$. If $t \in A \cup B$, then $[t, m] \in M \cap H = 1$. Thus $C_G(M) \geq \langle A, B \rangle = G$. But then $M = Z(G)$, a contradiction. □

We shall conclude this section by establishing the relation between connected transversals and loop theory. If $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are the sets of left and right translations, then A and B are $I(Q)$-connected transversals in $M(Q)$. Since $M(Q)$ is transitive on Q, it follows that the core of $I(Q)$ in $M(Q)$ is trivial. Kepka and Niemenmaa proved the following theorem in 1990 [5, Theorem 4.1].

Theorem 2.7. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H of G satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B \rangle$.*

If Q is a commutative loop, then $A = B$. Furthermore, if Q is a commutative inverse property loop, then $(L_a)^{-1} = L_{a^{-1}}$ and thus $A = A^{-1}$.

3. Main theorems

In this section we consider the situation that $A = B$, $A = A^{-1}$ and H is a nonabelian dihedral 2-group. We first consider the case where $|H| = 8$.

Theorem 3.1. *Let $H \leq G$ be a dihedral group of order 8. If A is a self-connected transversal to H in G, $G = \langle A \rangle$ and $A = A^{-1}$, then $H_G > 1$.*

Proof: Assume that $H_G = 1$. From Theorem 2.3 it follows that $Z(G) > 1$ and by Lemma 2.6, the core of $HZ(G)$ in G is equal to $KZ(G)$, where $1 < K \leq H$. If $|K| \geq 4$, then $HZ(G)/KZ(G)$ is cyclic and by Theorem 2.4, $G' \leq HZ(G)$. This means that $HZ(G)$ is normal in G. As $(HZ(G))' = H' \neq 1$, we get $H_G > 1$, a contradiction.

Thus we may assume that $|K| = 2$. Clearly, $K = Z(H) = H'$. We then consider $G/KZ(G)$ and the subgroup $HZ(G)/KZ(G)$. Now $HZ(G)/KZ(G) \cong C_2 \times C_2$. By Theorem 2.5 and Lemma 2.2,

$$(G/KZ(G))' \leq N_{G/KZ(G)}(HZ(G)/KZ(G)) = HZ(G)/KZ(G) \times Z(G/KZ(G)).$$

It follows that $G' \leq HM$, where $M/KZ(G) = Z(G/KZ(G))$. Clearly, HM and M are normal in G and $H \cap M = K$. We may also conclude that $M = CK$, where $Z(G) \subseteq C \subseteq A$. By using Lemma 2.1, we get

$$AKZ(G)/KZ(G) \cdot CKZ(G)/KZ(G) \subseteq AKZ(G)/KZ(G).$$

It follows that $CA \subseteq AK$. Then let $c \in C$ and $a \in A$. Now $ca = bh$, where $b \in A$ and $h \in K$. Thus $h = b^{-1}ca$ and $h^{a^{-1}} = (b^{-1}ca)^{a^{-1}} = ab^{-1}caa^{-1} = ab^{-1}a^{-1}bb^{-1}ac = [a^{-1}, b]b^{-1}cal = [a^{-1}, b]hl$, where $l \in H$. As $A = A^{-1}$ and

$KZ(G)$ is normal in G, we conclude that $h^{a^{-1}} \in K^{a^{-1}} \cap H \leq KZ(G) \cap H = K$. If $h \neq 1$, then $h^{a^{-1}} = h$ (as $|K| = 2$) and $a \in N_G(K)$.

If $d \in A$, then $(ca)^d = (bh)^d$, hence $ckag = bfh^d = cah^{-1}fh^d$, where $k \in K$ and $g, f \in H$. It follows that $h^d = f^{-1}hk^a g \in H$. This is true for every $d \in A$, so $h \in \bigcap_{d\in A} H^{d^{-1}} = H_G = 1$.

We may now conclude that $ca = b \in A$. As $A = A^{-1}$, we also have $c^{-1}a^{-1} = w \in A$. Thus $wb = c^{-1}a^{-1}ca \in H$. But then $w = b^{-1}$ and $c^{-1}a^{-1}ca = 1$. This means that $c \in Z(\langle A\rangle) = Z(G)$ and $M = CK \leq KZ(G)$. But then $G' \leq HM \leq HZ(G)$ and $HZ(G)$ is normal in G. Now $(HZ(G))' = H' \neq 1$ and thus $H_G > 1$, a contradiction. □

We shall next consider the general situation where H is any nonabelian dihedral 2-group.

Theorem 3.2. *Let $H \leq G$ be a dihedral 2-group of order 2^n, $n \geq 3$. If A is a selfconnected transversal to H in G, $G = \langle A\rangle$ and $A = A^{-1}$, then $H_G > 1$.*

Proof: Assume that G is a minimal counterexample. By Theorem 3.1 we may assume that $n \geq 4$. As in the proof of Theorem 3.1, it follows that the core of $HZ(G)$ in G is equal to $KZ(G)$, where $1 < K \leq H$. Likewise, we may conclude that $HZ(G)/K(ZG)$ is not cyclic. If $HZ(G)/KZ(G)$ is dihedral of order 2^m, $n > m \geq 3$, then the core of $HZ(G)/KZ(G)$ in $G/KZ(G)$ is not trivial, a contradiction.

Thus we may assume that $HZ(G)/KZ(G) \cong C_2 \times C_2$ and K is cyclic of order 2^{n-2}, where $n \geq 4$. Let $T = \langle x \in KZ(G)\colon x^2 = 1\rangle$. As $KZ(G)$ is normal in G, it follows that T is normal in G. Then consider G/T and its subgroup HT/T. As HT/T is a nonabelian dihedral 2-group, we conclude that the core of HT in G properly contains T. We denote this core by E and conclude that $E = FM$, where $F \leq K$, $|F| \geq 4$ and $M \leq Z(G)$ is an elementary abelian 2-group. Then $\langle x^2\colon x \in E\rangle$ is a nontrivial normal subgroup of G contained in H and thus $H_G > 1$. The proof is complete. □

By combining Theorem 2.7 with Theorem 3.2 we get

Theorem 3.3. *Let Q be a finite commutative inverse property loop. Then the inner mapping group $I(Q)$ cannot be a nonabelian dihedral 2-group.*

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